What Is the Role of Algebra in Applied Mathematics?

David A. Cox

When I first encountered abstract algebra in the late 1960s, I was drawn to its power and beauty. One of the liberating ideas of the subject is that in dealing with algebraic structures, you do not need to worry about what the objects are; rather, it is how they behave that is important. Algebra is a wonderful language for describing the behavior of mathematical objects.

My fascination with algebra led me to algebraic geometry, which was then among the most abstract areas of pure mathematics. At the time, I would never have predicted that twenty-five years later I would be writing papers with computer scientists, where we use algebraic geometry and commutative algebra to solve problems in geometric modeling. The algebra that I learned as pure and abstract has come to have significant applications.

What do these and other applications imply about the relation between algebra and applied mathematics? The purpose of this essay is to explore some aspects of this relation in the hope of provoking useful discussions between pure and applied mathematics. Here, "applied mathematics" includes not only what students learn in mathematics and applied mathematics departments, but also the mathematics learned in computer science, engineeering, and operations research departments.

I begin with examples from geometric modeling, economics, and splines to illustrate the possible applications of algebra. I then discuss computer algebra and conclude with remarks on the role of algebra in the applied mathematics curriculum.

Geometric Modeling, Cramer's Rule, and Modules

My first example concerns an unexpected application of Cramer's Rule and the Hilbert-Burch Theorem on the structure of certain free resolutions. While Cramer's Rule should be familiar, free resolutions (whatever they are) may sound rather abstract. As we will see, there are questions in geometric modeling where these topics arise naturally.

My part of the story begins with the research of Tom Sederberg and Falai Chen on parametric curves in the plane. If we are given relatively prime polynomials $a(t)$, $b(t)$, and $c(t)$ of degree n, then the parametric equations

$$x = \frac{a(t)}{c(t)} \quad \text{and} \quad y = \frac{b(t)}{c(t)} \tag{1}$$

describe a curve in the plane. To think about this geometrically, Sederberg and Chen [23] considered lines defined by

$$A(t)x + B(t)y + C(t) = 0, \tag{2}$$

where $A(t), B(t),$ and $C(t)$ are polynomials depending on the parameter t. As we vary t, the line varies also, hence the name *moving line*. A moving line *follows* a parametrization if for every value of the

David A. Cox is the Thalheimer Professor of Mathematics at Amherst College. His email address is `dac@cs.amherst.edu`.

This article is a slightly edited version of an article that appeared in the Gazette des Mathématiciens *(April 2005, no. 104, p. 13–22). It is also to appear in a future issue of the* Gaceta de la Real Sociedad Matemática Española.

parameter, the point lies on the corresponding line. In other words, (1) is a solution of (2) for all t. When we perform this substitution and clear denominators, we get the equation

$$A(t)a(t)+B(t)b(t)+C(t)c(t)\equiv 0, \tag{3}$$

where $\equiv$ indicates that equality holds for all t, i.e., $A(t)a(t)+B(t)b(t)+C(t)c(t)$ is the zero polynomial.

What algebraic structures are being used here? Since we are working over the real numbers $\mathbb{R}$, we have the polynomial ring $R=\mathbb{R}[t]$. Then, moving lines $A(t)x+B(t)y+C(t)=0$ that follow (1) correspond to elements of the set

$$\begin{aligned}&\{(A(t),B(t),C(t))\in R^3 \mid\\ &A(t)a(t)+B(t)b(t)+C(t)c(t)\equiv 0\}.\end{aligned} \tag{4}$$

This is an R-submodule of the free R-module R^3. In commutative algebra, we say that (3) is a *syzygy* and that (4) is a *syzygy module*.

In the mid-1990s, Sederberg asked me if I had seen equation (3). I said, "Yes, this equation defines the syzygy module of $a(t), b(t)$, and $c(t)$." As I expected, Tom asked me what "syzygy" means. I explained that in astronomy, a syzygy refers to the alignment of three celestial bodies along a straight or nearly straight line. The term was introduced into mathematics by Sylvester in 1853 [27]. These days, a syzygy can refer to either a polynomial relation between invariants or a linear relation with polynomial coefficients as in (3).

