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RIGHT SELF-INJECTIVE RINGS IN WHICH EVERY ELEMENT IS A SUM OF TWO UNITS

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A classical result of Zelinsky states that every linear transformation on a vector space V, except when V is one-dimensional over $\mathbb{Z}_2$, is a sum of two invertible linear transformations. We extend this result to any right self-injective ring R by proving that every element of R is a sum of two units if no factor ring of R is isomorphic to $\mathbb{Z}_2$.

Keywords: Right self-injective; rings generated by units; sum of two units.

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1. Introduction

In 1954 Zelinsky [16] proved that every element in the ring of linear transformations of a vector space V over a division ring D is a sum of two units unless dim $V = 1$ and $D = \mathbb{Z}_2$. Because $\mathrm{End}_D(V)$ is a (von-Neumann) regular ring, Zelinsky's result generated quite a bit of interest in regular rings that have the property that every element is a sum of (two) units. Clearly, a ring R, having $\mathbb{Z}_2$ as a factor ring, cannot have every element as a sum of two units. In 1958 Skornyakov [12, Problem 31, p. 167] asked: *Is every element of a regular ring (which does not have $\mathbb{Z}_2$ as a factor ring) a sum of units?* This question of Skornyakov was settled by Bergman (see [7]) in negative who gave an example of a directly-finite, regular ring with 2 invertible, in which not all elements are the sum of units. It is easy to see that if R is a unit regular ring with 2 invertible, then every element can be written as a sum of two units (see [3]). A number of authors have also considered arbitrary rings in which elements are the sum of units. For instance, Henriksen in [8, Theorem 3] proved that, an arbitrary ring R, every element of $M_n(R)$, $n > 1$, is a sum of three units. Henriksen also gave an example of a ring R such that not every element of $M_2(R)$ is a sum of two units [8, Example 10].

Since $\text{End}_D(V)$ is a right self-injective ring, the other natural question which arises from Zelinsky's result is the following: *Which (regular)*[a] *right self-injective rings have the property that every element is a sum of two units*? In this direction Utumi [13, Theorem 2] proved that in a regular right self-injective ring having no ideals with index of nilpotence 1, every element is a sum of units. In [11, Proposition 11] Raphael proved that if in a regular right self-injective ring R every idempotent is a sum of two units, then every element can be written as a sum of even number of units. In [2] it was proved that in a right self-injective ring with 2 invertible, every element can be written as a sum of a unit and a square root of 1. Recently, Vámos in [15, Theorem 21] proved that every element of a regular right self-injective ring is a sum of two units if the ring has no non-zero corner ring which is Boolean.[b] In this paper, we prove that every element of a right self-injective ring is a sum of two units if and only if it has no factor ring isomorphic to $\mathbb{Z}_2$. We extend this result to endomorphism rings of right quasi-continuous modules with finite exchange property (Theorem 3). Some consequences of our results are also given. For instance, it is shown that every element of the endomorphism ring of a flat cotorsion module is a sum of two units if no factor ring of the endomorphism ring is isomorphic to $\mathbb{Z}_2$. In Proposition 7 we give an interesting application of our result for group rings. The proof of our main result uses Type theory of regular right self-injective rings introduced by Kaplansky [9].

2. Definitions and Notations

All rings considered in this paper have unity and all modules are right unital. A ring R is called right self-injective if every R-homomorphism from a right ideal of R into R can be extended to an endomorphism of R. A ring R is called directly finite if $xy = 1$ implies $yx = 1$, for all $x, y \in R$. A ring R is called von-Neumann regular if every principal right (left) ideal of R is generated by an idempotent. A regular ring is called abelian if all its idempotents are central. An idempotent e in a regular ring R is called abelian idempotent if the ring eRe is abelian. An idempotent e in a regular right self-injective ring is called faithful idempotent if 0 is the only central idempotent orthogonal to e, that is, $ef = 0$ implies $f = 0$, where f is a central idempotent. A regular right self-injective ring is said to be of Type I provided it contains a faithful abelian idempotent. A regular right self-injective ring R is said to be of Type II provided R contains a faithful directly finite idempotent but R contains no non-zero abelian idempotents. A regular right self-injective ring is of Type III if it contains no non-zero directly finite idempotents. A regular right self-injective ring is of (i) Type I_f if R is of Type I and is directly finite, (ii) Type I_∞ if R is of Type I and is purely infinite i.e., $R_R \cong (R \oplus R)_R$, (iii) Type II_f if R is of Type II and is directly finite, (iv) Type II_∞ if R is of Type II and is purely infinite

[a]We are writing "regular" in brackets because for a right self-injective ring R, $R/J(R)$ is regular right self-injective, and clearly an element is a sum of k units in R if and only if it is so in $R/J(R)$.
[b]This condition is weaker than $1/2 \in R$.

