Chapter 1
Spanning and Independence Properties of Finite Frames

Peter G. Casazza and Darrin Speegle

Abstract The fundamental notion of frame theory is *redundancy*. It is this property which makes frames invaluable in so many diverse areas of research in mathematics, computer science and engineering because it allows accurate reconstruction after transmission losses, quantization, the introduction of additive noise and a host of other problems. This issue also arises in a number of famous problems in pure mathematics such as the Bourgain-Tzafriri Conjecture and its many equivalent formulations. As such, one of the most important problems in frame theory is to understand subsets the spanning and independence properties of sucsets of a frame. In particular, how many spanning sets does our frame contain? What is the smallest number of linearly independent subsets we can partition the frame into? What is the least number of Riesz basic sequences does the frame contain with universal lower Riesz bounds? Can we partition a frame into subsets which are nearly tight? This last question is equivalent to the infamous Kadison-Singer Problem. In this section we will present the state of the art on partitioning frames into linearly independent and spanning sets. A fundamental tool here is the famous Rado-Horn Theorem. We will give a new recent proof of this result along with some non-trivial generalizations of the theorem.

Key words: Spanning sets, independent sets, redundancy, Riesz sequence, Rado-Horn Theorem, spark, maximally robust, matroid, K-ordering of dimensions

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Peter G. Casazza
University of Missouri, Department of Mathematics, Columbia, MO 65211, USA, e-mail: casazzap@missouri.edu

Darrin Speegle,
Saint Louis University, Department of Mathematics and Computer Science, 221N. Grand Blvd., St. Louis, MO 63103 USA, e-mail: speegled@slu.edu

1.1 Introduction

The primary focus of this chapter is independence and spanning properties of finite frames. More specifically, we will be looking at partitioning frames into sets $\{A_k\}_{k=1}^K$ which are linearly independent, spanning, or both. Since increasing the number of sets in the partition makes it easier for each set to be independent, and harder to span, we will be looking for the smallest K needed to be able to choose independent sets, and the largest K allowed and still have each set of vectors spanning. In order to fix notation, let $\Phi = (\varphi_i)_{i=1}^M$ be a set of vectors in $\mathcal{H}^N$, not necessarily a frame. It is clear from dimension counting that if A_i is linearly independent for each $1 \le i \le K$, then $K \ge \lceil M/N \rceil$. It is also clear from dimension counting that if A_i spans $\mathcal{H}^N$ for each $1 \le i \le K$, then $K \le \lfloor M/N \rfloor$. So, in terms of linear independence and spanning properties, Φ is most "spread-out" if it can be partitioned into $K = \lceil M/N \rceil$ linearly independent sets, $\lfloor M/N \rfloor$ of which are also spanning sets.

This important topic of spanning and independence properties of frames was not developed in frame theory until recently. In [11] we see the first results on decompositions of frames into linearly independent sets. Recently, a detailed study of spanning and independence properties of frames was made in [5]. Also, in [6] we see a new notion of *redundancy* for frames which connects the number of linearly independent and spanning sets of a frame of non-zero vectors $(\varphi_i)_{i=1}^M$ to the largest and smallest eigenvalues of the frame operator of the normalized frame $\left(\frac{\varphi_i}{\|\varphi_i\|}\right)_{i=1}^M$. In this chapter we will see the state of the art on this topic which will also point out the remaining deep, important, open problems left on this topic.

Spanning and independence properties of frames are related to several important themes in frame theory. First, a fundamental open problem in frame theory is the Kadison-Singer problem in the context of frame theory, which was originally called the Feichtinger conjecture [11, 12, 16, 17]. The Kadison-Singer problem asks whether for every frame $\Phi = (\varphi_i)_{i\in I}$, not necessarily finite, that is norm bounded below, there exists a finite partition $\{A_j : j = 1, \dots, J\}$ such that for each $1 \le j \le J$, $(\varphi_i)_{i\in A_j}$ is a Riesz sequence. Since every Riesz sequence is, in particular, a linearly independent set, it is natural to study partitions of frames into linearly independent sets in order to better understand the Kadison-Singer problem in frame theory.

A second notion related to spanning and independence properties of frames is that of redundancy. Frames are sometimes described as "redundant" bases, and a theme throughout frame theory is to make the notion of redundancy precise. Two properties that have been singled out as desirable properties of redundancy are: redundancy should measure the maximal number of disjoint spanning sets, and redundancy should measure the minimal number of disjoint linearly independent sets [6]. Of course, these two numbers are not usually the same, but nonetheless, describing in an efficient way the maximal number of spanning sets and the minimal number of linearly independent sets is a useful goal in quantifying the redundancy of a frame.

