MEYER TYPE WAVELET BASES IN $\mathbb{R}^2$

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Abstract. It is shown that for any expansive, integer valued two by two matrix, there exists a (multi)-wavelet whose Fourier transform is compactly supported and smooth. A key step is showing that for almost every equivalence class of integrally similar matrices there is a representative A which is strictly expansive in the sense that there is a compact set K which tiles the plane by integer translations and such that $K \subset A(K^\circ)$, where K° is the interior of K.

1. Introduction and Preliminaries

A common thread in the theory of wavelets has been to ask for which dilations A do there exist (multi)-wavelets $\Psi = \{\psi^1, \dots, \psi^l\}$, usually with some additional properties. Gröchenig, Haas, and Madych [GH, GM], and Lagarias and Wang [LW1–LW6] studied, for example, which dilations A yield Haar type wavelets. Strichartz [S] was able to show that for each dilation which admits a Haar type wavelet, there is also an r-regular wavelet, a result that was extended to all integer valued, expansive matrices in [B2]. On the Fourier transform side, Dai and Larson [DL] and Hernandez, Wang and Weiss [HWW1, HWW2] initiated the study of minimally supported frequency wavelets, which were also studied in [DLS1, DLS2], [BL], [SW], and [FW]. It is known that all expansive, integer valued matrices admit minimally supported frequency wavelets. Gu and Han [GH] proved that all determinant two integer valued expansive matrices admit MRA wavelets, a result that was extended to arbitrary expansive, integer valued matrices in [BRS], [BM]. More recently, the question of extending Daubechies [D1, D2] construction to higher dimensions has been considered by Ayache [A] and independently by Belogay and Wang [BW]. Calogero [C1, C2] has studied the construction of Meyer type wavelets for the quinconx matrix in $\mathbb{R}^2$. The purpose of this paper is to solve the general existence problem for Meyer type wavelets in two dimensions. That is, we will show that for all expansive, integer valued two by two matrices, there exists a (multi)-wavelet Ψ such that for each i, $\hat{\psi}^i$ is smooth and compactly supported.

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A matrix is said to be *expansive* if all of its eigenvalues have modulus bigger than one. Such a matrix is often referred to as a *dilation*. We restrict our attention to dilations A which preserve a lattice $\Gamma = P\mathbb{Z}^n$, i.e., $A\Gamma \subset \Gamma$, where P is some $n \times n$ non degenerate matrix. By standard considerations, see [LW1], we will assume that $\Gamma = \mathbb{Z}^n$ and hence A has integer entries. We say that the matrices A and B are *integrally similar* if there is an integer matrix C of determinant ± 1 such that $A = CBC^{-1}$.

Given an expansive matrix A, a (multi)-wavelet (with respect to A) is a collection of square integrable functions $\Psi = \{\psi^1, \dots, \psi^l\}$ such that $\{\psi^i_{j,k} : i = 1, \dots, l, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. Here, for $\psi \in L^2(\mathbb{R}^n)$ we let

$$\psi_{j,k}(x) = D_{A^j} T_k \psi(x) = |\det A|^{j/2} \psi(A^j x - k) \qquad j \in \mathbb{Z}, k \in \mathbb{Z}^n,$$

where $T_y f(x) = f(x - y)$ is a translation operator by the vector $y \in \mathbb{R}^n$, and $D_A f(x) = \sqrt{|\det A|} f(Ax)$ is a dilation by the matrix A.

To fix notation, the Fourier transform we will use in this paper is

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx.$$

Given a subset $X \subset \mathbb{R}^n$, $\operatorname{conv} X$ denotes the convex hull of X, $\operatorname{sym\,conv} X = \operatorname{conv}(-X \cup X)$, and X° is the interior of X.

A *minimally supported frequency (MSF)* wavelet is a wavelet ψ such that $|\hat{\psi}| = \mathbf{1}_W$, for some measurable set W. For the purposes of this paper, we will say that the matrix A admits a Meyer type wavelet if there is a (multi)-wavelet Ψ such that each $\hat{\psi}^i$ is smooth and compactly supported.

It is easy to see that whenever $A = CBC^{-1}$ with C an integer matrix of determinant ± 1 and Ψ is a (multi)-wavelet with respect to B, then $\{\psi^1(Cx), \dots, \psi^l(Cx)\}$ is a (multi)-wavelet with respect to A. Moreover, if A and B are integrally similar, then A admits a Meyer type wavelet if and only if B admits a Meyer type wavelet.

Finally, for any measurable set $W \subset \mathbb{R}^n$ that satisfies

$$\sum_{j \in \mathbb{Z}} \mathbf{1}_W(A^j \xi) = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{1.1}$$

(in particular, for the support of any MSF wavelet associated to A^T), we define the *dilation projection* $d_A : \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(W)$ by

$$d_A(S) = \cup_{j=-\infty}^{\infty} (A^j(S) \cap W) \qquad \text{for } S \in \mathcal{P}(\mathbb{R}^n), \tag{1.2}$$

where $\mathcal{P}(W)$ is the power set of W.

The remainder of the terminology in this paper is standard, as can be found in [HW], [M] or [W].

2. The determinant two case

In this section, we prove that every expansive, two by two matrix of determinant ± 2 admits a Meyer type wavelet. Most of the work for this has been done by previous authors, so this section will consist of theorem quoting and one example.

The following follows from a theorem due to Latimer and MacDuffee (see [N]), or an elementary argument as presented in [LW1].

Proposition 2.1. *Let A be an expansive matrix with determinant ± 2. Let $C_0 = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$, $C_1 = \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix}$, $C_2 = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}$, and $D = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$. Then A is integrally similar to one of the following six matrices: D, C_0, $\pm C_1$, $\pm C_2$.*

From Proposition 2.1, it follows that in order to show that all integral matrices of determinant ± 2 admit Meyer type wavelets, it suffices to show that Meyer type wavelets exist for the six matrices listed in Proposition 2.1. The matrices D and C_0 follow from taking tensor products of one dimensional wavelets as explained, for example, in [W]. The existence of Meyer type wavelets for the quinconx matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ was shown by Calogero in [C1, C2]. Since the quinconx matrix is integrally similar to C_2 and the same argument works for $-C_2$, it suffices to prove that Meyer type wavelets exist for $\pm C_1$. Since our construction is going to be symmetric with respect to the origin, we will focus solely on C_1 (actually, we will focus on the matrix $\begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}$, which is integrally similar to C_1). We begin with an easy

Proposition 2.2. *Let A be an $n \times n$ expansive matrix and $W \subset \mathbb{R}^n \setminus \{0\}$ be a compact set that is bounded away from the origin and satisfies (1.1). Let m_0 be a function with $L = \mathrm{supp}(m_0) = \{\xi \in \mathbb{R}^n : m_0(\xi) \neq 0\}$ and $K = \mathbb{R}^n \setminus L$. If there exist $\epsilon, N > 0$ such that $\mathbf{B}(0,\epsilon) \subset L$ and $d_A(\mathbf{B}(0,N) \cap K) = W$, then $\Pi_{j=1}^{\infty} m_0(A^{-j}\xi)$ converges and is compactly supported. Moreover, if m_0 is smooth and $m_0(0) = 1$ then $\Pi_{j=1}^{\infty} m_0(A^{-j}\xi)$ converges to a smooth, compactly supported function.*

Proof. Note that since W and $K \cap \mathbf{B}(0,N)$ are bounded and bounded away from the origin, there is an $R \in \mathbb{N}$ such that whenever $|j| > R$, $A^j(K \cap \mathbf{B}(0,N)) \cap W = \emptyset$. For $|j| \leq R$ let $E_j = K \cap \mathbf{B}(0,N) \cap A^j(W)$. Consider the set $E := A^R(\cup_{j=-R}^{R} A^{-j} E_j)$ and note that $d_A(E) = A^{-R}(E) = W$. Moreover, if $j > R$ and $\xi \in A^j(W)$, then $\xi \in A^j(A^{-R}(E)) = \cup_{k=-R}^{R} A^{j-k}(E_k)$. Therefore, $\xi \in A^l(K)$ for some $l > 0$ and $\Pi_{j=1}^{\infty} m_0(A^{-j}\xi) = 0$. Since W is compact and bounded away from zero and A is expansive, $\cup_{j\in\mathbb{Z}} A^j(W) \cup \{0\}$ is closed. By (1.1) this implies that $\cup_{j\in\mathbb{Z}} A^j(W) = \mathbb{R}^n \setminus \{0\}$ and thus $\mathrm{supp}(\Pi_{j=1}^{\infty} m_0(A^{-j}\xi)) \subset \cup_{k=-\infty}^{R} A^k(W) \cup \{0\}$. Since A is expansive, $\cup_{k=-\infty}^{R} A^k(W)$ is bounded.

