ON WAVELETS INTERPOLATED
FROM A PAIR OF WAVELET SETS

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September 1, 2000

ABSTRACT. We show that any wavelet, with the support of its Fourier transform small enough, can be interpolated from a pair of wavelet sets. The interpolation procedure, which was introduced by X. Dai and D. Larson in [DL], allows us also to prove the extension property, a result already announced in [BGRW].

1. INTRODUCTION

The L^2 theory of orthonormal wavelets is not fully developed yet. There are three ways of understanding the whole set of wavelets in $L^2(\mathbb{R})$. The first one relies on characterization equations introduced by Y. Meyer (see [Le], [BSW], [Gr], [HW]) or those conjectured by G. Weiss (see [B], [WW]). The second one is based on the idea of a generalized multiresolution analysis introduced by L. Baggett, H. Medina and K. Merrill in [BMM]. The final one, described by X. Dai and D. Larson in [DL], approaches wavelets from operator theory point of view by treating them as wandering vectors. One of the advantages of this approach is the interpolation procedure, which allows one for a construction of "new" wavelets from "old" wavelets. The procedure is particularly simple in the case when the "old" wavelets are of the MSF type. As a result, these elementary MSF wavelets can be viewed as "building blocks" of a huge class of wavelets. Even the situation when we use only two wavelet sets to construct new wavelets is very interesting. [DL] contains a detailed study of this basic interpolation. For example, it is shown that Meyer's original construction of wavelets follows naturally from this point of view. The construction of Meyer was based on characterization equations mentioned above. In this paper we establish a further connection between these equations and wavelets interpolated from a pair of wavelet sets. To be more precise, we use the equations to prove that if the support of $\hat{\psi}$ is "small", then the wavelet ψ can be interpolated from a pair of wavelet sets. To exhibit the usefulness of this basic interpolation we apply it to prove that the local behaviour of the Fourier transform $\hat{\psi}$ of a wavelet ψ is as arbitrary as possible in any point on the real line, except the origin. This is closely related to [BGRW], where it was shown, that the origin is a point of a special meaning for $\hat{\psi}$.

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2. INTERPOLATION PAIRS OF WAVELET SETS

Following [DL] we shall define an interpolation pair of wavelet sets. Let E and F be wavelet sets. Since both these sets partition $\mathbb{R}$ under translations by $2k\pi$, $k \in \mathbb{Z}$, we can consider a natural mapping $\sigma : E \to F$, such that $\sigma(\xi) = \xi + 2k\pi$, where $k \in \mathbb{Z}$. The *interpolation map* σ_E^F is the unique extension of σ to $\mathbb{R}$, such that

$$(2.1) \qquad \sigma_E^F(\xi) = 2^j \sigma_E^F(2^{-j}\xi) \quad \text{for} \quad \xi \in 2^j E, \quad j \in \mathbb{Z}.$$

That the extension exists and is unique, follows from the fact, that both E and F partition $\mathbb{R}$ under dilations by 2^j, $j \in \mathbb{Z}$. The ordered pair (E, F) is called an *interpolation pair* if $(\sigma_E^F)^2$ is equal to identity. If that is the case, then every function ψ given by

$$\hat{\psi} = h_1 \mathbf{1}_E + h_2 \mathbf{1}_F,$$

where h_1 and h_2 are 2–dilation–periodic, is an orthonormal wavelet[1] provided that the matrix

$$(2.2) \qquad \begin{bmatrix} h_1 & h_2 \\ h_2 \circ \sigma_E^F & h_1 \circ \sigma_E^F \end{bmatrix}$$

is unitary a.e. on E. We refer to such wavelets as to wavelets interpolated from a pair (E, F).

The question we are concerned about is: "How big is the class of wavelets interpolated from a pair of wavelet sets?" Although this question is pretty vague we shall try to give a precise answer to it.