A third place where spanning and independence properties of frames are vital, concerns *erasures*. (See the erasure section of the applications chapter). During transmission, it is possible that frame coefficients are lost (erasures) or corrupted. Then we have to try to do accurate reconstruction after losses of frame coefficients. This can be done if the remaining frame vectors still span the space. So, for example, if a frame contains at least two spanning sets, then we can still do perfect reconstruction after the loss of one frame vector.

A fundamental tool for working with spanning and independence properties of frames is the celebrated *Rado-Horn Theorem* [20, 24]. This theorem gives a necessary and sufficient condition for a frame to be partitioned into K disjoint linearly independent sets. The terminology *Rado-Horn Theorem* was introduced in the paper [7]. The Rado-Horn Theorem is a problem for frame theory in that it is impractical in applications. In particular, it requires doing a computation on every subset of the frame. What we want, is to be able to identify the minimal number of linearly independent sets we can partition a frame into by using *properties of the frame* such as the eigenvalues of the frame operator, the norms of the frame vectors etc. To do this, we will develop a sequence of deep refinements of the Rado-Horn Theorem [6, 15] which are able to determine the number of linearly independent and spanning sets of a frame in terms of the properties of the frame. There are at least four proofs of the Rado-Horn Theorem today [5, 15, 18, 20, 24]. The original proof is delicate. The recent refinements [5, 15] are even more so. So we will develop these refinements slowly throughout various sections of this chapter to make this understandable.

Finally, let us recall that any frame $\Phi = (\varphi_i)_{i=1}^M$ with frame operator S is isomorphic to a Parseval frame $S^{-1/2}\Phi = \left(S^{-1/2}\varphi_i\right)_{i=1}^M$ and these two frames have the same linearly independent and spanning sets. So in our work we will mostly be working with Parseval frames.

1.1.1 Full Spark Frames

There is one class of frames for which the answers to our questions concerning the partition of the frame into independent and spanning sets are obvious. These are the *full spark frames.*

Definition 1. The spark of a frame $(\varphi_i)_{i=1}^M$ in $\mathcal{H}_N$ is cardinality of the smallest linearly dependent subset of the frame. We say the frame is full spark if every N-element subset of the frame is linearly independent.

Full spark frames have appeared in the literature under the name *generic frames* [8] and *maximally robust to erasures* [13] since these frames have the property that the loss (erasure) of any $M-N$ of the frame elements still leaves a frame. For a full spark frame $(\varphi_i)_{i=1}^M$, any partition $\{A_j\}_{j=1}^K$ of $[1,M]$ into $K=\lceil \frac{M}{N} \rceil$ sets with $|A_j|=N$ for $j=1,2,\ldots,K-1$ and A_K the remaining elements has the property that $(\varphi_i)_{i\in A_k}$ is a linearly independent spanning set for all $1\le k\le K$ and $(\varphi_i)_{i\in A_K}$ is linearly independent (and also spanning if $M=KN$).

It appears that full spark frames are quite specialized and perhaps do not occur very often. But, it is known that every frame is arbitrarily close to a full spark frame. In [8] it is shown that this result holds even for Parseval frames. That is, the full spark frames are dense in the class of frames and the full spark Parseval frames are dense in the class of Parseval frames.

To prove these results, we do some preliminary work. For a frame $\Phi=(\varphi_i)_{i=1}^M$ with frame operator S, it is known that $\left(S^{-1/2}\varphi_i\right)_{i=1}^M$ is the closest Parseval frame to Φ [2, 3, 10, 14, 21]. Recall (See the chapter on the Kadison-Singer Problem and the Paulsen Problem) that a frame Φ for $\mathcal{H}^N$ is ϵ-nearly Parseval if the eigenvalues of the frame operator of the frame $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_N$ satisfy $1-\epsilon\le\lambda_N\le\lambda_1\le 1+\epsilon$.

Proposition 1. *Let $(\varphi_i)_{i=1}^M$ be an ϵ-nearly Parseval frame for $\mathcal{H}^N$, with frame operator S. Then $\left(S^{-1/2}\varphi_i\right)_{i=1}^M$ is the closest Parseval frame to $(\varphi_i)_{i=1}^M$ and*

$$\sum_{i=1}^M \|S^{-1/2}\varphi_i-\varphi_i\|^2 \le N(2-\epsilon-2\sqrt{1-\epsilon})\le N\frac{\epsilon^2}{4}.$$

Proof. See the section on applications of frames to problems in pure mathematics for a proof.