Now, assume that m_0 is smooth and $m_0(0) = 1$. Since $m(\xi) = 1 + O(\xi)$ as $\xi \to 0$ the product $\Pi_{j=1}^{\infty} m_0(A^{-j}\xi)$ converges pointwise. Furthermore, by standard considerations involving the infinite product rule, see the proof of Theorem 3 in [B1], this product defines a smooth function.

Example 2.3. *There exists a Meyer type wavelet for the matrix* $A = \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}$.

Proof. We note here that the set

$$W = \text{conv}\{(0, \frac{1}{2}), (-1, \frac{1}{2}), (-1, -\frac{1}{2})\} \cup \text{conv}\{(0, -\frac{1}{2}), (1, \frac{1}{2}), (1, -\frac{1}{2})\} \tag{2.1}$$

is the support of an MSF wavelet with low pass filter the $\mathbb{Z}^2$ periodization of $\mathbf{1}_T$, where $T = \text{sym conv}\{(0, \frac{1}{2}), (\frac{1}{2}, 0)\}$. The scaling set for W is $\text{sym conv}\{(-1, -\frac{1}{2}), (0, -\frac{1}{2})\}$.

We will now smooth the low pass filter associated with W. Let $\mathcal{D} = \{(0,0), (1,0)\}$ be the set of representatives of different cosets of $\mathbb{Z}^2/A\mathbb{Z}^2$. Hence the Smith-Barnwell equation for A is $|m_0(\xi)|^2 + |m_0(\xi + (\frac{1}{2}, \frac{1}{2}))|^2 = 1$. We smooth the low pass filter in such a way that the support of the new filter is the interior of the $\mathbb{Z}^2$ periodization of $\text{sym conv}\{(0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{8}), (\frac{1}{2}, -\frac{1}{8})\})$.

Let $g : \mathbb{R} \to [0, \infty)$ be a C^∞ function such that

$$\text{supp}\, g := \{\eta \in \mathbb{R} : g(\eta) \neq 0\} = (-\infty, 1/4).$$

Let $v_1 = (3/8, 1/2)$, $v_2 = (3/8, -1/2)$, $v_3 = (-3/8, 1/2)$, $v_4 = (-3/8, -1/2)$. Define a C^∞ function $h : \mathbb{R}^2 \to [0, \infty)$ by

$$h(\xi) = \prod_{j=1}^{4} g(\langle \xi, v_j \rangle).$$

Clearly

$$\text{supp}\, h = (\text{conv}\{(\pm 2/3, 0), (0, \pm 1/2)\})^\circ.$$

Let f be $\mathbb{Z}^2$ periodization of h, i.e., $f(\xi) = \sum_{k \in \mathbb{Z}^2} f(\xi + k)$. Finally define the function m_0 by

$$m_0(\xi) = \sqrt{f(\xi)/(f(\xi) + f(\xi + (1/2, 1/2)))}.$$

Since the denominator is always positive and f "vanishes strongly" (see the proof of Claim 3.3), m_0 is C^∞, $\mathbb{Z}^2$ periodic function satisfying $|m_0(\xi)|^2 + |m_0(\xi + (1/2, 1/2))|^2 = 1$ for all $\xi \in \mathbb{R}^2$, and $m_0(0) = 1$.

Now, we show that Cohen's condition is satisfied for this set. The natural first guess for K is the scaling set $K_1 = \text{sym conv}\{(-1, -\frac{1}{2}), (0, -\frac{1}{2})\}$. This set almost works; the points $\pm(1, \frac{1}{2})$ get mapped under A^{-1} to the points $\pm(0, \frac{1}{2})$, which are not in the support of m_0. However, letting B_1 be a small neighborhood of $(1, \frac{1}{2})$ intersected with K_1, the set $K = K_1 \setminus \big(B_1 \cup -B_1\big) \cup \big(\overline{B_1} + (-1, 0)\big) \cup \big(-\overline{B_1} + (1, 0)\big)$ satisfies Cohen's condition.

Therefore $\hat{\varphi}(\xi) = \Pi_{j=1}^{\infty} m_0(A^{-j}\xi)$ is the scaling function for the multiresolution analysis $(V_j)_{j\in\mathbb{Z}}$ associated to the dilation A^T defined by

$$V_j = \overline{\operatorname{span}}\{D_{(A^T)^j} T_l \varphi : l \in \mathbb{Z}^n\} \qquad \text{for } j \in \mathbb{Z}.$$

Finally, it suffices to show that $\hat{\varphi}$ is compactly supported by verifying the hypotheses of Proposition 2.2 with W given by (2.1). A direct computation shows that the images of the zero set in Figure 1 under various powers A^j line up as pictured below in Figure 2.

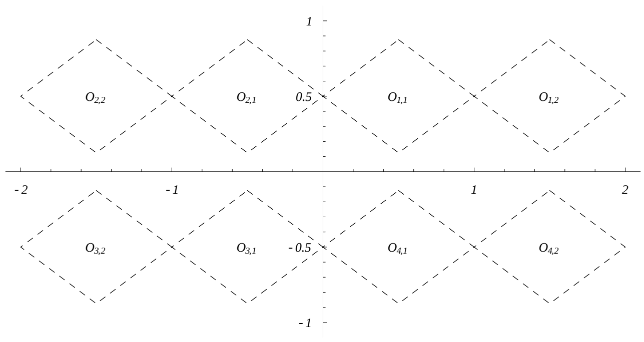

Figure 1. Zero set of m_0.

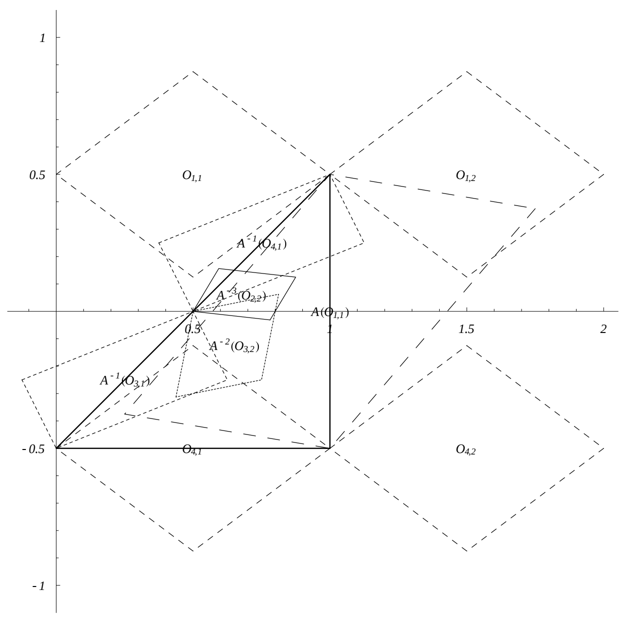

Figure 2. Verification of Proposition 2.2

Clearly, all of W is covered by the union of these sets. Furthermore, ψ given by

$$\hat{\psi}(\xi) = m_0(A^{-1}\xi + (1/2, 1/2))e^{\pi i(\xi_1+\xi_2)}\hat{\varphi}(A^{-1}\xi) \qquad \text{for } \xi = (\xi_1, \xi_2) \in \mathbb{R}^2,$$

is a Meyer type wavelet associated to the dilation A^T.