In [La] it is shown that if (E, F) is an interpolation pair and ψ is a wavelet such that the support of $\hat{\psi}$ is contained in $E \cup F$, then ψ can be interpolated from the pair (E, F). Therefore, at least in the case of interpolation pairs, it is enough to concentrate on the support of $\hat{\psi}$ to find out if a given wavelet ψ comes from an interpolation; however, left open is the question of how to know whether the support of $\hat{\psi}$ is contained in the union of an interpolation pair of wavelet sets. This motivates the following approach. Let $\psi \in L^2(\mathbb{R})$ be a wavelet. Denote the support of $\hat{\psi}$ by K and define two maps

$$T_K(\xi) = \sum_{k \in \mathbb{Z}} \mathbf{1}_K(\xi + 2k\pi)$$

and

$$D_K(\xi) = \sum_{j \in \mathbb{Z}} \mathbf{1}_K(2^j \xi),$$

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[1]We use the Fourier transform defined by

$$\hat{\psi}(\xi) = \int_{\mathbb{R}} \psi(x) e^{-ix\xi} \, dx.$$

where $\xi \in \mathbb{R}$. The maps are defined a.e. and the supremum norms $\|T_K\|_\infty$, $\|D_K\|_\infty$ determine the "size" of the set K. To see how this notion operates let us recall Meyer's equations characterizing wavelets (see [HKLS]):

$$(2.3) \qquad \sum_{k \in \mathbb{Z}} |\hat{\psi}(\xi + 2k\pi)|^2 = 1,$$

$$(2.4) \qquad \sum_{k \in \mathbb{Z}} \hat{\psi}(\xi + 2k\pi)\overline{\hat{\psi}(2^j(\xi + 2k\pi))} = 0 \quad \text{for} \quad j \geq 1,$$

$$(2.5) \qquad \sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \xi)|^2 = 1,$$

$$(2.6) \qquad t_q(\xi) := \sum_{j=0}^{\infty} \hat{\psi}(2^j \xi)\overline{\hat{\psi}(2^j(\xi + 2q\pi))} = 0, \quad \text{for} \quad q \in 2\mathbb{Z} + 1\,.$$

From equations (2.3) and (2.5) it follows that the case $\|T_K\|_\infty = \|D_K\|_\infty = 1$ gives us the most elementary wavelets, that is MSF wavelets (even if only one of the norms is equal to 1, then ψ must be an MSF wavelet). We will prove that the second simplest case, that is $\|T_K\|_\infty = \|D_K\|_\infty = 2$ corresponds exactly to wavelets interpolated from a pair of wavelet sets. Of course, in light of [La], we can achieve this by focusing on the support of the Fourier transform of a wavelet.

Theorem 2.1. *Let $\psi \in L^2(\mathbb{R})$ be an orthonormal wavelet and K denote the support of $\hat{\psi}$. There exists an interpolation pair (E, F) such that $K = E \cup F$ if and only if $\|T_K\|_\infty = \|D_K\|_\infty = 2$.*

Proof. Only one implication of the above statement requires a proof (the other one follows immediately from the fact that E and F are wavelet sets). The real task is to decompose K into a sum of two wavelet sets, provided that K is the support of the Fourier transform of a wavelet ψ and T_K and D_K are bounded by 2 (proving that these wavelet sets will form an interpolation pair will not be difficult).

The proof consists of several, easy to justify, claims.

Claim A. $T_K(\xi) = D_K(\xi)$ *for a.e. $\xi \in K$.*

Proof A. To prove Claim A we observe that T_K and D_K can only attain values 1 and 2 on K (value 0 is restricted because of (2.3) and (2.5)). From equation (2.3) it follows that $\{\xi \in \mathbb{R} \colon |\hat{\psi}(\xi)| = 1\} = \{\xi \in K \colon T_K(\xi) = 1\}$. Similarly, equation (2.5) asserts that $\{\xi \in \mathbb{R} \colon |\hat{\psi}(\xi)| = 1\} = \{\xi \in K \colon D_K(\xi) = 1\}$. Thus, Claim A is true. ♣A