But then Tom did something unexpected: he asked me what "module" means! At first, I was shocked that he didn't know such a basic term, but then I remembered that civil engineers don't take courses in abstract algebra. (Tom's degree is in civil engineering, though he is now in computer science.) Vectors of polynomials occur frequently in Tom's research, and modules over polynomial rings are the natural language for discussing such objects. Yet Tom had never heard the term "module" until I mentioned it to him.

This made me realize that our conception of "applied algebra" may need to be enlarged. I will say more about this below, but first let me finish the story of moving lines. Sederberg and Chen had the insight that when two moving lines follow the parametrization (1), their common point of intersection describes the curve. Looking deeper, they also noticed that there are always two moving lines

$$\begin{aligned}p&:=A_1(t)x+B_1(t)y+C_1(t)=0,\\ q&:=A_2(t)x+B_2(t)y+C_2(t)=0\end{aligned} \tag{5}$$

that follow the parametrization and have the following special properties:

- All moving lines that follow the parametrization come from polynomial linear combinations of p and q, i.e., moving lines of the form $h_1p+h_2q=0$, where h_1 and h_2 are polynomials in t.
- If a, b, and c have degree n in t, then p has degree μ in t and q has degree $n-\mu$, where μ is an integer satisfying $0\le\mu\le\lfloor n/2\rfloor$.
- Up to a constant and appropriate signs, the polynomials $a(t)$, $b(t)$, and $c(t)$ are the 2×2 minors of the matrix
$$\begin{pmatrix}A_1(t) & B_1(t) & C_1(t)\\ A_2(t) & B_2(t) & C_2(t)\end{pmatrix}.$$
- The equation of the curve is given by the resultant
$$\text{Resultant}(p,q,t)=0.$$

The moving lines p and q are called a *μ-basis*. Thus the μ-basis determines both the parametrization (via the third bullet) and the equation of the curve (via the fourth). See [7] for the details.

From the point of view of commutative algebra, the first bullet says that the syzygy module is free with basis $(A_t(i), B_i(t), C_i(t))$, $i=1,2$. For anyone who knows the theory of modules over a principal ideal domain, this is no surprise. But the degree restriction in the second bullet is quite different: it tells us that we are really working in the homogeneous situation, where we replace $a(t), b(t)$, and $c(t)$ with homogeneous polynomials $a(s,t), b(s,t)$, and $c(s,t)$ of degree n. This gives the ideal

$$I=\langle a(s,t),b(s,t),c(s,t)\rangle\subset S=\mathbb{R}[s,t],$$

and as shown in [6] and [7], the first two bullets translate into the following free resolution of the ideal I:

$$0\to S(-n+\mu)\oplus S(-2n+\mu)\xrightarrow{\begin{pmatrix}A_1 & A_2\\ B_1 & B_2\\ C_1 & C_2\end{pmatrix}} S(-n)^3\xrightarrow{\begin{pmatrix}a & b & c\end{pmatrix}} I\to 0 \tag{6}$$

(the notation $S(-n)$ keeps track of degrees), and the third bullet implies that a, b, and c are the 2×2 minors of the 3×2 matrix in (6). This is a special case of the Hilbert-Burch Theorem (see [6]).[1]

While this might seem sophisticated, some parts make perfect sense. If you take the moving line