We have thus proven

Theorem 2.4. *Let A be a two by two expansive integer matrix of determinant ± 2. Then, A admits a Meyer type wavelet.*

3. General facts

The main goal of this section is to give a simple condition on a dilation A which guarantees the existence of Meyer type wavelets associated with A. Since this condition is meaningful regardless of the dimension, we will work on $\mathbb{R}^n$. An application of this condition to $\mathbb{R}^2$ is given by Corollary 3.6.

Definition 3.1. We say that an $n \times n$ integral dilation B is *strictly expansive* if there exists a compact set $K \subset \mathbb{R}^n$ such that

$$\sum_{k \in \mathbb{Z}^n} \mathbf{1}_K(\xi + k) = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{3.1}$$

$$K \subset BK^\circ, \qquad \text{where } K^\circ \text{ is the interior of } K. \tag{3.2}$$

When (3.1) and (3.2) hold, we say that B is strictly expansive with respect to K.

We shall prove the following existence theorem.

Theorem 3.2. *Suppose A is a $n \times n$ integral dilation matrix. If $B = A^T$ is strictly expansive then there exists a multiresolution analysis with a scaling function and an associated wavelet family of $(|\det A| - 1)$ functions in the Schwartz class.*

Proof. Suppose the compact set K satisfies (3.1) and (3.2). Given $\varepsilon > 0$ we define

$$K^{-\varepsilon} = \{\xi \in \mathbb{R}^n : \mathbf{B}(\xi, \varepsilon) \subset K\}$$
$$K^{+\varepsilon} = \{\xi \in \mathbb{R}^n : \mathbf{B}(\xi, \varepsilon) \cap K \neq \emptyset\}$$

Note that $K^{-\varepsilon}$ is closed, $K^{+\varepsilon}$ is open, and the interior of K satisfies $K^\circ = \bigcup_{\varepsilon > 0} K^{-\varepsilon}$. Hence there exists $\varepsilon > 0$ such that

$$K^{+\varepsilon} \subset B(K^{-\varepsilon}). \tag{3.3}$$

Pick a function $g : \mathbb{R}^n \to [0, \infty)$ in the class C^∞ such that $\int_{\mathbb{R}^n} g = 1$ and

$$\operatorname{supp} g := \{\xi \in \mathbb{R}^n : g(\xi) \neq 0\} = \mathbf{B}(0, \varepsilon).$$

Define a function f by

$$f(\xi) = (\mathbf{1}_K * g)(\xi).$$

Clearly f is in the class C^∞, $0 \le f(\xi) \le 1$, and

$$\operatorname{supp} f = \{\xi \in \mathbb{R}^n : f(\xi) \neq 0\} \subset K^{+\varepsilon}, \tag{3.4}$$

$$\{\xi \in \mathbb{R}^n : f(\xi) = 1\} = K^{-\varepsilon}. \tag{3.5}$$

Moreover, by (3.1)

$$\sum_{k\in\mathbb{Z}^n} f(\xi + k) = \sum_{k\in\mathbb{Z}^n} (\mathbf{1}_K * g)(\xi + k) = 1 \qquad \text{for all } \xi \in \mathbb{R}^n. \tag{3.6}$$

Finally define a function $m : \mathbb{R}^n \to [0,1]$ by

$$m(\xi) = \sqrt{\sum_{k\in\mathbb{Z}^n} f(B(\xi + k))}. \tag{3.7}$$

Claim 3.3. *The function m given by (3.7) is C^∞, $\mathbb{Z}^n$-periodic, and*

$$\sum_{d\in\mathcal{D}} |m(\xi + B^{-1}d)|^2 = 1 \qquad \textit{for all } \xi \in \mathbb{R}^n, \tag{3.8}$$

$$m(\xi) > 0 \implies \xi \in \mathbb{Z}^n + B^{-1}(K^{+\varepsilon}), \tag{3.9}$$

$$m(\xi) = 0 \qquad \textit{for } \xi \in (B^{-1}\mathbb{Z}^n \setminus \mathbb{Z}^n) + B^{-1}(K^{-\varepsilon}), \tag{3.10}$$

where $B = A^T$, and $\mathcal{D} = \{d_1, \dots, d_b\}$ is the set of representatives of different cosets of $\mathbb{Z}^n/B\mathbb{Z}^n$, where $b = |\det A|$.

Proof of Claim 3.3. To guarantee that m is C^∞, the function f must "vanish strongly", i.e., if $f(\xi_0) = 0$ for some ξ_0 then $\partial^\alpha f(\xi_0) = 0$ for any multi-index α. It is clear that if nonnegative function f in C^∞ "vanishes strongly" then $\sqrt{f}$ is also C^∞.

The condition (3.8) is a consequence of

$$\sum_{d\in\mathcal{D}} |m(\xi + B^{-1}d)|^2 = \sum_{k\in\mathbb{Z}^n}\sum_{d\in\mathcal{D}} f(B(\xi + B^{-1}d + k)) = \sum_{k\in\mathbb{Z}^n}\sum_{d\in\mathcal{D}} f(B\xi + d + Bk) = 1,$$

by (3.6).

To see (3.9), take ξ such that $m(\xi) > 0$. By (3.4) and (3.7), $B(\xi + k) \in K^{+\varepsilon}$ for some $k \in \mathbb{Z}^n$, and hence (3.9) holds.

We claim that (3.10) follows from (3.8) and

$$m(\xi) = 1 \qquad \text{for } \xi \in \mathbb{Z}^n + B^{-1}(K^{-\varepsilon}). \tag{3.11}$$

Indeed, if $\xi \in B^{-1}d + k + B^{-1}(K^{-\varepsilon})$ for some $d \in \mathcal{D} \setminus B\mathbb{Z}^n$ and $k \in \mathbb{Z}^n$, then by (3.11) we have $m(\xi - B^{-1}d) = 1$. Hence by (3.8) $m(\xi) = 0$ and (3.10) holds. Finally, (3.11) is the immediate consequence of (3.5) and (3.7). This ends the proof of the claim.

We can write m in the Fourier expansion as

$$m(\xi) = \frac{1}{\sqrt{|\det A|}} \sum_{k \in \mathbb{Z}^n} h_k e^{-2\pi i \langle k, \xi \rangle}, \tag{3.12}$$

where we include the factor $|\det A|^{-1/2}$ outside the summation as in [B1]. Since m is C^∞, the coefficients h_k decay polynomially at infinity, that is for all $N > 0$ there is $C_N > 0$ so that

$$|h_k| \le C_N |k|^{-N} \qquad \text{for } k \in \mathbb{Z}^n \setminus \{0\}.$$

Since m satisfies (3.8) and $m(0) = 1$, m is a low-pass filter which is regular in the sense of the definition following Theorem 1 in [B1]. By Theorem 5 in [B1] $\varphi \in L^2(\mathbb{R}^n)$ defined by

$$\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m(B^{-j}\xi), \tag{3.13}$$

has orthogonal translates, i.e.,

$$\langle \varphi, T_l \varphi \rangle = \delta_{l,0} \qquad \text{for } l \in \mathbb{Z}^n,$$

if and only if m satisfies the Cohen condition, that is there exists a compact set $\tilde{K} \subset \mathbb{R}^n$ such that

- $\tilde{K}$ contains a neighborhood of zero,
- $|\tilde{K} \cap (l + \tilde{K})| = \delta_{l,0}$ for $l \in \mathbb{Z}^n$,
- $m(B^{-j}\xi) \neq 0$ for $\xi \in \tilde{K},\ j \ge 1$.