Let $R := \{\xi \in K \colon T_K(\xi) = D_K(\xi) = 1\}$ (we know already that $R = \{\xi \in \mathbb{R} \colon |\hat{\psi}(\xi)| = 1\}$). From Claim A it follows that if $\xi \in K \setminus R$, then there is a unique integer $k \neq 0$ such that $\xi + 2k\pi \in K \setminus R$ and also a unique integer $j \neq 0$ such that $2^j\xi \in K \setminus R$. Thus, we can define a map $\tau_K \colon K \setminus R \to K \setminus R$ by setting

$$\tau_K(\xi) = \xi + 2k\pi,$$

where $k \in \mathbb{Z} \setminus \{0\}$ (k depends on ξ) and a map $d_K \colon K \setminus R \to K \setminus R$ by setting

$$d_K(\xi) = 2^j\xi,$$

where $j \in \mathbb{Z} \setminus \{0\}$ (j depends on ξ).

Claim B. τ_K and d_K commute.

Proof B. We prove Claim B by applying equation (2.6). Let $\xi \in K \setminus R$, $\tau_K(\xi) = \xi + 2k\pi$ and $d_K(\xi) = 2^j\xi$. Clearly, $k = 2^n q$ for some $n \in \mathbb{N} \cup \{0\}$ and $q \in 2\mathbb{Z} + 1$. Since, by (2.6), $t_q(2^{-n}\xi) = 0$ we have

$$(2.7) \qquad \sum_{l=0}^{\infty} \hat{\psi}(2^{l-n}\xi)\overline{\hat{\psi}(2^{l-n}(\xi + 2k\pi))} = 0.$$

It is easy to see that the above sum contains a non–zero term. Indeed, the term corresponding to $l = n$ is simply $\hat{\psi}(\xi)\overline{\hat{\psi}(\xi + 2k\pi)}$, which can not be zero, since both ξ and $\xi + 2k\pi$ are in K. Thus, to make the whole sum equal to 0 we need to have at least one more non–zero term. On the other hand, since $\xi \in K$ and $D_K \leq 2$, we conclude that we can have at most one more non–zero term, the one corresponding to $l = j + n$. Therefore, (2.7) reduces to

$$\hat{\psi}(\xi)\overline{\hat{\psi}(\xi + 2k\pi)} + \hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi + 2k\pi))} = 0.$$

The above allows us to make two conclusions: $2^j(\xi + 2k\pi)$ belongs to $K \setminus R$ and $2^j k = 2^l q$ is a non–zero integer. Thus, we have

$$d_K \circ \tau_K(\xi) = d_K(\xi + 2k\pi) = 2^j(\xi + 2k\pi) = 2^j\xi + 2^l q\pi = \tau_K(2^j\xi) = \tau_K \circ d_K(\xi),$$

which concludes the proof of the claim.   ♣**B**

The fact that the maps τ_K and d_K commute can be stated as: "the dilation brother of my translation brother is the translation brother of my dilation brother" and represented by the following diagram

$$(2.8) \qquad \begin{array}{ccc} \xi & \xrightarrow{\ \tau_K\ } & \xi + 2k\pi \\ d_K \downarrow & & d_K \downarrow \\ 2^j\xi & \xrightarrow{\ \tau_K\ } & 2^j(\xi + 2k\pi)\,, \end{array}$$

where $2^j k \in \mathbb{Z} \setminus \{0\}$. The above diagram, or rather its simplicity, is the main reason why we can decompose K into a sum of two wavelet sets. Of course, the set R must be a part of these wavelet sets, thus we only need to investigate the remaining set $K \setminus R$.