[1] *This special case was discovered by Franz Meyer in 1887 [19], who conjectured similar results for the syzygy module of $a_1,\dots,a_n\in S$. Hilbert, in his great paper of 1890 [16], created the basic theory of what we now call commutative algebra. His first application was Meyer's conjecture, which he proved via the original version of the Hilbert-Burch Theorem.*

equations (5) and solve for x and y via Cramer's Rule, you get

$$(7)\qquad \begin{aligned} x &= \frac{\det\begin{pmatrix} -C_1 & B_1 \\ -C_2 & B_2 \end{pmatrix}}{\det\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}} = \frac{\det\begin{pmatrix} B_1 & C_1 \\ B_2 & C_2 \end{pmatrix}}{\det\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}}, \\ y &= \frac{\det\begin{pmatrix} A_1 & -C_1 \\ A_2 & -C_2 \end{pmatrix}}{\det\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}} = \frac{-\det\begin{pmatrix} A_1 & C_1 \\ A_2 & C_2 \end{pmatrix}}{\det\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}}. \end{aligned}$$

Comparing this to the original parametrization (1), it should be no surprise that a, b, and c are the 2×2 minors of the matrix formed by the μ-basis. When Sederberg and Chen saw this consequence of Cramer's Rule, they knew they were on the right track; Hilbert may have been led to his version of the Hilbert-Burch Theorem by similar reasoning.

Remember that when Sederberg and Chen conjectured this in the late 1990s, Sederberg did not know what a module was. To me, this suggests that modules may have a place in the applied mathematics curriculum.

Economics, Cramer's Rule, and Symbolic Linear Algebra

Recently I had a conversation with a colleague about the role of Cramer's Rule in linear algebra. Although Cramer's Rule played an important role in the historical development of linear algebra, it seems less relevant these days, especially with the emphasis on solving equations by Gaussian elimination. Moreover, Cramer's Rule is useless when a system of equations has an ill-conditioned coefficient matrix. For these reasons, my colleague was wondering if the topic should be omitted—why burden students with an unnecessary formula? I've always loved Cramer's Rule for its intrinsic beauty, but this approach doesn't always work for those students who prefer to view mathematics through its applications. After some thought, I realized that while Cramer's Rule may be useless for *numerical* linear algebra, it has a valid place in the larger world of *applied* linear algebra.

To see the difference, note that the use of Cramer's Rule in (7) is applied—it's part of geometric modeling—and is nonnumerical. For another nonnumerical application of Cramer's Rule, consider the following elementary example from economics. The *IS-LM model* analyzes the interaction between total national income and the money supply, following ideas of John Maynard Keynes. As explained in [24, pp. 115–7], we want to understand

$$Y = \text{total national product}$$

and

$$r = \text{interest rate}$$

in terms of policy parameters (e.g., the money supply) and behavioral parameters (e.g., the marginal propensity to save). When linearized at an equilibrium point, the IS-LM model gives the equations

$$\begin{aligned} sY + ar &= I^o + G \\ mY - hr &= M_s - M^o, \end{aligned}$$

where $s, a, m, h, I^o, G, M_s, M^o$ are positive parameters (for example, M_s is the money supply, and s is the marginal propensity to save). The goal is to see how Y and r vary when we vary the parameters. This is where Cramer's Rule shines: it gives *formulas* for Y and r that make such questions easy to answer.

The link between this example and (7) is that in both cases we applied Cramer's Rule to equations depending on *parameters*. It follows that applied linear algebra has both *numerical* and *symbolic* aspects. Do our current linear algebra courses do justice to both? Right now, the main place a symbolic parameter occurs is in the expression $\det(A - \lambda I_n)$ for the characteristic polynomial of an $n \times n$ matrix A. As the above examples indicate, determinants with symbolic parameters have a more substantial role to play, even at the elementary level.

The link becomes even deeper when we think about what it means algebraically to do linear algebra with parameters. For simplicity, assume that the independent parameters $t_1, \ldots, t_n$ appear rationally in the equations (e.g., no square roots or exponentials of parameters). Since linear algebra needs a field, it is natural to work over the rational function field $K = \mathbb{R}(t_1, \ldots, t_n)$. However, there are many situations where the parameters appear polynomially and denominators cause problems. This means doing linear algebra over the polynomial ring $R = \mathbb{R}[t_1, \ldots, t_n]$. Here, "vectors" become vectors of polynomials, i.e., elements of a free module over R. For example, given $a, b, c \in R = \mathbb{R}[t]$, the polynomial solutions $(A, B, C) \in R^3$ of the linear equation $Aa + Bb + Cc = 0$ form the syzygy module (4).