The first guess for $\tilde{K}$ to be K is in general incorrect, e.g. if K has isolated points. Instead we claim that there is $0 < \delta < 1$ so that

$$\tilde{K} = \{\xi \in K : |\mathbf{B}(\xi, \varepsilon) \cap K| \ge \delta |\mathbf{B}(\xi, \varepsilon)|\} \tag{3.14}$$

does the job. Clearly, if $\xi \in \tilde{K}$ then $f(\xi) \neq 0$ and thus $m(B^{-1}\xi) \neq 0$. By (3.3) $B^{-1}\tilde{K} \subset B^{-1}K^{+\varepsilon} \subset K^{-\varepsilon} \subset \tilde{K}$ and thus $m(B^{-j}\xi) \neq 0$ for all $j \ge 1$. Since $0 \in K^\circ$ by (3.2) thus $0 \in (K^{-\varepsilon})^\circ$ by (3.3) and hence $0 \in \tilde{K}^\circ$. Finally, it suffices to check that

$$\sum_{k \in \mathbb{Z}^n} \mathbf{1}_{\tilde{K}}(\xi + k) = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{3.15}$$

By the compactness of K there is a finite index set $I \subset \mathbb{Z}^n$ such that

$$\sum_{k\in I} \mathbf{1}_K(\xi + k) \geq 1 \qquad \text{for all } \xi \in [-1,1]^n. \tag{3.16}$$

Take any $\xi \in [-1/2, 1/2]^n$ and integrate (3.16) over $\mathbf{B}(\xi, \varepsilon)$ to obtain

$$\sum_{k\in I} |\mathbf{B}(\xi + k, \varepsilon) \cap K| \geq |\mathbf{B}(\xi, \varepsilon)|.$$

Therefore, if we take $\delta = 1/\#I$ then there is $k \in I$ such that $|\mathbf{B}(\xi + k, \varepsilon) \cap K| \geq \delta |\mathbf{B}(\xi, \varepsilon)|$ and hence $\xi + k \in \tilde{K}$. Thus (3.15) holds and $\tilde{K}$ given by (3.14) satisfies the Cohen condition. Therefore φ is a scaling function for the multiresolution analysis $(V_j)_{j\in\mathbb{Z}}$ defined by

$$V_j = \overline{\operatorname{span}}\{D_{A^j} T_l \varphi : l \in \mathbb{Z}^n\} \qquad \text{for } j \in \mathbb{Z}.$$

It remains to show that φ is in the Schwartz class. We are going to prove that φ is band-limited, i.e., $\hat{\varphi}$ is compactly supported. By (3.9) and (3.13)

$$\hat{\varphi}(\xi) \neq 0 \implies \xi \in B\mathbb{Z}^n + K^{+\varepsilon}. \tag{3.17}$$

On the other hand, by (3.10) $m(B^{-j}\xi) = 0$ for $\xi \in B^{j-1}\mathbb{Z}^n \setminus B^j\mathbb{Z}^n + B^{j-1}(K^{-\varepsilon})$. Since

$$\bigcup_{j=2}^{\infty} (B^{j-1}\mathbb{Z}^n \setminus B^j\mathbb{Z}^n) = B\mathbb{Z}^n \setminus \{0\},$$

and

$$K^{+\varepsilon} \subset B(K^{-\varepsilon}) \subset B^{j-1}(K^{-\varepsilon}) \qquad \text{for } j \geq 2,$$

we have

$$\hat{\varphi}(\xi) = 0 \qquad \text{for } \xi \in B\mathbb{Z}^n \setminus \{0\} + K^{+\varepsilon}. \tag{3.18}$$

Combining (3.17) and (3.18) we have $\hat{\varphi}(\xi) = 0$ for $\xi \in (K^{+\varepsilon})^{\mathbf{c}}$. Therefore $\operatorname{supp} \hat{\varphi} \subset K^{+\varepsilon}$ and therefore φ is in the Schwartz class. To conclude the proof of Theorem 3.2 it suffices to use Proposition 3.4 due to Wojtaszczyk, see [W, Corollary 5.17] which also holds for $r = \infty$.

Remark. It is widely known (see [S2]) that for every dilation B there is an ellipsoid Δ and an $s > 1$ such that $\Delta \subset s\Delta \subset B\Delta$. If one uses this fact to construct wavelets, the wavelets obtained do not have compactly supported Fourier transforms, see [B2] for details. It is thus crucial to our considerations that the set K in the definition of strictly expansive matrices be compact and tile the plane by translations.

Proposition 3.4. *Assume that we have a multiresolution analysis on* $\mathbb{R}^n$ *associated with an integral dilation* A*. Assume that this MRA has a scaling function* $\varphi(x)$ *in the Schwartz class such that* $\hat{\varphi}(\xi)$ *is real. Then there exists a wavelet family associated with this MRA consisting of* $(|\det A| - 1)$ *Schwartz class functions.*

The remainder of this section consists of finding a computationally convenient form of Theorem 3.2 in the two-dimensional case.

Proposition 3.5. *A two by two integer matrix* A *is expansive if and only if (a)* $|\det(A)| \geq 2$*, (b)* $|\mathrm{tr}(A)| \leq \det(A)$ *when the determinant is positive, and (c)* $|\mathrm{tr}(A)| \leq -\det(A) - 2$ *when the determinant of* A *is negative.*

Proof. We split into cases. First, we assume the eigenvalues are real. If $d = \det(A) > 0$ and $t = \mathrm{tr}(A) > 0$, then the eigenvalues $t \pm \sqrt{t^2 - 4d}$ have the same sign, and their absolute values are bigger than one if and only if $t - \sqrt{t^2 - 4d} > 2$. This holds if and only if $t^2 - 4t + 4 > t^2 - 4d$; that is, if and only if $t - 1 < d$, which is equivalent to $|t| \leq d$ in the case under consideration. If $t < 0$, then the eigenvalues are both negative, and A is expansive if and only if $t + \sqrt{t^2 - 4d} < -2$, which holds if and only if $t^2 + 4t + 4 > t^2 - 4d$; that is, $t > -d - 1$ which is equivalent to $|t| \leq |d|$ in the case under consideration.

If $d < 0$ and $t > 0$, then the eigenvalues have opposite signs, so both have absolute value bigger than one if and only if $t - \sqrt{t^2 - 4d} < -2$. This is easily seen to be equivalent to $t < -d - 1$ which is equivalent to $|t| \leq -d - 2$ in the case under consideration. If $t < 0$, then A is expansive if and only if $t + \sqrt{t^2 - 4d} > 2$, which is again easily seen to be equivalent to $t > d + 1$, which is equivalent to $|t| \leq -d - 2$ in the case under consideration.

If $t = 0$, it is easy to see that A is expansive if and only if $|d| \geq 2$, whether the eigenvalues are real or imaginary.

Finally, for complex eigenvalues, if $|t| = 1$, then A is expansive if and only if $1 - 4d < -3$; that is, $d \geq 2$. If $|t| \geq 2$, then A is expansive whenever $t^2 - 4d < 0$; that is, whenever $d > t^2/4$. Since d must also be an integer, this means A is expansive whenever $d \geq |t|$. Since the reverse implication falls under the previous cases considered (namely, when $d < |t|$), this completes the proof of the proposition.