For every $j, k \in \mathbb{Z} \setminus \{0\}$ we define $K_{j,k} := \{\xi \in K : \xi + 2k\pi \in K, 2^j\xi \in K\}$, that is

$$(2.9) \qquad K_{j,k} = K \cap (K - 2k\pi) \cap 2^{-j}K.$$

Since T_K and D_K are bounded by 2, it follows that the sets $K_{j,k}$ are mutually disjoint and

$$(2.10) \qquad \bigcup_{j,k \in \mathbb{Z} \setminus \{0\}} K_{j,k} = K \setminus R.$$

Claim C. *We have the following equalities*

$$K_{j,k} + 2k\pi = K_{j,-k},$$

$$2^j K_{j,k} = K_{-j,2^j k},$$

$$2^j (K_{j,k} + 2k\pi) = K_{-j,-2^j k}.$$

Proof C. Let $\xi \in K_{j,k}$. By diagram (2.8) we have $2^j(\xi + 2k\pi) \in K$. Therefore,

$$(2.11) \qquad\qquad K_{j,k} \subset 2^{-j} K - 2k\pi.$$

From (2.9) and (2.11) it follows that

$$(2.12) \qquad\qquad K_{j,k} = K \cap (K - 2k\pi) \cap 2^{-j} K \cap (2^{-j} K - 2k\pi).$$

Thus,

$$K_{j,k} + 2k\pi = (K + 2k\pi) \cap K \cap (2^{-j} K + 2k\pi) \cap 2^{-j} K = K_{j,-k},$$

which proves the first equality of the claim. Applying (2.12) in the same manner yields two remaining equalities.   ♣**C**

Claim C allows us to obtain an analogous version of diagram (2.8)

$$(2.13) \qquad
\begin{array}{ccc}
K_{j,k} & \xrightarrow[+2k\pi]{\tau_K} & K_{j,-k} \\[4pt]
d_K \downarrow {\cdot 2^j} & & d_K \downarrow {\cdot 2^j} \\[4pt]
K_{-j,2^j k} & \xrightarrow[+2^j 2k\pi]{\tau_K} & K_{-j,-2^j k},
\end{array}$$

where $2^j k \in \mathbb{Z} \setminus \{0\}$. Let $A_{j,k}$ denote the sum of all the sets in the above diagram, that is,

$$A_{j,k} := K_{j,k} \cup K_{j,-k} \cup K_{-j,2^j k} \cup K_{-j,-2^j k},$$

where, to avoid redundancy, we consider only $j, k \in \mathbb{N}$. ($\mathbb{N}$ does not contain 0).

Claim D. *The sets $A_{j,k}$ are mutually disjoint for all $j, k \in \mathbb{N}$ and*

$$\bigcup_{j,k \in \mathbb{N}} A_{j,k} = K \setminus R.$$

Proof D. The disjointness follows from the fact that the sets $K_{j,k}$ are mutually disjoint. By (2.10) we have

$$(2.14) \qquad K \setminus R = \bigcup_{j,k \in \mathbb{N}} K_{j,k} \cup \bigcup_{j,k \in \mathbb{N}} K_{j,-k} \cup \bigcup_{j,k \in \mathbb{N}} K_{-j,k} \cup \bigcup_{j,k \in \mathbb{N}} K_{-j,-k}.$$

However, if we consider the set $K_{-j,k}$, where $j \in \mathbb{N}$ and $k \in \mathbb{Z} \setminus \{0\}$, then according to diagram (2.13) we must have $2^{-j} k \in \mathbb{Z} \setminus \{0\}$. Therefore, it follows that either $K_{-j,k}$ is

empty or $k = 2^j k'$ for some non–zero integer k'. This observation allows us to reduce two last sums in (2.14), which leads to

$$K \setminus R = \bigcup_{j,k \in \mathbb{N}} K_{j,k} \cup \bigcup_{j,k \in \mathbb{N}} K_{j,-k} \cup \bigcup_{j,k' \in \mathbb{N}} K_{-j,2^j k'} \cup \bigcup_{j,k' \in \mathbb{N}} K_{-j,-2^j k'} = \bigcup_{j,k \in \mathbb{N}} A_{j,k}$$