Splines and Modules

For another example of how linear algebra relates to modules, we turn to the study of multivariate splines. Consider the spline illustrated in Figure 1.

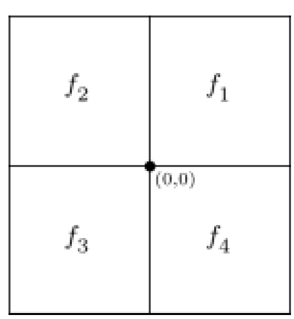

Figure 1. A polynomial spline.

Here, we have a square centered at the origin, together with a spline built using the polynomials $f_1, f_2, f_3, f_4 \in R = \mathbb{R}[x, y]$ indicated in the figure. In practice, the degrees of the polynomials are fixed in advance, but we will postpone doing so in order to reveal the underlying algebraic structure. To obtain a C^1 spline, the polynomials must satisfy

$$\begin{aligned} f_1 - f_2, f_3 - f_4 &\in \langle x^2 \rangle, \\ f_2 - f_3, f_1 - f_4 &\in \langle y^2 \rangle, \end{aligned} \tag{8}$$

where $\langle x^2 \rangle$ is the ideal of $R = \mathbb{R}[x, y]$ generated by x^2, and similarly for $\langle y^2 \rangle$. Then

$$\{(f_1, \ldots, f_4) \in R^4 \mid f_1, f_2, f_3, f_4 \text{ satisfy (8)}\} \tag{9}$$

is easily seen to be a submodule of R^4. Furthermore, one can show (see Exercise 5 of [6, Sec. 8.3]) that this submodule is free of rank four with basis

$$(1, 1, 1, 1),\ (0, x^2, x^2, 0),\ (y^2, y^2, 0, 0),\ (x^2, y^2, 0, 0).$$

In other words, every spline in (9) can be written uniquely as a polynomial linear combination of the above four splines.

To see how this relates to linear algebra, consider C^1 splines of degree $\le k$. It is straightforward to reduce (8) to a system of linear equations involving the coefficients of f_1, f_2, f_3, f_4. Solving these equations by standard techniques gives a good method for finding splines. But since we understand the module (9), we can answer this question instantly: the splines of degree $\le k$ are given uniquely by

$$\begin{aligned} &g_1 \cdot (1, 1, 1, 1) + g_2 \cdot (0, x^2, x^2, 0) \\ &\quad + g_3 \cdot (y^2, y^2, 0, 0) + g_4 \cdot (x^2, y^2, 0, 0), \end{aligned}$$

where g_1, g_2, g_3, and g_4 have degrees at most $k, k-2, k-2$, and $k-4$, respectively. Since the space of polynomials in x, y of degree $\le k$ has dimension $\binom{k+2}{2}$, it follows that the space of C^1 polynomial splines for Figure 1 of degree $\le k$ has dimension

$$\binom{k+2}{2} + 2\binom{k}{2} + \binom{k-2}{2}. \tag{10}$$

More generally, in 1973 Strang [26] conjectured a formula for the dimension of the space of C^r polynomial splines for a given triangulation of the plane. This was proved in 1988 by Billera [1] using the above methods. An introduction to splines from the module point of view can be found in [6].

In geometric modeling, the idea of fixing the degree of a moving line or surface has also proved to be useful; see [5] for a survey. In general, there is a lovely interplay between working with polynomials of fixed degree (studied via linear algebra) and polynomials of arbitrary degree (studied via modules). This leads to the notions of *Hilbert functions* and *Hilbert polynomials*. For example, the spline module (9) has Hilbert function (10), which is a polynomial in k for $k \ge 2$.