Corollary 3.6. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an expansive, integer valued matrix. If there exists* $u \in \mathbb{R}$ *such that*

$$\eta(A) := \max\{|-uc + a|, |-(u+2)c + a|, |ud - b + (-1-u)(-uc + a)|,\\ |(u+2)d - b + (-1-u)(-(u+2)c + a)|\} < |\det(A)|,$$

then A *is strictly expansive.*

Proof. Let X be the (2-dimensional) Banach space with unit ball

$$B_X = \mathrm{conv}\{\pm(u, 1), \pm(u+2, 1)\}.$$

Consider $A : \mathbb{R}^2 \to \mathbb{R}^2$ as a linear map on X. We show that $|\det(A)| \|A^{-1}\| = \eta(A)$. Thus, under the hypotheses of the corollary, $\|A^{-1}\| < 1$ and $B_X \subset A(B_X)^\circ$. Hence A is strictly expansive with respect to the compact set $K = \frac{1}{2} B_X$ which satisfies (3.1) and (3.2).

First, note that $(x, y) \in X^* = \mathbb{R}^2$ has norm less than or equal to 1 if and only if $|ux + y| \le 1$ and $|(u+2)x + y| \le 1$, which holds if and only if $-1 - ux \le y \le 1 - ux$ and $-1 - (u+2)x \le y \le 1 - (u+2)x$. Hence, the extreme points of the unit ball of X^* are $\{\pm(0,1), \pm(1, -1-u)\}$. So, since

$$\det(A) A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

it follows that

$$\begin{aligned} |\det(A)| \|A^{-1}\| &= \max\left(\left\| A^{-1} \begin{pmatrix} u \\ 1 \end{pmatrix} \right\|_X, \left\| A^{-1} \begin{pmatrix} u+2 \\ 1 \end{pmatrix} \right\|_X \right) \\ &= \max\left(\left\| \begin{pmatrix} ud - b \\ -uc + a \end{pmatrix} \right\|_X, \left\| \begin{pmatrix} (u+2)d - b \\ -(u+2)c + a \end{pmatrix} \right\|_X \right) = \eta(A). \end{aligned}$$

Remark. *If we set $u = -1$ in Corollary 3.6, then we obtain that the matrix A is strictly expansive if the ℓ_1 norm of each column is less than the determinant of A.*

4. A reduction

In this section, we prove that every expansive two by two integer matrix admits a Meyer type wavelet if every expansive matrix of the form $\begin{pmatrix} 0 & 1 \\ -d & t \end{pmatrix}$ admits a Meyer type wavelet.

Suppose we are given an arbitrary dilation matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Although there are algorithms [V] for determining when A is integrally similar to $\begin{pmatrix} 0 & 1 \\ * & * \end{pmatrix}$ based on the theorem of Latimer and MacDuffee, these techniques do not seem to be well-suited for our purposes. Therefore, we will devise an *ad hoc* procedure of reducing a dilation matrix to a matrix which is either strictly expansive or integrally similar to a matrix of the form $\begin{pmatrix} 0 & 1 \\ * & * \end{pmatrix}$. (Note that by the theorem of Latimer and MacDuffee, there are (many) 2×2 dilation matrices which are not integrally similar to matrices of the form $\begin{pmatrix} 0 & 1 \\ * & * \end{pmatrix}$.)

Theorem 4.1. *Suppose A is a 2×2 dilation matrix with integer entries which is not integrally similar to the matrix of the form $\begin{pmatrix} 0 & 1 \\ * & * \end{pmatrix}$. Then A is integrally similar to a matrix which is strictly expansive with respect to a centrally symmetric set.*

Before starting the proof of Theorem 4.1 we need some prerequisites. We are going to employ the elementary transformations

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}, \tag{4.1}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}, \text{ and} \tag{4.2}$$

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+\lambda c & -\lambda^2 c + \lambda(d-a) + b \\ c & -\lambda c + d \end{pmatrix}. \tag{4.3}$$

Lemma 4.2. *Every integer matrix is integrally similar to a matrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with*

$$|b|, |c| \geq |a-d|. \tag{4.4}$$

Proof. Suppose A is a given dilation matrix. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an integrally similar matrix to A with the minimal sum of the diagonal entries, i.e., a^2+d^2. Since the sum of the diagonal entries of a matrix obtained by the transformation (4.3) is equal to $a^2+d^2+2\lambda c(\lambda c+a-d)$ we must have

$$\lambda c(\lambda c + a - d) \geq 0 \qquad \text{for all } \lambda \in \mathbb{Z}. \tag{4.5}$$

By taking $\lambda = \pm 1$ we see that $|c| < |a-d|$ would contradict (4.5). Therefore $|c| \geq |a-d|$. Using transformations (4.2) and (4.3) we can also conclude that $|b| \geq |a-d|$.

Therefore, without loss of generality, we can assume that for a given dilation A, (4.4) holds.

We now turn to showing that we can assume without loss of generality that $\operatorname{tr} A \geq 0$.

Lemma 4.3. *(a) A dilation matrix A is integrally similar to a matrix that is strictly expansive with respect to a centrally symmetric set if and only if $-A$ is integrally similar to a matrix that is strictly expansive with respect to a centrally symmetric set.*
(b) A dilation matrix A is integrally similar to $\begin{pmatrix} 0 & 1 \\ * & * \end{pmatrix}$ *if and only if $-A$ is integrally similar to* $\begin{pmatrix} 0 & 1 \\ * & * \end{pmatrix}$.

Proof. To see part (a), note that if A is similar to B, then $-A$ is similar to $-B$. Moreover, if B is strictly expansive with respect to the centrally symmetric set K, then $-B$ is also strictly expansive with respect to the centrally symmetric set K.

To see (b), if A is similar to $\begin{pmatrix} 0 & 1 \\ * & * \end{pmatrix}$, then $-A$ is similar to $\begin{pmatrix} 0 & -1 \\ * & * \end{pmatrix}$, which is similar to $\begin{pmatrix} 0 & 1 \\ * & * \end{pmatrix}$ by (4.1).

Thus, since A is a dilation matrix if and only if $-A$ is a dilation matrix, it suffices to prove Theorem 4.1 in the case that $\operatorname{tr} A \geq 0$.

Proof of Theorem 4.1. Suppose the dilation A satisfies (4.4) and $\operatorname{tr} A = a + d \geq 0$.
1° case. Suppose that

$$|a| < |b| \quad \text{and} \quad |d| < |c|. \tag{4.6}$$

Geometrically, (4.6) says that the vertices of the square $[-1,1]^2$, i.e., $(\pm 1, \pm 1)$ are mapped by the dilation A into different quadrants of the plane. Furthermore, the mapped vertices $A(\pm 1, \pm 1)$ can not lie on the axes.

We need to consider several subcases. Assume that exactly two of the mapped vertices $A(\pm 1, \pm 1)$ are of the form $(\pm 1, \pm 1)$. By the symmetry of the vertices $(\pm 1, \pm 1)$ and $|\det A| \geq 2$, the only remaining possibility is that none of the mapped vertices $A(\pm 1, \pm 1)$ is of the form $(\pm 1, \pm 1)$. Furthermore, if the dilation $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ maps exactly two of the vertices $(\pm 1, \pm 1)$ into each other, so does a dilation obtained by the transformation (4.1) or (4.2). Hence by applying (4.1) and (4.2) we can also assume that $a, b \geq 0$. By (4.6) this uniquely determines then the first row, and A must be one of the matrices below

$$\begin{aligned} &\begin{pmatrix} a & a+1 \\ d+1 & d \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a & a+1 \\ -d-1 & d \end{pmatrix} \qquad \text{if } d \geq 0, \\ &\begin{pmatrix} a & a+1 \\ d-1 & d \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a & a+1 \\ -d+1 & d \end{pmatrix} \qquad \text{if } d \leq 0. \end{aligned} \tag{4.7}$$

Note that we must have $a, |d| \geq 1$, since our dilation is not integrally similar to $\begin{pmatrix} 0 & 1 \\ * & * \end{pmatrix}$. The first matrix in (4.7) can not be even expansive by Proposition 3.5. The second and the fourth matrix are strictly expansive by the remark following Corollary 3.6, since their determinants are $2ad + a + d + 1$ and $2ad - a + d - 1$, respectively. Finally, using (4.4) for the third dilation in (4.7) we must necessarily have that $a = 1$ and $d = -1$. It is then easy to show that $\begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}$ is strictly expansive by Corollary 3.6 with $u = -1.1$.