and proves our claim.   ♣D

At this point we are ready to construct the wavelet sets. The intersection with $K \setminus R$ of one of them is obtained by summing all the terms on the diagonal of diagram (2.13) for $j, k \in \mathbb{N}$ and the intersection of the other one by summing all the terms on the antidiagonal. Thus, we define

$$E' := \bigcup_{j,k \in \mathbb{N}} (K_{j,k} \cup K_{-j,-2^j k}), \qquad F' := \bigcup_{j,k \in \mathbb{N}} (K_{j,-k} \cup K_{-j,2^j k}).$$

and

$$E := E' \cup R, \qquad F := F' \cup R.$$

Claim D simply assures, that

(2.14) $$K = E' \cup F' \cup R$$

and that the sum is disjoint.

Claim E. *E and F are wavelet sets.*

Proof E. From diagram (2.13) it follows that both sets are translation and dilation equivalent to each other, therefore, it is enough to prove that E is a wavelet set.

From diagram (2.13) it follows that

(2.15) $$K \cap \bigcup_{n \in \mathbb{Z} \setminus \{0\}} (E + 2n\pi) = F' \quad \text{and} \quad K \cap \bigcup_{n \in \mathbb{Z} \setminus \{0\}} 2^n E = F'.$$

Since F' and E are disjoint, this proves that 2π–translations, as well as 2–dilations, of E are disjoint. Moreover, by (2.14) and (2.15) we obtain

$$K \cap \bigcup_{n \in \mathbb{Z}} (E + 2n\pi) = K \quad \text{and} \quad K \cap \bigcup_{n \in \mathbb{Z}} 2^n E = K.$$

Since T_K and D_K are bounded below by 1, this proves that 2π–translations, as well as 2–dilations, of E cover the real line.   ♣E

Claim F. *(E, F) is an interpolation pair.*

Proof F. We need to prove that the interpolation map σ_E^F composed with itself gives us identity. From (2.1) it follows that it is enough to show, that $(\sigma_E^F)^2(\xi) = \xi$ for a.e. $\xi \in E$. Since σ_E^F restricted to R is identity, we can focus on the case when $\xi \in E'$. Clearly, $\sigma_E^F(\xi) = \tau_K(\xi)$ for $\xi \in E'$. Moreover, the same is true if $\xi \in F'$. Indeed, from diagram (2.13) it follows, that for such ξ's we have $\sigma_E^F(\xi) = d_K \circ \tau_K \circ d_K(\xi) = \tau_K(\xi)$. Thus, σ_E^F restricted to $K \setminus R$ is equal to τ_K, which shows that $(\sigma_E^F)^2 = id$.   ♣F

Since we proved that (E, F) is an interpolation pair and, by (2.14), $K = E \cup F$, this concludes the proof of the theorem.   ♣

As an immediate consequence of Theorem 2.1 we obtain the following characterization

Corollary 2.2. *If $\psi \in L^2(\mathbb{R})$ is an orthonormal wavelet and K denotes the support of $\hat{\psi}$ then ψ is interpolated from a pair of wavelet sets (E, F) if and only if $\|T_K\|_\infty = \|D_K\|_\infty = 2$. Moreover, if this is the case, then $K = E' \cup F'$, where (E', F') is an interpolation pair.*

3. Extension property

The behaviour at the origin of the Fourier transform of wavelets was studied in [BGRW]. The starting point of this investigation was an observation, that if ψ is a wavelet and we set $\hat{\psi}(0) = 0$, then the origin becomes a Lebesgue point of the function $\hat{\psi}$. It turns out that the origin is the only point on the real line at which the behaviour of $\hat{\psi}$ is limited. We shall prove that for any other point $\xi_0 \in \mathbb{R}$ there exists its neighborhood U such that every measurable function b on U with $|b| \leq 1$ can be extended to the real line in such a way that the extension is the Fourier transform of a wavelet (this result was already announced in [BGRW] as Theorem (1.6), the case $|b| = 1$ was proven in [IP]). This will be achieved by finding an open interval $U \ni \xi_0$ small enough to be a subset of a wavelet set E, which will form an interpolation pair with another wavelet set F. When this is done, we shall use interpolation based on (E, F) to find a wavelet ψ such that $\hat{\psi}|_U = b$. This last step of the proof will be familiar to the reader who is accustomed to Meyer's construction of wavelets, which was mentioned before, or its generalizations, presented both in [HWW] and [DL].