We also need to check that a frame which is close to a Parseval frame is itself close to being Parseval.

Proposition 2. *Let $\Phi=(\varphi_i)_{i=1}^M$ be a Parseval frame for $\mathcal{H}^N$ and let $\Psi=(\psi_i)_{i=1}^M$ be a frame for $\mathcal{H}^N$ satisfying*

$$\sum_{i=1}^M \|\varphi_i-\psi_i\|^2<\epsilon<\frac{1}{9}.$$

Then Ψ is a $3\sqrt{\epsilon}$ nearly Parseval frame.

Proof. Given $x\in\mathcal{H}^N$ we compute

$$\begin{aligned}\left(\sum_{i=1}^M|\langle x,\psi_i\rangle|^2\right)^{1/2} &\le \left(\sum_{i=1}^M|\langle x,\varphi_i-\psi_i\rangle|^2\right)^{1/2}+\left(\sum_{i=1}^M|\langle x,\varphi_i\rangle|^2\right)^{1/2}\\ &\le \|x\|\left(\sum_{i=1}^M\|\varphi_i-\psi_i\|^2\right)^{1/2}+\|x\|\\ &\le \|x\|(1+\sqrt{\epsilon}).\end{aligned}$$

The lower frame bound is similar.

The final result needed is that if a Parseval frame Φ is close to a frame Ψ with frame operator S, then Φ is close to $S^{-1/2}\Psi$.

Proposition 3. *If $\Phi = (\varphi_i)_{i=1}^M$ is a Parseval frame for $\mathcal{H}^N$ and $\Psi = (\psi_i)_{i=1}^M$ is a frame with frame operator S satisfying*

$$\sum_{i=1}^{M} \|\varphi_i - \psi_i\|^2 < \epsilon < \frac{1}{9},$$

then

$$\sum_{i=1}^{M} \|\varphi_i - S^{-1/2}\psi_i\|^2 < 2\epsilon \left[1 + \frac{9}{4}N\right].$$

Proof. We compute

$$\begin{aligned}\sum_{i=1}^{M} \|\varphi_i - S^{-1/2}\psi_i\|^2 &\le 2\left[\sum_{i=1}^{M} \|\varphi_i - \psi_i\|^2 + \sum_{i=1}^{M} \|\psi_i - S^{-1/2}\psi_i\|^2\right] \\ &\le 2\left[\epsilon + N\frac{(3\sqrt{\epsilon})^2}{4}\right] \\ &= 2\epsilon\left[1 + \frac{9}{4}N\right],\end{aligned}$$

where in the second inequality we applied Proposition 1 to the frame $(\psi_i)_{i=1}^M$ which is $3\sqrt{\epsilon}$ nearly Parseval by Proposition 2.

Now we are ready for the main theorem. We will give a new elementary proof of this result.

Theorem 1. *Let $\Phi = (\varphi_i)_{i=1}^M$ be a frame for $\mathcal{H}^N$ and let $\epsilon > 0$. Then there is a full spark frame $\Psi = (\psi_i)_{i=1}^M$ so that*

$$\|\varphi_i - \psi_i\| < \epsilon, \text{ for all } i = 1, 2, \ldots, M.$$

Moreover, if Φ is a Parseval frame, then Ψ may be chosen to be a Parseval frame.

Proof. Since Φ must contain a linearly independent spanning set, we may assume that $(\varphi_i)_{i=1}^N$ is such a set. We let $\psi_i = \varphi_i$ for $i = 1, 2, \ldots, N$. The complement of the union of all hyperplanes spanned by subsets of $(\varphi_i)_{i=1}^N$ is open and dense in $\mathcal{H}^N$ and so there is a vector ψ_{N+1} in this open set with $\|\varphi_{N+1} - \psi_{N+1}\| < \epsilon$. By definition, $(\psi_i)_{i=1}^{N+1}$ is full spark. Now we continue this argument. The complement of the union of all hyperplanes spanned by subsets of $(\psi_i)_{i=1}^{N+1}$ is an open dense set in $\mathcal{H}^N$ and so we can choose a vector

ψ_{N+2} from this set with $\|\varphi_{N+2}-\psi_{N+2}\| < \epsilon$. Again, by construction, $(\psi_i)_{i=1}^{N+2}$ is full spark. Iterating this argument we construct $(\psi_i)_{i=1}^M$.