Therefore, we can assume that none of the vertices $(\pm 1, \pm 1)$ is mapped by A into $(\pm 1, \pm 1)$. By (4.6) we have that $[-1,1]^2 \subset A[-1,1]^2$. If $[-1,1]^2 \subset A(-1,1)^2$ then A is strictly expansive. By a simple geometry, the last inclusion may fail only if at least one (and thus two by the symmetry) sides of the square $[-1,1]^2$ are contained in the boundary of the parallelogram $A[-1,1]^2$. This means that the vertices of $A[-1,1]^2$ must be of the form $\pm(x_1,1),\pm(x_2,1)$ or $\pm(1,y_1),\pm(1,y_2)$. But then the matrix A must be of the form $\begin{pmatrix} * & * \\ \pm 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & \pm 1 \\ * & * \end{pmatrix}$, respectively. This would mean that A is integrally similar to $\begin{pmatrix} 0 & 1 \\ * & * \end{pmatrix}$ — a contradiction. This ends the proof of case 1°.

We remark that we can now assume that the diagonal entries $a, d \ge 0$. Indeed, if one element in the diagonal is negative, the other must be positive (by $\operatorname{tr} A \ge 0$) and we automatically have (4.6) by (4.4).

2° case. Suppose that $b = 0$ or $c = 0$. By (4.2) we can assume that $c = 0$. By applying (4.3) to A we see that we can reduce the upper right corner of A in the absolute value to $\min_{\lambda\in\mathbb{Z}} |\lambda(d-a)+b| \le |d-a|/2$. Hence, it suffices to consider the case $|b| \le |a-d|/2$ and $c = 0$. Moreover, since (4.3) does not modify a or d when $c = 0$ and A is a dilation we must have that $a, d \ge 2$. By the remark following Corollary 3.6, A is strictly expansive since

$$|b| + |d| \le |a-d|/2 + |d| < 2\max(|a|,|d|) \le |ad| = |\det A|.$$

This shows the case 2°.

Suppose next that neither case 1° nor 2° holds. Since (4.6) fails then we either have $a \ge |b|$ or $d \ge |c|$. Without loss of generality, we can assume that $d \ge |c|$ by applying (4.2) and $c \ge 0$ by (4.1). Since 2° does not hold, c must be positive. Moreover, $c \ne 1$. Indeed, for $c = 1$ the matrix A is integrally similar to $\begin{pmatrix} 0 & 1 \\ * & * \end{pmatrix}$ by the application of (4.3) with $\lambda = -d$ and (4.2). Therefore we are left with the possibility that our dilation $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies

$$a, c, d \ge 0, \quad 2 \le c \le d, \quad c, |b| \ge |a-d|. \tag{4.8}$$

The analysis of classes of dilations satisfying (4.8) is quite intricate and technical. We split the argument into positive and negative determinant case.

3° case. Suppose that A satisfies (4.8) and $\det A > 0$, and hence $\det A \ge 2$. We claim that A is strictly expansive by the virtue of Corollary 3.6 with $u = -d/c + \delta$ for sufficiently small $\delta > 0$, unless A is $\begin{pmatrix} 0 & -2 \\ 2 & 2 \end{pmatrix}$. This last dilation is also strictly expansive by Corollary 3.6 with $u = -1.1$.

First take $u = -d/c$. We claim that the last three expressions appearing in the definition of $\eta(A)$ are (strictly) less than $|\det A|$. If this is the case then for sufficiently small $\delta > 0$ these three inequalities will continue to hold with $u = -d/c + \delta$ and moreover

$$|-uc + a| = |-\delta c + a + d| < \operatorname{tr} A \le \det A.$$

Hence A is strictly expansive by Corollary 3.6 with $u = -d/c + \delta$. Finally, to show the claim, take $u = -d/c$. The second expression appearing in $\eta(A)$ satisfies by (4.8)

$$|-(u+2)c + a| = |-2c + a + d| \leq \operatorname{tr} A \leq \det A.$$

Furthermore, at least one of the above inequalities has to be strict. If we had an equality in the first inequality then $a = 0$ and $c = d$ which would imply $b \leq -d$ and thus $\operatorname{tr} A < \det A$. The third expression in $\eta(A)$ satisfies

$$|ud - b + (-1-u)(-uc + a)| = |-b - d - a + ad/c| = |\det A/c - \operatorname{tr} A| < \det A.$$

Finally, we need to show that the fourth expression in $\eta(A)$ satisfies

$$\begin{aligned} |(u+2)d - b + (-1-u)(-(u+2)c + a)| &= |-b - d - a + 2c + ad/c| \\ &= |\det A/c - \operatorname{tr} A + 2c| < \det A. \end{aligned} \tag{4.9}$$

Indeed,

$$\det A(1 - 1/c) + \operatorname{tr} A - 2c \geq \operatorname{tr} A(2 - 1/c) - 2c = (2 - 1/c)(a + d - c) - 1 > 0,$$

unless $a + d - c = 0$, that is $a = 0$ and $c = d$ which would imply $b \leq -d$ by (4.8) (b is negative since $\det A > 0$). If this is the case then (4.9) reduces to $-b + d < -bd$, i.e., $b < -d/(d-1)$ which clearly holds unless $d = 2$ and $b = -2$. Therefore, A must be $\begin{pmatrix} 0 & -2 \\ 2 & 2 \end{pmatrix}$ which was excluded from our considerations in the beginning of case 3°. This shows the claim and ends the proof of 3°.

4° case. Suppose that A satisfies (4.8) and $\det A < 0$, and hence $\det A \leq -2$. We claim that A is strictly expansive by the virtue of Corollary 3.6 with

$$u = \frac{(a-d) - \sqrt{(a-d)^2 + 4bc}}{2c}.$$

Let $\lambda_1 < 0$ be the negative eigenvalue of A and $\lambda_2 > 0$ be the positive eigenvalue of A. Note that by expansiveness $\lambda_1 < -1$ and $\lambda_2 > 1$. We need to check that the four expressions appearing in the definition of $\eta(A)$ are (strictly) less than $|\det A|$.

$$-uc + a = \frac{(d+a) + \sqrt{(a+d)^2 - 4(ad - bc)}}{2} = \frac{\operatorname{tr} A + \sqrt{(\operatorname{tr} A)^2 - 4 \det A}}{2} = \lambda_2 < |\det A|,$$

since $|\lambda_1| > 1$. Note that the above implies that

$$cu = -\lambda_2 + a. \tag{4.10}$$

Hence we also need to show that

$$|-(u+2)c+a| = |\lambda_2 - 2c| < |\det A|.$$

Since $c > 0$ and $1 < \lambda_2 < |\det A|$, it suffices to show that

$$\lambda_2 - 2c > \det A = \lambda_1\lambda_2.$$

Since $c \leq d$

$$\lambda_2 - 2c - \lambda_1\lambda_2 \geq \lambda_2 - 2d - \lambda_1\lambda_2 \geq \lambda_2 - 2\operatorname{tr} A - \lambda_1\lambda_2 = -2\lambda_1(-\lambda_1 - 1)\lambda_2 > 0.$$