We start by exhibiting a special class of interpolation pairs which we shall use for our construction; similar classes were considered in [Gu].

Proposition 3.1. *Let T and U be measurable subsets of $\mathbb{R}$ and both $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ be given. If $E := U \cup (2^{-n}U - 2k\pi) \cup T$ is a wavelet set and the sum is disjoint, then $F := 2^{-n}U \cup (U - 2^n 2k\pi) \cup T$ is a wavelet set and the sum is disjoint. Moreover, (E, F) is an interpolation pair.*

Proof. If the sets U, $2^{-n}U - 2k\pi$ and T are pairwise disjoint and their sum is a wavelet set, then the sets $2^{-n}U$, $U - 2^n 2k\pi$ and T must be mutually disjoint as well. Therefore, F is translation and dilation equivalent to E, which shows that F is a wavelet set. We shall prove that (E, F) is an interpolation pair. As we mentioned before, it is enough to show that the interpolation map σ_E^F satisfies $(\sigma_E^F)^2(\xi) = \xi$ for $\xi \in E$. Computing σ_E^F on E yields

$$
\sigma_E^F(\xi) = \begin{cases} \xi & \text{for } \xi \in T \\ \xi - 2^n 2k\pi & \text{for } \xi \in U \\ \xi + 2k\pi & \text{for } \xi \in 2^{-n}U - 2k\pi \,. \end{cases}
$$

Therefore, on F, σ_E^F has the form

$$
\sigma_E^F(x) = \begin{cases} \xi & \text{for } \xi \in T \\ 2^{-n}\sigma_E^F(2^n\xi) = \xi - 2k\pi & \text{for } \xi \in 2^{-n}U \\ 2^n\sigma_E^F(2^{-n}\xi) = \xi + 2^n 2k\pi & \text{for } \xi \in U - 2^n 2k\pi \end{cases}
$$

So, it is easily checked that for $\xi \in E$, $(\sigma_E^F)^2(\xi) = \xi$. ♣

Although k given in the above proposition can be any integer we will only need to consider $k = 1$ to proceed with our construction. As a technical lemma we shall prove that if $\xi_0 > 0$, then there is a wavelet set E given as in Proposition 3.1, such that U is an open interval containing ξ_0. We also would like to point out that, according to [BGRW], if ψ is an MRA wavelet and m is an arbitrary integer, then $4m\pi$ is a Lebesgue point of $\hat{\psi}$, provided $\hat{\psi}(4m\pi) = 0$. This indicates, that the case when $\xi_0 = 4m\pi$ may be harder to consider. Indeed, as we will see, the most general case $\xi_0 \neq 2m\pi$ can be treated in a simple way based on the approach offered in [S]. The special case $\xi_0 = 2m\pi$ requires more advanced techniques described in [IP]. Both of these methods rely on introducing the maps $\tau : \mathbb{R} \to [-\pi, \pi)$ defined by $\tau(\xi) = \xi + 2k\pi$ (where k is the unique integer such that $\xi + 2k\pi \in [-\pi, \pi)$) and $d : \mathbb{R} \setminus \{0\} \to [-2\pi, -\pi) \cup [\pi, 2\pi)$ defined by $d(\xi) = 2^j\xi$ (where j is the unique integer such that $2^j\xi \in [-2\pi, -\pi) \cup [\pi, 2\pi)$). Ionascu and Pearcy [IP] proved that a measurable set V is a subset of a wavelet set E if and only if there exist $A \supset V$ and $B \supset V$ such that $\tau|_A, \tau|_B, d|_A$ and $d|_B$ are injections, $\tau(A) = [-\pi, \pi)$ and $d(B) = [-2\pi, -\pi) \cup [\pi, 2\pi)$. Moreover, in this case, the wavelet set E may be chosen as a subset of $A \cup B$. A sufficient condition for a set V to be contained in a wavelet set was previously noted in [S], namely, $\tau|_V$ and $d|_V$ are injective, $d(V) \neq [-2\pi, -\pi) \cup [\pi, 2\pi)$ and there exists an $\delta > 0$ such that $\tau(V) \cap [-\delta, \delta] = \emptyset$. We shall use these two results to prove the technical lemma.