The examples discussed so far were chosen to illustrate how

the theory of modules over polynomial rings
= linear algebra with parameters
= symbolic linear algebra

arises naturally in a variety of applied contexts. This subject also has a rich theoretical side that uses sophisticated tools from algebraic geometry and commutative algebra, yet its central problem is solving linear equations over polynomial rings. A clear statement of this philosophy appears in the introduction to Eisenbud's recent book *The Geometry of Syzygies* [12, p. xii].

Computer Algebra and Applied Algebra

The last forty years have witnessed the emergence of substantial computational power and the discovery (in some cases, rediscovery) of fundamental algorithms for symbolic computation. These interlinked developments have resulted in a tremendous burst of research, both pure and applied. Books in this area range from undergraduate textbooks to technical monographs, some aimed at researchers in algebraic geometry and commutative algebra, others intended for more diverse audiences.

Computer algebra encompasses a wide variety of subjects, including coset enumeration, Galois theory, modular arithmetic, symbolic integration, symbolic summation, difference equations, power series, and special functions, among others. As one might expect, polynomials play an important role in computer algebra, where one finds algorithms for greatest common divisors, factorization, Gröbner bases, resultants, characteristic sets, quantifier elimination, and cylindrical algebraic decomposition. Introductions to computer algebra include the survey [2] and the texts [3], [4], [9], [13], [14], [20], [28], and [29]. Computer algebra can be used to solve problems in robotics, splines, integer programming, differential equations, statistics, coding theory, computational chemistry, computer-aided geometric design, geometric theorem proving, and systems of polynomial equations. These and other applications are described in [2], [6], [8], [10], [15], and [28]. The bibliographies of these books reveal a vast literature of applications of computer algebra.

How does computer algebra relate to the applications of polynomial algebra described earlier in the article? For those parts of the applied mathematics community where the computational *tools* are of paramount importance, the computer algebra books mentioned above may be the solution. (Not yet the perfect solution, since the books don't always make full use of the language of abstract algebra, and those that do sometimes assume too much or require a steep learning curve.) But for many applied communities, computational tools are not the central issue; rather, the focus is on understanding the overall structure of the problems being studied. This is where the *language* of algebra can be helpful. This is why applied mathematics may want to think about how to give its students better access to algebra.

The Algebra Curriculum

Where does a student of applied mathematics learn algebra? This question can have many different answers:

- Some students learn algebra as an undergraduate mathematics major.
- Some learn algebra in a graduate algebra course.
- Some learn algebra in specialized courses in an area of applied mathematics where algebra is especially relevant.
- Some learn algebra later in their research career.
- Some never need algebra.

The last bullet reflects the reality that applied mathematics is an immense enterprise. In spite of my belief in the value of abstract algebra, it is not essential that all applied mathematicians learn the subject.

For the first four bullets, let me offer some reflections on the role of algebra in each situation.

Virtually all undergraduate mathematics majors study abstract algebra. While this seems to solve the problem for these students, let me point out that in the U.S., such courses rarely discuss modules, and they often don't say much about polynomial rings in several variables (the focus is more on the univariate case). Furthermore, the abstraction can make the course seem less relevant to students whose interests lie in applied mathematics. In awareness of this problem, some algebra texts focus on applications (see [17] for a recent example), and some "pure" texts include numerous applied examples and a few (such as [18] and [22]) have sections on Gröbner bases. But the biggest weakness of any undergraduate algebra course is that many applied mathematicians have backgrounds in subjects like physics, engineering, or operations research. They never see such a course.