For the *moreover* part, we choose $\delta > 0$ so that $\delta < \frac{1}{9}$ and

$$2\delta \left[1 + \frac{9}{4}N\right] < \epsilon^2.$$

By the first part of the theorem, we can choose a full spark frame $(\psi_i)_{i=1}^M$ so that

$$\sum_{i=1}^M \|\varphi_i - \psi_i\|^2 < \delta.$$

Letting S be the frame operator for $(\psi_i)_{i=1}^M$, we have that $\left(S^{-1/2}\psi_i\right)_{i=1}^M$ is a full spark frame and by Proposition 3 we have that

$$\sum_{i=1}^M \|\varphi_i - S^{-1/2}\psi_i\|^2 < 2\delta \left[1 + \frac{9}{4}N\right] < \epsilon^2.$$

We end this section with an open problem:

Problem 1. If $(\varphi_i)_{i=1}^M$ is an equal norm Parseval frame for $\mathcal{H}^N$ and $\epsilon > 0$, is there a full spark equal norm Parseval frame $\Psi = (\psi_i)_{i=1}^M$ so that

$$\|\psi_i - \varphi_i\| < \epsilon, \text{ for all } i = 1, 2, \ldots, M?$$

We refer the reader to [1] for a discussion of this problem and its relationship to algebraic geometry.

1.2 Spanning and independence properties of finite frames

The main goal of this section is to show that equal norm Parseval frames of M vectors in $\mathcal{H}^N$ can be partitioned into $\lfloor M/N \rfloor$ bases and one additional set which is linearly independent. In particular, equal norm Parseval frames will contain $\lfloor M/N \rfloor$ spanning sets and $\lceil M/N \rceil$ linearly independent sets.

We begin by relating the algebraic properties of spanning and linear independence to the analytical properties of frames and Riesz sequences.

Proposition 4. *Let $\Phi = (\varphi_i)_{i=1}^M \subset \mathcal{H}^N$. Then, Φ is a frame for $\mathcal{H}^N$ if and only if* span $\Phi = \mathcal{H}^N$.

Proof. If Φ is a frame for $\mathcal{H}^N$ with frame operator S then $A \cdot Id \leq S$ for some $0 < A$. So Φ must span $\mathcal{H}^N$.

The converse is a standard compactness argument. If Φ is not a frame then there are vectors $x_n \in \mathcal{H}^N$ with $\|x_n\| = 1$ and satisfying:

$$\sum_{i=1}^{M} |\langle x_n, \varphi_i \rangle|^2 \leq \frac{1}{n}, \text{ for all } n = 1, 2, \ldots.$$

Since we are in a finite dimensional space, by switching to a subsequence of $\{x_n\}_{n=1}^{\infty}$ if necessary we may assume that $\lim_{n\to\infty} x_n = x \in \mathcal{H}^N$. Now we have

$$\begin{aligned} \sum_{i=1}^{M} |\langle x, \varphi_i \rangle|^2 &\leq 2 \left[\sum_{i=1}^{M} |\langle x_n, \varphi_i \rangle|^2 + \sum_{i=1}^{M} |\langle x - x_n, \varphi_i \rangle|^2 \right] \\ &\leq 2 \left[\frac{1}{n} + \sum_{i=1}^{M} \|x - x_n\|^2 \|\varphi_i\|^2 \right] \\ &= 2 \left[\frac{1}{n} + \|x - x_n\|^2 \sum_{i=1}^{M} \|\varphi_i\|^2 \right] \end{aligned}$$

As $n \to \infty$, the right-hand-side of the above inequality goes to zero. Hence,

$$\sum_{i=1}^{M} |\langle x, \varphi_i \rangle|^2 = 0,$$

and so $x \perp \varphi_i$ for all $i = 1, 2, \ldots, M$. That is, Φ does not span $\mathcal{H}^N$.

Proposition 5. *Let $\Phi = (\varphi_i)_{i=1}^{M} \subset \mathcal{H}^N$. Then, Φ is linearly independent if and only if Φ is a Riesz sequence.*

Proof. If Φ is a Riesz sequence then there is a constant $0 < A$ so that for all scalars $\{a_i\}_{i=1}^{M}$ we have

$$A \sum_{i=1}^{M} |a_i|^2 \leq \| \sum_{i=1}^{M} a_i \varphi_i \|^2.$$

Hence, if $\sum_{i=1}^{M} a_i \varphi_i = 0$, then $a_i = 0$ for all $i = 1, 2, \ldots, M$.

Conversely, if Φ is linearly independent then (See the Introduction) the lower Riesz bound of Φ equals the lower frame bound and so Φ is a Riesz sequence.

Notice that in the two propositions above, we do not say anything about the frame bounds or the Riesz bounds of the sets Φ. The following examples show that the lower frame bounds and Riesz bounds can be close to zero.