Lemma 3.2. *For every $\xi_0 > 0$ there exists an open interval $U \ni \xi_0$, a measurable set $T \subset \mathbb{R}$ and some natural n, such that $E := U \cup (2^{-n}U - 2\pi) \cup T$ is a wavelet set and the sum is disjoint.*

Proof. Case 1: $\xi_0 \neq 2m\pi$ for any $m \in \mathbb{N}$. Let U be an open neighborhood of ξ_0 such that $2U \cap U = \emptyset$ and $2k\pi \notin U$ for any $k \in \mathbb{Z}$. Write $U = (2k\pi + \epsilon_1, 2(k+1)\pi - \epsilon_2)$, where $\epsilon_1, \epsilon_2 > 0$ and $k \in \mathbb{Z}$. Pick $n \in \mathbb{N}$ such that $2^{-n}U \subset [0, \epsilon_1] \cap [0, \pi/2]$ and denote the disjoint sum $U \cup (2^{-n}U - 2\pi)$ by V. Notice that $\tau|_V$ is 1-1, $d|_V$ is 1-1, $\tau(V) \cap [-\delta, \delta] = \emptyset$ for $\delta = \min(2^{-n}(2k\pi + \epsilon_1), \epsilon_2)$, and $d(V) \neq [-2\pi, -\pi) \cup [\pi, 2\pi)$. Hence, by [S], there is a $T \subset \mathbb{R}$ such that $V \cup T$ is a wavelet set, as desired.

Case 2: $\xi_0 = 2m\pi$ for some $m \in \mathbb{N}$. For $m = 1$ we can choose $U = (4\pi/3, 8\pi/3)$, $T = \emptyset$ and $n = 1$. For $m \neq 1$, choose $n \in \mathbb{N}$ such that $2^{-n}\xi \in (2\pi/3, 4\pi/3)$, and choose U such that $2^{-n}U \subset (2\pi/3, 4\pi/3)$ and $U \subset (2m\pi - \pi/3, 2m\pi + \pi/3)$. Notice that $\tau(2^{-n}U) \subset (2\pi/3, \pi) \cup [-\pi, -2\pi/3)$ and $\tau(U) \subset (-\pi/3, \pi/3)$, so setting $V = U \cup (2^{-n}U - 2\pi)$ we have $\tau|_V$ and $d|_V$ are 1-1.

We wish to apply the theorem from [IP] which characterizes sets which are subsets of wavelet sets. In order to do so, we must find sets T_1 and T_2 (both disjoint from V) such that the restrictions of τ and d to $T_1 \cup V$ and $T_2 \cup V$ are 1–1, $\tau(T_1 \cup V) = [-\pi, \pi)$ and $d(T_2 \cup V) = [-2\pi, -\pi) \cup [\pi, 2\pi)$.