A graduate course in abstract algebra is more likely to cover modules and polynomial rings in several variables, and graduate texts like [11] and [21] have sections on Gröbner bases. But does such a course seem relevant to students in applied mathematics? Or is it just something they need for the qualifying exams?[2] Another complication is that there is more algebra that people need to know. For example, given the increasing importance of noncommutative structures (e.g., vertex algebras), there is the worry that algebra texts may devote too much space to the commutative case. So there might not be time to discuss topics like Gröbner bases, even if they are in the book. My guess is that in order for algebra courses to serve the needs of *all* students, there should be a greater emphasis on examples that illustrate the usefulness of algebra as a *language* for describing mathematical objects in pure and applied situations. But as above, the biggest weakness is that applied mathematics students will never see such a course unless they are in a department that includes both pure and applied mathematics. In applied mathematics, engineering, or operations research departments, students are unlikely to encounter a general course in abstract algebra. Furthermore, when such students try to take abstract algebra in a pure mathematics department, they are likely to encounter a version of the course aimed at future algebraists.

Specialized courses in areas of applied mathematics may be one of the best ways to introduce students to the algebra relevant for their research. This works well in areas of applied mathematics where the use of algebra has a clear utility and a good track record. But who was the first person to teach such a course? Where did that person learn algebra? Also, if such courses become the main source of algebra in applied mathematics, then we run the risk of limiting the scope of how algebra can be used.

Some of the most interesting uses of algebra are the unexpected ones. This is part of what makes mathematics so much fun. As happened with Sederberg, applied mathematicians sometimes find that algebra is relevant to the problem they are working on. Sometimes the tools of algebra solve the problem, while other times the language of algebra clarifies the issues and structures involved and helps the researcher focus on what is essential (which in turn can generate juicy problems for the algebraists to explore). But how does the researcher come to realize that algebra is relevant? I see two ways this can happen:

- Talk to a mathematician who knows algebra. Here, the challenge is on the pure side of mathematics: are algebraists trained in such a way that they can talk to nonexperts and make sense? Given the importance of what is called "technology transfer" in the U.S., how do we make students in pure mathematics better able to communicate with people far outside their speciality?
- Remember some algebra learned earlier. Aside from the algebra courses or specialized courses discussed above, applied mathematicians could also be exposed to algebra by having certain applied courses take the opportunity to use the

[2] *One answer is that in the U.S., many applied mathematicians end up at four-year colleges, where they teach a wide variety of courses, often including abstract algebra.*

language of algebra. This could happen in an applied linear algebra course that addresses the numerical and symbolic aspects of the subject or in a numerical analysis course that discusses some topics from Stetter's recent book *Numerical Polynomial Algebra* [25].[3]

In some ways, it all comes down to communication between pure and applied mathematics. There is already a lot, but there needs to be more.

The questions and suggestions posed in this article are preliminary and should be taken with a grain of salt. Their main purpose is to stimulate conversations about the proper role of algebra in applied mathematics. I am firmly convinced that algebra has a lot to offer, once we collectively figure out the best way to make use of this amazing language.

Acknowledgements

I am grateful to D. Barbezat, E. Lamagna, J. Reyes, and T. Leise for helpful suggestions and conversations. Thanks also go to M.-F. Roy for her encouragement and to G. Avrunin, F. Chen, R. Goldman, P. Lax, T. Sederberg, and B. Turkington for useful comments on early drafts of this essay. Finally, I owe a great debt to my collaborators in geometric modeling for helping me to see the wonderful consequences of the algebra I love.

References

[1] L. BILLERA, Homology of smooth splines: Generic triangulations and a conjecture of Strang, *Trans. Amer. Math. Soc.* **310** (1988), 325–40.

[2] A. M. COHEN, H. CUYPERS, and H. STERK (eds.), *Some Tapas of Computer Algebra*, Springer-Verlag, Berlin, Heidelberg, and New York, 1999.

[3] J. S. COHEN, *Computer Algebra and Symbolic Computation: Elementary Algorithms*, A K Peters, Wellesley, MA, 2002.

[4] ____ , *Computer Algebra and Symbolic Computation: Mathematical Methods*, A K Peters, Wellesley, MA, 2003.

[5] D. COX, *Curves, Surfaces, and Syzygies*, in [15, 131–50].

[6] D. COX, J. LITTLE, and D. O'SHEA, *Using Algebraic Geometry*, second ed., Springer-Verlag, New York, Berlin, and Heidelberg, 2005.