The third inequality

$$|ud - b + (-1-u)(-uc+a)| < |\det A|,$$

is a consequence of

$$ud - b + (-1-u)(-uc+a) = uc - a = -\lambda_2. \tag{4.11}$$

Indeed, if we solve the quadratic equation above we have

$$u = \frac{(a-d) \pm \sqrt{(a-d)^2 + 4bc}}{2c}.$$

Finally we need to show

$$|(u+2)d - b + (-1-u)(-(u+2)c+a)| < |\det A|.$$

Note that by (4.10) and (4.11)

$$\begin{aligned}(u+2)d - b + (-1-u)(-(u+2)c+a) &= -\lambda_2 + 2d + 2c + 2cu \\ &= -\lambda_2 + 2c + 2d + 2a - 2\lambda_2 = -\lambda_2 + 2\lambda_1 + 2c.\end{aligned}$$

To see that $-\lambda_2 + 2\lambda_1 + 2c < -\det A$ note that since $c \leq d \leq \operatorname{tr} A$,

$$-\lambda_2 + 2\lambda_1 + 2c + \lambda_1\lambda_2 \leq -\lambda_2 + 2\lambda_1 + 2(\lambda_1 + \lambda_2) + \lambda_1\lambda_2 = 4\lambda_1 + \lambda_2(1 + \lambda_1) < 0.$$

It remains to show

$$2\lambda_1 - \lambda_2 + 2c - \det A > 0.$$

If $\lambda_1 \leq -3$ then

$$2\lambda_1 - \lambda_2 + 2c - \lambda_1\lambda_2 = 2\lambda_1 - (1 + \lambda_1)\lambda_2 + 2c \geq 2\lambda_1 + 2\lambda_2 + 2c = 2(\operatorname{tr} A + c) > 0.$$

If $-3 < \lambda_1 < -1$ then using

$$\sqrt{(\operatorname{tr} A)^2 - 4\det A} < 1 - \det A, \tag{4.12}$$

we have

$$2\lambda_1 - \lambda_2 + 2c - \det A = \lambda_1 - \sqrt{(\operatorname{tr} A)^2 - 4\det A} + 2c - \det A > \lambda_1 - 1 + 2c \geq 3 + \lambda_1 > 0,$$

since $c \geq 2$. Finally, (4.12) is equivalent to $(\operatorname{tr} A)^2 < (1 + \det A)^2$ which is a consequence of $|\operatorname{tr} A| \leq |2 + \det A|$ by Proposition 3.5. This ends the proof of case 4° and thus shows Theorem 4.1.

5. Special cases

In this section, we prove that there exist Meyer type wavelets for all expansive matrices of the form $\begin{pmatrix} 0 & 1 \\ -d & t \end{pmatrix}$. The determinant two case was completed in section 2, our main concern in this section is to prove

Theorem 5.1. *Every two by two dilation A is integrally similar to a strictly expansive matrix with respect to a centrally symmetric set unless $|\det(A)| = 2$ or $\det(A) = 3$ and $\mathrm{tr}(A) = 0$.*

Proof. By Theorem 4.1, it suffices to prove Theorem 5.1 for matrices of the form $\begin{pmatrix} 0 & 1 \\ -d & t \end{pmatrix}$. By Lemma 4.3 and equation (4.1), we can assume that $t \geq 0$.

Lemma 5.2. *Let A be an integer valued, expansive matrix of the form $A = \begin{pmatrix} 0 & 1 \\ -d & t \end{pmatrix}$ with $t \geq 0$. Then A is integrally similar to a strictly expansive matrix with respect to a centrally symmetric set if either (a) $d > 3$ and $t > 2$ or (b) $d < -3$ and $t \neq -d - 2$.*

Proof. First, suppose that $d > 3$ and $t > 3$. (Since A is expansive, we can rewrite as $3 < t \leq d$.) Then,

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -d & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -4 - d + 2t & -2 + t \end{pmatrix}.$$

Now, by Proposition 3.6 with $u = -1$, it suffices to show that $1 + |-2 + t| < d$ and $2 + |-4 - d + 2t| < d$.

To this end, note that $-d + 2 < t - 2 \leq d - 2$ implies that $1 + |t - 2| \leq d - 1$. For the second inequality, notice that $2t - 4 \geq 4$, so $2t - 4 - d \geq 4 - d > 2 - d$. Also, note that $2t - 4 \leq 2d - 4$, so $2t - 4 - d \leq d - 4 < d - 2$. Hence, $|2t - 4 - d| < d - 2$, as desired.

Now, if $d > 3$ and $t = 3$, then

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -d & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 - d + t & 1 + t \end{pmatrix}.$$

That the resulting matrix is strictly expansive follows from Proposition 3.6 with $u = -1$ by noting that $1 + |t - 1| = 3 < d$ and $1 + |-1 - d + t| = 1 + |2 - d| < d$.

Suppose that $d < 0$ and $t \neq -d - 2$ and $t \neq 0$. Then,

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -d & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 - d - t & 1 + t \end{pmatrix}.$$

We proceed with checking the ℓ_1 norms of the columns. First, $1 + |1 + t| = t + 2 < |d|$ by assumptions. Second, note that $1 + |-1 - d - t| \leq \max\{1 + |-1 - d - 1|, 1 + |-1 - d - (-d - 2)|\} = \max\{-d - 1, 2\} < -d$.

Finally, when $d < 0$ and $t = 0$, we can (as above) that $\begin{pmatrix} 0 & 1 \\ -d & t \end{pmatrix}$ is integer equivalent to $\begin{pmatrix} 2 & 1 \\ -4-d & -2 \end{pmatrix}$. Then, $1+|-2| = 3 < |d|$ and $2+|-4-d| = -d-4+2 = -d-2 < |d|$, as desired.

The remainder of this section focuses on the special cases which are not covered in Lemma 5.2; namely, $t \leq 2$ and $d > 3$, $t = |d| - 2$ and $d < -3$, and $|d| = 3$.

Lemma 5.3. *All matrices of the form* $\begin{pmatrix} 0 & 1 \\ -d & 2 \end{pmatrix}$ *are integrally similar to strictly expansive matrices with respect to centrally symmetric sets, where* $d \geq 3$.

Proof. Apply the transformation

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -d & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -d+1 & 1 \end{pmatrix}.$$

We wish to show that there is a u as in Theorem 3.2 that makes our matrix strictly expansive. We claim that there is an $\epsilon = \epsilon(d) > 0$ such that $-1 < u < -1 + \epsilon$ works.

For the first equation, note that if $u < 0$, then $|u(-d+1)+1| = u(-d+1)+1$, which is less than d if and only if $u > -1$. For the second equation, if we plug in $u = -1$, then we obtain $|(u+2)(-d+1)+1| = d-2 < d$. So, by continuity, there is an $1 > \epsilon_1 > 0$ such that the second equation is satisfied when $-1 < u < -1 + \epsilon_1$.

For the third equation, again we plug in $u = -1$ to get $|u-1+(-1-u)(-u(1-d)+1)| = 2 < d$, so as above we obtain $0 < \epsilon_2 < \epsilon_1$ such that the second and third equations are satisfied when $-1 < u < -1 + \epsilon_2$.

Finally, for the last equation, we plug in $u = -1$ to get $|(u+2) - 1 + (-1-u)(-(u+2)(1-d)+1))| = 0 < d$. By continuity, get $\epsilon_3 < \epsilon_2$ such that all four equations are satisfied when $-1 < u < -1 + \epsilon_3$.

Lemma 5.4. *The matrices* $\begin{pmatrix} 0 & 1 \\ -d & 1 \end{pmatrix}$ *are integrally similar to strictly expansive matrices with respect to centrally symmetric sets for* $d \geq 3$.