We simply note that $T_1 = [2(m-1)\pi + \pi/3, 2m\pi + \pi/3) \cap U^c \cap (2^{-n}U + 2(m-1)\pi)^c$ and $T_2 = 1/2([2\pi/3, 4\pi/3) \cap (2^{-n}U)^c) \cup 1/2([-4\pi/3, -2\pi/3) \cap (2^{-n}U - 2\pi)^c)$ satisfy the conditions above. Therefore, by Ionascu and Pearcy's characterization, there is a set $T \subset T_1 \cup T_2$ such that the disjoint sum $U \cup (2^{-n}U - 2\pi) \cup T$ is a wavelet set.   ♣

From Proposition 3.1 it follows that the set $E = U \cup (2^{-n}U - 2\pi) \cup T$ from Lemma 3.2 and the set $F = 2^{-n}U \cup (U - 2^n2\pi) \cup T$ form an interpolation pair. The standard interpolation procedure allows us to prove the main result of this section, which generalizes the special case $|b| = 1$ proven in [IP].

Theorem 3.3. *Suppose $\xi_0 \neq 0$ then there exists a neighborhood U of ξ_0, with the property that if b is any measurable function on U with $|b| \leq 1$, then we can find a wavelet ψ such that $\hat{\psi}(\xi) = b(\xi)$ for all $\xi \in U$.*

Proof. First we observe that if ψ is a wavelet, then $\hat{\psi}(-\cdot)$ is the Fourier transform of a wavelet (this follows from Meyer's equations). Therefore, we can restrict our attention to the case $\xi_0 > 0$.

Let U, T and n be as in Lemma 3.2. We will interpolate from the pair (E, F), where $E = U \cup (2^{-n}U - 2\pi) \cup T$ and $F = 2^{-n}U \cup (U - 2^n 2\pi) \cup T$. In order to do so we define b_1, b_2, b_3 and b_4 by

$$b_1(\xi) = \sqrt{1 - |b(2^n\xi)|^2} \text{ for } \xi \in 2^{-n}U \text{ and } 0 \text{ otherwise},$$

$$b_2(\xi) = \sqrt{1 - b_1^2(\xi + 2\pi)} \text{ for } \xi \in 2^{-n}U - 2\pi \text{ and } 0 \text{ otherwise},$$

$$b_3(\xi) = \sqrt{1 - b_2^2(2^{-n}\xi)} \text{ for } \xi \in U - 2^n 2\pi \text{ and } 0 \text{ otherwise},$$

$$b_4(\xi) = 1 \text{ for } \xi \in T \text{ and } 0 \text{ otherwise}.$$

A simple calculation shows that if we set $h_1(\xi) = e^{i\xi/2}(|b(\xi)| + b_2(\xi) + b_4(\xi))$ on E and $h_2(\xi) = e^{i\xi/2}(b_1(\xi) + b_3(\xi))$ on F and extend 2–dilation–periodically, then matrix (2.2) is unitary on E and, therefore, ψ given by $\hat{\psi} = h_1 \mathbf{1}_E + h_2 \mathbf{1}_F$ is a wavelet. To finish the proof we observe that there exists a unimodular function ν defined on U such that $b(\xi) = \nu(\xi)e^{i\xi/2}|b(\xi)|$. Moreover, since U is an open interval whose measure does not exceed 2π we can extend ν to a 2π–periodic unimodular function $\tilde{\nu}$. Then, by Meyer's equations, $\tilde{\nu}(\xi)\hat{\psi}(\xi)$ is the Fourier transform of a wavelet and $\tilde{\nu}(\xi)\hat{\psi}(\xi) = b(\xi)$ for all $\xi \in U$, as desired. ♣

4. CONCLUSION

We answered the question: "How big is the class of wavelets interpolated from a pair of wavelet sets?" in two ways. The proof of the extension property indicated that, at least locally, every wavelet can be approximated by those wavelets in the frequency domain. The characterization theorem, presented as Corollary 2.2, asserted that a minimal necessary condition on the support of the Fourier transform of an interpolated wavelet is also sufficient to assure that a wavelet comes from interpolation. In this sense the class is as big as possible. The main difficulty in proving this characterization was in finding a wavelet set contained in the support of the Fourier transform of a given wavelet. The problem of doing so in general was already formulated by D. Larson in [La] and its solution does not follow immediately from the present L^2 theory of wavelets.

ACKNOWLEDGMENTS

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