Example 1. Given $\epsilon > 0, N \in \mathbb{N}$, there is a linearly independent set containing N norm one vectors in $\mathcal{H}^N$ with lower frame bound (and hence lower Riesz bound) less than ϵ. To see this, let $(e_i)_{i=1}^N$ be an orthonormal basis for $\mathcal{H}^N$ and define a unit norm linearly independent set

$$\Phi = (\varphi_i)_{i=1}^N = \left(e_1, \frac{e_1 + \sqrt{\epsilon} e_2}{\sqrt{1+\epsilon}}, e_3, \ldots, e_N \right).$$

Now,

$$\sum_{i=1}^N |\langle e_2, \varphi_i \rangle|^2 = \frac{\epsilon}{1+\epsilon} < \epsilon.$$

1.2.1 Applications of the Rado-Horn Theorem I

Returning to the main theme of this chapter, we ask: When is it possible to partition a frame of M vectors for $\mathcal{H}^N$ into K linearly independent [resp, spanning] sets? The main combinatorial tool that we have to study this question is the Rado-Horn Theorem.

Theorem 2 (Rado-Horn Theorem I). *Let $\Phi = (\varphi_i)_{i=1}^M \subset \mathcal{H}^N$ and $K \in \mathbb{N}$. There exists a partition $\{A_1, \ldots, A_K\}$ of $[1, M]$ such that for each $1 \le k \le K$, the set $(\varphi_i : i \in A_k)$ is linearly independent if and only if for every non-empty $J \subset [1, M]$,*

$$\frac{|J|}{\dim \operatorname{span}\{\varphi_i : i \in J\}} \le K.$$

This theorem was proven in more general algebraic settings in [19, 20, 24], as well as later rediscovered in [18]. We delay the discussion of the proof of this theorem to Section 1.3. We content ourselves now with noting that the forward direction of the Rado-Horn Theorem I is essentially obvious. It says that in order to partition Φ into K linearly independent sets, there can not exist a subspace S which contains more than $K \dim(S)$ vectors. The reverse direction indicates that there are no obstructions to partitioning sets of vectors into linearly independent sets other than dimension counting obstructions.

We wish to use the Rado-Horn Theorem I to partition frames into linearly independent sets. Proposition 4 tells us that every spanning set is a frame, so it is clear that in order to get strong results we are going to need to make some assumptions about the frame. A natural extra condition is that of an *equal-norm Parseval frame*. Intuitively, equal norm Parseval frames have no preferred directions, so it seems likely that one should be able to partition them into a small number of linearly independent sets. We will be able to do better than that; we will relate the minimum norm of the vectors in the Parseval frame to the number of linearly independent sets that the frame can be partitioned into.

Proposition 6. *Let* $0 < C < 1$ *and let* Φ *be a Parseval frame with* M *vectors for* $\mathcal{H}^N$ *such that* $\|\varphi\|^2 \geq C$ *for all* $\varphi \in \Phi$. *Then,* Φ *can be partitioned into* $\lceil \frac{1}{C} \rceil$ *linearly independent sets.*

Proof. We show that the hypotheses of the Rado-Horn Theorem are satisfied. Let $J \subset [1, M]$. Let $S = \operatorname{span}\{\varphi_j : j \in J\}$, and let P denote the orthogonal projection of $\mathcal{H}^N$ onto S. Since the orthogonal projection of a Parseval frame is again a Parseval frame and the sum of the norms squared of the vectors of the Parseval frame is the dimension of the space, we have

$$\begin{aligned}\dim S &= \sum_{j=1}^{M} \|P_S \varphi_j\|^2 \geq \sum_{j \in J} \|P_S \varphi_j\|^2 \\ &= \sum_{j \in J} \|\varphi_j\|^2 \geq |J| C.\end{aligned}$$

Therefore,

$$\frac{|J|}{\dim \operatorname{span}\{\varphi_j : j \in J\}} \leq \frac{1}{C},$$

and Φ can be partitioned into $\lceil \frac{1}{C} \rceil$ linearly independent sets by the Rado-Horn Theorem.

We now present a trivial way of constructing an equal norm Parseval frame of M vectors for $\mathcal{H}^N$ when N divides M. Let $(e_i)_{i=1}^N$ be an orthonormal basis for $\mathcal{H}^N$ and let $\Phi = (Ce_1, \ldots, Ce_1, Ce_2, \ldots, Ce_2, \ldots, Ce_N, \ldots, Ce_N)$ be the orthonormal basis repeated M/N times, where $C = \sqrt{N/M}$. Then, it is easy to check that Φ is a Parseval frame. Another, slightly less trivial example is to union M/N orthonormal bases with no common elements and to normalize the vectors of the resulting set. In each of these cases, the Parseval frame can be trivially decomposed into M/N bases for $\mathcal{H}^N$. The following corollary can be seen as a partial converse.