(see [5, pp. 111–115]). $N \subseteq_e M$ ($N \subseteq^{\oplus} M$) will denote that N is an essential submodule (summand) of M.

For additional notations and terminology we refer the reader to [5] and [10].

3. Main Results

The following result characterizes the right self-injective rings with the property that every element is a sum of two units.

Theorem 1. *For a right self-injective ring R, the following conditions are equivalent*:

(1) *Every element of R is a sum of two units.*
(2) *Identity of R is a sum of two units.*
(3) *R has no factor ring isomorphic to $\mathbb{Z}_2$.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are obvious.

Now, we proceed to show (3) ⇒ (1).

By [14] we know that $R/J(R)$ is a regular, right self-injective ring. Since every element of R is a sum of two units if and only if every element $R/J(R)$ is a sum of two units, we may assume that R is regular. By [5, Proposition 10.21], $R \cong S \times T$, where S is purely infinite and T is directly finite. Since S is purely infinite, $S_S \cong (S \oplus S)_S$ (see [5, Theorem 10.16], and so $S \cong M_2(S)$. Now we show that every element in $M_2(S)$, where S is a regular right self-injective ring is a sum of two units. By [1, Corollary 2.6], every $A \in M_2(S)$ admits a diagonal reduction, i.e. there exist invertible matrices P and Q in $M_2(S)$ such that PAQ is a diagonal matrix, say $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$. Then $PAQ = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 1 \\ 1 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & -1 \\ -1 & b \end{smallmatrix}\right)$ is a sum of two units and so A is a sum of two units. So, every element of $M_2(S)$ and hence every element of S is a sum of two units. Since, T is a directly finite, regular, right self-injective ring, $T \cong R_1 \times R_2$ where R_1 is Type I_f and R_2 is Type II_f [5, Theorem 10.22].

First, we show that every element of R_1 is a sum of two units. Since by [5, Theorem 10.24], $R_1 \cong \Pi M_n(S_i)$ where each S_i is an abelian regular right self-injective ring, it is enough to show that each element of $M_n(S_i)$ is a sum of two units. But, if $n > 1$, then as argued above, we are through. So, it is enough to show that every element in an abelian regular ring S_i, which has no factor isomorphic to $\mathbb{Z}_2$, is a sum of two units. Let $a \in S_i$. Suppose, to the contrary, that a is not a sum of two units. Let $\Omega = \{I : I$ is an ideal of S_i and $a + I$ is not a sum of two units in $S_i/I\}$. Clearly, Ω is non-empty and it can be easily checked that Ω is inductive (for example see [4, Theorem 2]). So, by Zorn's Lemma, Ω has a maximal element, say, I. Clearly, S_i/I is an indecomposable ring and hence has no central idempotent. But, as S_i/I is abelian regular, S_i/I must be a division ring. Since, $a + I$ is not a sum of two units in S_i/I, it follows that $S_i/I \cong \mathbb{Z}_2$, a contradiction. Thus, each element of R_1 is a sum of two units.

Finally, we show that every element of R_2 is a sum of two units. Since R_2 is Type II_f, it has no non-zero abelian idempotents. Therefore, by [5, Proposition 10.28] there exists an idempotent $e \in R_2$ such that $(R_2)_{R_2} \cong (eR_2 \oplus eR_2)_{R_2}$ and so $R_2 \cong M_2(eR_2e)$, and as every element of $M_2(eR_2e)$ is a sum of two units as seen above, every element of R_2 is a sum of two units. Therefore, each element of T is a sum of two units. Hence, every element of R is a sum of two units. This completes the proof. □

Let V be a right vector space over a division ring D, then $\mathrm{End}_D(V)$ is a regular, right self-injective ring. It is easy to see that the identity of $\mathrm{End}_D(V)$ is a sum of two units, except when $\dim(V_D) = 1$ and $D = \mathbb{Z}_2$ (see [16, Lemma]). As a consequence, we get the following result.

Corollary 2 (Zelinsky, [16]). *Every element of* $\mathrm{End}_D(V)$ *is a sum of two units, except when* $\dim(V_D) = 1$ *and* $D = \mathbb{Z}_2$.

Because every right self-injective ring is an exchange right quasi-continuous ring, the following result is a generalization of Theorem 1.