Proof. Consider

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -d & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -d & 0 \end{pmatrix}.$$

Let

$$B = \operatorname{sym\,conv}\{(1,1),(1,-1)\}.$$

Then,

$$A(B) = \operatorname{sym\,conv}\{(2,-d),(0,-d)\},$$

as in Figure 3.

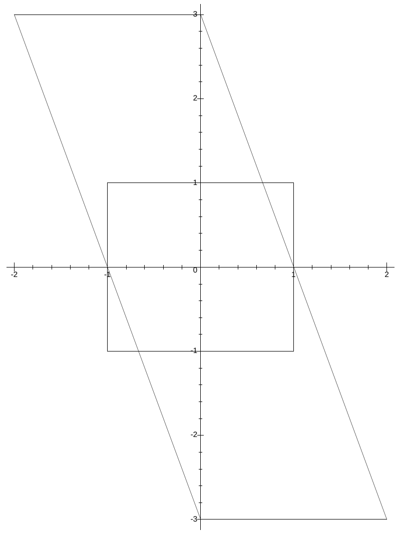

Figure 3. First Try For Strict Expansiveness

We need to (carefully) move the pieces of B which are not also in $A(B)$. Let $T = \text{conv}\{(1,0),(1,1),(1-1/d,1)\}$ and $T_\epsilon = \text{conv}\{(1,-\epsilon),(1,1),(1-1/d-\epsilon,1)\}$. Then, $A(T_\epsilon) = \text{conv}\{(1-\epsilon,-d),(2,-d),(2-1/d-\epsilon,-d+1+d\epsilon)\}$. So, for $\epsilon > 0$ and small enough, if we let $B' = B \setminus (T_\epsilon \cup (-T_\epsilon)) \cup (T_\epsilon - (0,2)) \cup (-T_\epsilon + (0,2))$, then $\overline{B'} \subset (A(B'))^\circ$ as pictured in Figure 4.

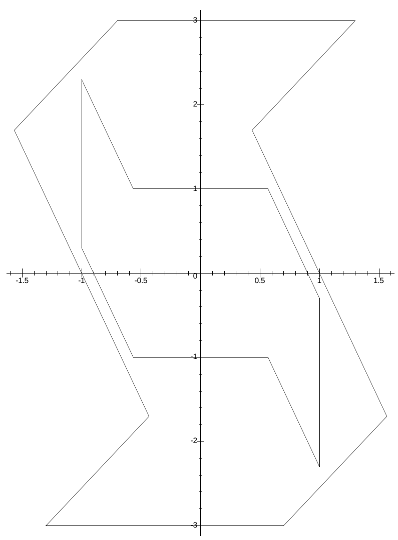

Figure 4. Strict Expansiveness for Trace 1

Lemma 5.5. *Every matrix of the form* $\begin{pmatrix} 0 & 1 \\ -d & -d-2 \end{pmatrix}$ *is integrally similar to a strictly expansive matrix with respect to a centrally symmetric set when* $d \leq -3$.

Proof. Note that A is integrally similar to $\begin{pmatrix} -1 & 1 \\ 1 & -d-1 \end{pmatrix}$, which satisfies Corollary 3.6 with $u = -1.1$. The easy details of verification are omitted.

Lemma 5.6. *For $|d| \geq 4$ and $d = -3$, the matrices $\begin{pmatrix} 0 & 1 \\ -d & 0 \end{pmatrix}$ are integrally similar to strictly expansive matrices with respect to centrally symmetric sets.*

Proof. Consider

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -d & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1-d & -1 \end{pmatrix}.$$

Let $B = \operatorname{sym conv}\{(1,1),(1,-1)\}$. Then, $A(B) = \operatorname{sym conv}\{(2,-d-2),(0,-d)\}$.

Now, in the case that d is positive, let $T = \operatorname{conv}\{(1,-1),(1,1),(1-2/(d+1),1)\}$ and let

$$K_0 = \operatorname{sym conv}\{(1-2/(d+1),1),(1,-1)\} \cup (T+(0,-2)) \cup (-T+(0,2)).$$

Since $A(T) = \operatorname{conv}\{(0,-d),(2,-d-2),(2-2/(d+1),-d)\}$, we have $K_0 \subset A(K_0)$. Moreover, for $d \geq 4$, $K_0 \setminus A(K_0^\circ)$ consists of two line segments connecting $\pm(1-2/(d+1),1)$ and $\pm(1,-1)$. Finally, to show that A is strictly expansive it suffices to slightly modify the set K_0 into a compact set K satisfying $K \subset A(K^\circ)$ and $\sum_{k\in\mathbb{Z}^2} \mathbf{1}_K(\xi+2k) = 1$ for a.e. $\xi \in \mathbb{R}^2$. Given $\varepsilon \geq 0$ define points $v_1 = (1-2/(d+1)-\varepsilon, 1)$, $v_2 = (1,-1-(d+1)\varepsilon)$, $v_3 = (1,-3+(d+1)\varepsilon)$, $v_4 = (1+\varepsilon,-3)$, $v_5 = (1-\varepsilon,-3)$, $v_6 = (1-2/(d+1)-\varepsilon,-1)$, and $v_j = -v_{j-6}$ for $7 \leq j \leq 12$. Let K_ε be a polygon whose boundary consists of line segments connected by vertices $v_1, v_2, \ldots, v_{12}$. Note that for $\varepsilon = 0$, K_ε is just K_0 given as above. Finally, it follows by simple (but long) calculations, that $K = K_\varepsilon$ satisfies strict expansiveness condition for sufficiently small $\varepsilon = \varepsilon(d) > 0$. Figure 5 shows polygons K_ε and $A(K_\varepsilon)$ when $d = 4$ and $\varepsilon = 0.1$.

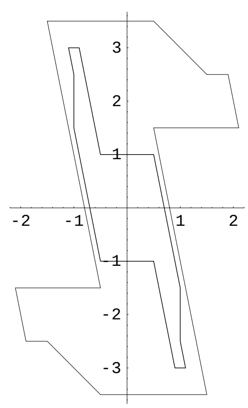

Figure 5. Strict Expansiveness for Trace 0, Det ≥ 4

In the case that d is negative, note that by (4.1) A is integrally similar to $\begin{pmatrix} 0 & -d \\ 1 & 0 \end{pmatrix}$. One easily checks that this matrix satisfies Corollary 3.6 with $u = -2-\delta$, for $\delta > 0$ small enough.

Proof of Theorem 5.1 (continued). The only matrices that are not covered by the above lemmas are the matrices $\begin{pmatrix} 0 & 1 \\ -3 & 3 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}$. Notice that $\begin{pmatrix} 0 & 1 \\ -3 & 3 \end{pmatrix}$ is similar to $A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$, which has $\eta(A) = 2.9$ when $u = -0.9$, proving Theorem 5.1.

The authors do not know whether the remaining matrix $\begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}$ is integrally similar to a strictly expansive matrix, nor whether any matrix of determinant two can be strictly expansive. However, it is easy to see that there is a Meyer type wavelet for the matrix $A = \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}$. Indeed, by Proposition 3.6, there are Meyer type wavelets ψ_1, ψ_2 with scaling function ϕ for dilation by -3 on the line. One can easily check that the functions $\psi_1 \otimes \phi$ and $\psi_2 \otimes \phi$ are wavelets for the dilation A. Thus, we have proven the following

Theorem 5.7. *Let A be an expansive, two by two integer matrix. Then, there exist Meyer type wavelets $\psi^1, \dots, \psi^l$, where $l = |\det A| - 1$.*

Proof. Combine Theorem 5.1 with Theorem 2.4, Theorem 4.1 and the example immediately preceding this theorem statement.

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