Corollary 1. *If* Φ *is an equal norm Parseval frame of* M *vectors for* $\mathcal{H}^N$, *then* Φ *can be partitioned into* $\lceil M/N \rceil$ *linearly independent sets. In particular, if* $M = kN$, *then* Φ *can be partitioned into* k *Riesz bases.*

Proof. This follows immediately from Proposition 6 and the fact that

$$\sum_{i=1}^{M} \|\varphi_i\|^2 = N,$$

which tells us that $\|\varphi_i\|^2 = M/N$ for all $i = 1, \ldots, M$.

The argument above does not give any information about the lower Riesz bounds of the k Riesz bases we get in Corollary 1. Understanding these bounds is an exceptionally difficult problem and is equivalent to solving the Kadison-Singer Problem (see the Chapter on *The Kadison-Singer and Paulsen Problems in Finite Frame Theory*).

1.2.2 Applications of the Rado-Horn Theorem II

The Rado-Horn Theorem I has been generalized in several ways. In this section, we present the generalization to matroid and two applications of this generalization to partitioning into spanning and independent sets. We refer the reader to [22] for an introduction to matroid theory.

A *matroid* is a finite set X together with a collection $\mathcal{I}$ of subsets of X, which satisfies three properties:

1. $\emptyset \in \mathcal{I}$
2. if $I_1 \in \mathcal{I}$ and $I_2 \subset I_1$, then $I_2 \in \mathcal{I}$, and
3. if $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then there exists $x \in I_2 \setminus I_1$ such that $I_1 \cup \{x\} \in \mathcal{I}$.

Traditionally, the sets $I \in \mathcal{I}$ are called *independent* sets, which can lead to some confusion. For this chapter, we will use *linearly independent* to denote linear independence in the vector space sense, and *independent* to denote independence in the matroid sense. The rank of a set $E \subset X$ is defined to be the cardinality of a maximal independent (in the matroid sense) set contained in E.

There are many examples of matroids, but perhaps the most natural one comes from considering linear independence. Given a frame (or other finite collection of vectors) Φ in $\mathcal{H}^N$, define

$$\mathcal{I} = \{I \subset \Phi : I \text{ is linearly independent}\}.$$

It is easy to see that $(\Phi, \mathcal{I})$ is a matroid.

Another, slightly more involved example is to let X be a finite set which spans $\mathcal{H}^N$, and

$$\mathcal{I} = \{I \subset X : \text{span}(X \setminus I) = \mathcal{H}^N\}.$$

Then, in the definition of matroid, properties (1) and (2) are immediate. To see property (3), let I_1, I_2 be as in (3). We have that $\text{span}(X \setminus I_1) = \text{span}(X \setminus I_2) = \mathcal{H}^N$. Let $E_1 = X \setminus I_1$ and $E_2 = X \setminus I_2$; then, we have $|E_1| > |E_2|$. Find a basis G_1 for $\mathcal{H}^N$ by first taking a maximal linearly independent subset F of $E_1 \cap E_2$, and adding elements from E_1 to form a basis. Then find another basis G_2 for $\mathcal{H}^N$ by taking F and adding elements from E_2. Since $|E_1| > |E_2|$, there must be an element $x \in E_1 \setminus E_2$ which was not chosen to be in G_1. Note that $x \in I_2 \setminus I_1$, and $I_1 \cup \{x\} \in \mathcal{I}$, since $X \setminus (I_1 \cup \{x\})$ contains G_1, which is a basis. Another important source of examples is graph theory.

There is a natural generalization of the Rado-Horn Theorem to the matroid setting.

Theorem 3 (Rado-Horn Theorem II). *[19] Let $(X, \mathcal{I})$ be a matroid, and let K be a positive integer. A set $J \subset X$ can be partitioned into K independent sets if and only if for every subset $E \subset J$,*

$$\frac{|E|}{\operatorname{rank}(E)} \leq K. \tag{1.1}$$

We will be applying the matroid version of the Rado-Horn Theorem to frames in Theorem 5 below, but first let us illustrate a more intuitive use. Consider the case of a collection Φ of M vectors where we wish to partition Φ into K linearly independent sets after discarding up to L vectors from Φ. It is natural to guess, based on our experience with the Rado-Horn Theorem, that this is possible if and only if for every non-empty $J \subset [1, M]$

$$\frac{|J| - L}{\dim \operatorname{span}\{\varphi_j : j \in J\}} \leq K.$$

However, it is not immediately obvious how to prove this from the statement of the Rado-Horn Theorem. In the following theorem, we prove that in some instances, the above conjecture is correct. Unfortunately, the general case will have to wait until we prove a different extension of the Rado-Horn Theorem in Theorem 6.

