Homework-7

1. Let R be a ring such that for each nonzero $a \in R$ there is a unique $b \in R$ such that $aba = a$. Prove:
(a) R has no zero divisors.
(b) $bab = b$.
(c) R is a division ring.

2. Let R be a ring such that for each $a \in R$ there exists $x \in R$ such that $a^2x = a$. Prove that:
(i) R has no nonzero nilpotent elements.
(ii) $axa = a$.

3. Let A and B be two ideals in a ring R such that $R = A + B$. Then show that $R/(A \cap B) \cong R/A \times R/B$.

4. Let R be the ring of 2×2 matrices over a field F of the form $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, $a, b \in F$. Prove that $M = \{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \in F\}$ is a maximal ideal in R.