[7] D. COX, T. SEDERBERG, and F. CHEN, The moving line ideal basis of planar rational curves, *Comput. Aided Geom. Des.* **15** (1998), 803–27.

[8] D. COX and B. STURMFELS (eds.), *Applications of Computational Algebraic Geometry*, Amer. Math. Soc., Providence, RI, 1998.

[9] J. H. DAVENPORT, Y. SIRET, and E. TOURNIER, *Computer Algebra*, second ed., Academic Press, London, 1993.

[10] A. DICKENSTEIN and I. Z. EMIRIS (eds.), *Solving Polynomial Equations: Foundations, Algorithms, and Applications*, Springer-Verlag, Berlin, Heidelberg, and New York, 2005.

[3] *This is easier said than done, since as in pure mathematics, there is more and more that students in applied mathematics need to know. However, it is worth thinking about.*

[11] D. DUMMIT and R. FOOTE, *Abstract Algebra*, third ed., John Wiley & Sons, New York, 2004.

[12] D. EISENBUD, *The Geometry of Syzygies*, Springer-Verlag, New York, Berlin, and Heidelberg, 2005.

[13] J. VON DER GATHEN and J. GERHARD, *Modern Computer Algebra*, Cambridge Univ. Press, Cambridge, 2003.

[14] K. O. GEDDES, S. R. CZAPOR, and G. LABAHN, *Algorithms for Computer Algebra*, Kluwer, Dordrecht, 1992.

[15] R. GOLDMAN and R. KRASAUSKAS (eds.), *Topics in Algebraic Geometry and Geometric Modeling*, Contemp. Math., Vol. 334, Amer. Math. Soc., Providence, RI, 2003.

[16] D. HILBERT, *Ueber die Theorie der algebraischen Formen*, Math. Annalen **36** (1890), 473–534.

[17] D. JOYNER, R. KREMINSKI, and J. TURISCO, *Applied Abstract Algebra*, Johns Hopkins Univ. Press, Baltimore, MD, 2004.

[18] N. LAURITZEN, *Concrete Abstract Algebra*, Cambridge Univ. Press, Cambridge, 2003.

[19] F. MEYER, *Zur Theorie der reducibeln ganzen Functionen von n Variabeln*, Math. Annalen **30** (1887), 30–74.

[20] B. MISHRA, *Algorithmic Algebra*, Springer-Verlag, New York, Berlin, and Heidelberg, 1993.

[21] J. J. ROTMAN, *Advanced Modern Algebra*, Prentice-Hall, Upper Saddle River, NJ, 2002.

[22] ____ , *A First Course in Abstract Algebra*, second ed., Prentice-Hall, Upper Saddle River, NJ, 2000.

[23] T. W. SEDERBERG and F. CHEN, Implicitization using moving curves and surfaces, in *Proceedings of SIGGRAPH*, 1995, 301–8.

[24] C. P. SIMON and L. BLUME, *Mathematics for Economists*, W. W. Norton, New York and London, 1994.

[25] H. STETTER, *Numerical Polynomial Algebra*, SIAM, Philadelphia, 2004.

[26] G. STRANG, The dimension of piecewise polynomial spaces, and one-sided approximation, in *Conference on the Numerical Solution of Differential Equations (Univ. Dundee, Dundee, 1987),* Lectures Notes in Math., Vol. 363, Springer-Verlag, New York, Berlin, and Heidelberg, 1974, pp. 144–52.

[27] J. J. SYLVESTER, On a theory of syzygetic relations of two rational integral functions, comprising an application of the theory of Sturm's functions, and that of the greatest algebraic common measure, *Philos. Trans. Roy. Soc. London* **143** (1853), 407-548.

[28] D. WANG, *Elimination Theory*, Springer-Verlag, Wien and New York, 2001.

[29] F. WINKLER, *Polynomial Algorithms in Computer Algebra*, Springer-Verlag, Wien and New York, 1996.