Proposition 7. *Let Φ be a collection of M vectors in $\mathcal{H}^N$ and $K, L \in \mathbb{N}$. If there exists a set H with $|H| \leq L$ such that the set $\Phi \setminus H$ can be partitioned into K linearly independent sets, then for every non-empty $J \subset [1, M]$*

$$\frac{|J| - L}{\dim \operatorname{span}\{\varphi_j : j \in J\}} \leq K.$$

Proof. If $J \subset [1, M] \setminus H$, then the Rado-Horn Theorem I implies

$$\frac{|J|}{\dim \operatorname{span}\{\varphi_j : j \in J\}} \leq K.$$

For general J with $|J| \geq L + 1$, notice that

$$\frac{|J| - L}{\dim \operatorname{span}\{\varphi_j : j \in J\}} \leq \frac{|J \setminus H|}{\dim \operatorname{span}\{\varphi_j : j \in J \setminus H\}} \leq K,$$

as desired.

Proposition 8. *Let Φ be a collection of M vectors in $\mathcal{H}^N$ indexed by $[1, M]$ and let $L \in \mathbb{N}$. Let $\mathcal{I} = \{I \subset [1, M] :$ there exists a set $H \subset I$ with $|H| \leq L$ such that $I \setminus H$ is linearly independent$\}$. Then $(\Phi, \mathcal{I})$ is a matroid.*

Proof. As usual, the first two properties of matroids are immediate. For the third property, let $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$. There exist H_1 and H_2 such that $I_j \setminus H_j$ is linearly independent and $|H_j| \leq L$ for $j = 1, 2$. If $|H_1|$ can be chosen so that $|H_1| < L$, then we can add any vector to I_1 and still have the new set linearly independent. If $|H_1|$ must be chosen to have cardinality L, then $|I_1 \setminus H_1| < |I_2 \setminus H_2|$ and both sets are linearly independent, so there is a vector $x \in (I_2 \setminus H_2) \setminus (I_1 \setminus H_1)$ so that $(I_1 \setminus H_1) \cup \{x\}$ is linearly independent. By the assumption that H_1 must be chosen to have cardinality L, $x \notin H_1$. Therefore, $x \notin I_1$ and $I_1 \cup \{x\} \in \mathcal{I}$, as desired.

Theorem 4. *Let $\Phi = (\varphi_i)_{i=1}^M$ be a collection of M vectors in $\mathcal{H}^N$. Let $K, L \in \mathbb{N}$. There exists a set H with $|H| \leq LK$ such that the set $\Phi \setminus H$ can be partitioned into K linearly independent sets if and only if for every non-empty $J \subset [1, M]$,*

$$\frac{|J| - LK}{\dim \operatorname{span}\{\varphi_j : j \in J\}} \leq K.$$

Proof. The forward direction is a special case of Proposition 7. For the reverse direction, define the matroid $(\Phi, \mathcal{I})$ as in Proposition 8. By the matroid version of the Rado-Horn Theorem, we can partition Φ into K independent sets if and only if for every non-empty $J \subset [1, M]$,

$$\frac{|J|}{\operatorname{rank}(\{\varphi_j : j \in J\})} \leq K.$$

We now show that this follows if for every non-empty $J \subset [1, M]$,

$$\frac{|J| - LK}{\dim \operatorname{span}\{\varphi_j : j \in J\}} \leq K.$$

Suppose we have for every non-empty $J \subset [1, M]$,

$$\frac{|J| - LK}{\dim \operatorname{span}\{\varphi_j : j \in J\}} \leq K.$$

Let $J \subset [1, M]$. Note that if we can remove fewer than L vectors from $(\varphi_j)_{j \in J}$ to form a linearly independent set, then $\operatorname{rank}(\{\varphi_j : j \in J\}) = |J|$, so

$$\frac{|J|}{\operatorname{rank}(\{\varphi_j : j \in J\})} = 1 \leq K.$$

On the other hand, if we need to remove at least L vectors from $(\varphi_j)_{j \in J}$ to form a linearly independent set, then $\operatorname{rank}(\{\varphi_j : j \in J\}) = \dim \operatorname{span}\{\varphi_j : j \in J\} + L$, so

$$\begin{aligned} |J| &\leq K \dim \operatorname{span}\{\varphi_j : j \in J\} + LK \\ &= K \operatorname{rank}(\{\varphi_j : j \in J\}), \end{aligned}$$

as desired. Therefore, if for every $J \subset [1, M]$,

$$\frac{|J| - LK}{\dim \operatorname{span}\{\varphi_j : j \in J\}} \leq K,$$

then there is a partition $\{A_i\}_{i=1}^K$ of $[1, M]$ such that $(\varphi_j : j \in A_i) \in \mathcal{I}$ for each $1 \leq i \leq K$. By the definition of our matroid, for each $1 \leq i \leq K$, there exists