Theorem 3. *Let M_S be a quasi-continuous module with finite exchange property and $R = \mathrm{End}_S(M)$. Then every element of R is a sum of two units if and only if no factor ring of R is isomorphic to $\mathbb{Z}_2$.*

Proof. Assume that no factor ring of R is isomorphic to $\mathbb{Z}_2$. Let $\Delta = \{f \in R\colon \ker f \subset_e M\}$. Then Δ is an ideal of R. By [10, Corollary 3.13], $\overline{R} = R/\Delta \cong R_1 \oplus R_2$, where R_1 is regular, right self-injective and R_2 is an exchange ring with no non-zero nilpotent element. We have already shown in Theorem 1 that each element of R_1 is a sum of two units. Since, R_2 has no non-zero nilpotent element, each idempotent in R_2 is central. Now, if any element $a \in R_2$ is not a sum of two units, then as in the proof of Theorem 1, we find an ideal I of R_2 such that $x = a + I \in R_2/I$ is not a sum of two units in R_2/I and R_2/I has no central idempotent. This implies that R_2/I is an exchange ring without any non-trivial idempotent, and hence it must be local. Let $T = R_2/I$. Then $x + J(T)$ is not a sum of two units in $T/J(T)$, which is a division ring. Therefore, $T/J(T) \cong \mathbb{Z}_2$, a contradiction. Hence, every element of R_2 is also a sum of two units. Therefore, every element of $\overline{R}$ is a sum of two units. Next, we observe that $\Delta \subseteq J(R)$. Suppose to the contrary that $\Delta \nsubseteq J(R)$, then Δ contains a non-zero idempotent, say e. But as $\ker(e) \subseteq_e M$, $\ker(e) = M$ and so $e = 0$, a contradiction. Thus $\Delta \subseteq J(R)$. Therefore, we may conclude that every element of R is a sum of two units. The converse is obvious. □

Remark 4. As continuous module is quasi-continuous and also has exchange property [10, Theorem 3.24], it follows that in the endomorphism ring of a continuous (and hence also of injective and quasi-injective) module, every element is a sum of two units if and only if no factor of the endomorphism ring is isomorphic to $\mathbb{Z}_2$.

A module M is called *pure-injective* if for any module A and any pure submodule B of A, every homomorphism $f: B \to M$ extends to a homomorphism $g: A \to M$. A module M is called *cotorsion* if $\mathrm{Ext}^1_R(F, M) = 0$ for every flat R-module F, equivalently if every short exact sequence $0 \to M \to E \to F \to 0$ with F flat, splits. The ring R is called right cotorsion (resp. right pure-injective) if R_R is cotorsion (resp. right pure-injective). By [6], if M is a flat cotorsion right R-module and $S = \mathrm{End}(M_R)$, then $S/J(S)$ is a regular, right self-injective ring.

Corollary 5. *Every element of the endomorphism ring of a flat cotorsion (in particular, pure injective) module is a sum of two units if and only if no factor ring is isomorphic to* $\mathbb{Z}_2$.

Vámos [15, Theorem 21] proves that if R is a regular right self-injective ring such that no non-zero corner ring of R is Boolean then every element of R is a sum of two units. But the condition that no non-zero corner ring is Boolean, is not necessary for every element in a regular right self-injective ring to be a sum of two units. For instance, if S is a self-injective Boolean ring, then every element of $R = M_n(S)$, $n > 1$, is a sum of two units (see the proof of Theorem 1) although R has non-zero Boolean corner rings. It may further be noted that the condition of Vámos, namely no non-zero corner ring is Boolean, is not sufficient even if we replace a right self-injective ring by a commutative continuous ring as is shown in the following example.

Example 6. Let F be a field with $\mathbb{Z}_2$ as proper prime subfield. Set $F_n = F$ and $K_n = \mathbb{Z}_2$ for each $n \in \mathbb{N}$ and set $Q = \Pi_{\mathbb{N}} F_n$. Set $R = \{x = (x_n)_{\mathbb{N}} \in Q$: $x_n \in K_n$ for all but finitely many $n\}$. Then R is continuous (see [5, Example 13.8]. As idempotents in R are just the elements with each component either 0 or 1, no non-zero corner ring of R is Boolean. But, clearly the identity of R is not a sum of two units.

We conclude by showing an application of our result for group rings.

Proposition 7. *If R is a right self-injective ring and G a locally finite group, then every element of RG is a sum of two units unless R has a factor ring isomorphic to* $\mathbb{Z}_2$.

Proof. Let α be any arbitrary element of RG then $\alpha = r_0 + r_1 g_1 + r_2 g_2 + \cdots + r_n g_n$. Let $H = \langle g_1, \ldots, g_n \rangle$ be the subgroup generated by $g_1, \ldots, g_n$. Since G is locally finite, H must be finite. Clearly, $\alpha \in RH$. Now, since R is right self-injective and H is a finite group, the group ring RH is right self-injective. Note that if R has no factor ring isomorphic to $\mathbb{Z}_2$ then the group ring RH also has no factor ring isomorphic to $\mathbb{Z}_2$. Therefore, by Theorem 1, $\alpha = u_1 + u_2$ where $u_1, u_2 \in RH$ are units. But then u_1, u_2 will be units in RG also. Hence, every element of RG is a sum of two units. □

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