Homework 1

1. Let R be a nonzero ring. Prove that the following statements are equivalent:

(a) R has a unique maximal right ideal.

(b) R has a unique maximal left ideal.

(c) $R \setminus U(R)$ is an ideal of R. (Note: $U(R)$ denotes the set of all units of R)

Note: A ring R is called a local ring if it satisfies any of the above equivalent conditions.

2. A ring R is called a von Neumann regular ring if for each element $x \in R$, there exists an element $y \in R$ such that $x = xyx$.

(a) Prove that for a von Neumann regular ring R, $J(R) = 0$.

(b) Prove that a ring R is a von Neumann regular ring if and only if each principal right ideal of R is generated by an idempotent.

(c) Let V be a vector space over a field F. Prove that the ring $End(V)$ consisting of all linear transformations on V is a von Neumann regular ring.

3. Let $x, y \in R$ with $xy = 1$.

(a) If $yx \neq 1$, then show that the right annihilator $ann_r(x) > 0$.
(b) Show that if $ann_r(x^{n-1}) < ann_r(x^n)$ with $n > 0$, then $ann_r(x^n) < ann_r(x^{n+1})$.
(c) Using (a) and (b) show that if R is right noetherian, then $yx = 1$.