$H_i \subset A_i$ with $|H_i| \leq L$ such that $(\varphi_j : j \in A_i \setminus H_i)$ is linearly independent. Let $H = \cup_{i=1}^{K} H_i$ and note that $|H| \leq LK$ and $J \setminus H$ can be partitioned into K linearly independent sets.

The matroid version of the Rado-Horn Theorem will be applied to finite frames in the following theorem.

Theorem 5. *Let $\delta > 0$. Suppose that $\Phi = (\varphi_i)_{i=1}^{M}$ is a Parseval frame of M vectors for $\mathcal{H}^N$ with $\|\varphi_i\|^2 \leq 1 - \delta$ for all $\varphi \in \Phi$. Let $R \in \mathbb{N}$ such that $R \geq \frac{1}{\delta}$. Then, it is possible to partition $[1, M]$ into R sets $\{A_1, \ldots, A_R\}$ such that for each $1 \leq r \leq R$, the family $(\varphi_j : j \notin A_r)$ spans $\mathcal{H}^N$.*

Proof. Let $\mathcal{I} = \{E \subset [1, M] : \operatorname{span}\{\varphi_j : j \notin E\} = \mathcal{H}^N\}$. Since any frame is a spanning set, we have that $([1, M], \mathcal{I})$ is a matroid. By the Rado-Horn Theorem II, it suffices to show (1.1) for each subset of $[1, M]$. Let $E \subset [1, M]$. Define $S = \operatorname{span}\{\varphi_j : j \notin E\}$, and let P be the orthogonal projection onto $S^\perp$. Since the orthogonal projection of a Parseval frame is again a Parseval frame, we have that $(P\varphi : \varphi \in \Phi)$ is a Parseval frame for $S^\perp$. Moreover, we have

$$\begin{aligned}\dim S^\perp &= \sum_{j=1}^{M} \|P\varphi_j\|^2 = \sum_{j \in E} \|P\varphi_j\|^2 \\ &\leq |E|(1 - \delta).\end{aligned}$$

Let M be the largest integer smaller than or equal to $|E|(1 - \delta)$. Since $\dim S^\perp \leq M$, we have that there exists a set $E_1 \subset E$ such that $|E_1| = M$ and $\operatorname{span}\{P\varphi_j : j \in E_1\} = S^\perp$. Let $E_2 = E \setminus E_1$. We show that E_2 is independent. For this, write $h \in \mathcal{H}^N$ as $h = h_1 + h_2$, where $h_1 \in S$ and $h_2 \in S^\perp$. We have that $h_2 = \sum_{j \in E_1} \alpha_j P f \varphi_j$ for some choice of $\{\alpha_j : j \in E_1\}$. Write $\sum_{j \in E_1} \alpha_j \varphi_j = g_1 + h_2$, where $g_1 \in S$. Then, there exist $\{\alpha_j : j \notin E\}$ such that $\sum_{j \notin E} \alpha_j \varphi_j = h_1 - g_1$. So,

$$\sum_{j \notin E_2} \alpha_j \varphi_j = h,$$

and thus E_2 is independent.

Now, since E contains an independent set of cardinality $|E| - M$, it follows that $\operatorname{rank}(E) \geq |E| - M \geq |E| - |E|(1 - \delta) = \delta|E|$. Therefore,

$$\frac{|E|}{\operatorname{rank}(E)} \leq \frac{1}{\delta} \leq R,$$

as desired.

1.2.3 Applications of the Rado-Horn Theorem III

Up to this point, we have mostly focused on linear independence properties of frames. We now turn to spanning properties. We present a more general form of the Rado-Horn Theorem, which describes what happens when the vectors cannot be partitioned into linearly independent sets.

The worst possible blockage that can occur preventing us from partitioning a frame $(\varphi_i)_{i=1}^M$ into K linearly independent sets would be the case where there are disjoint subsets (not necessarily a partition) $\{A_k\}_{k=1}^K$ of $[1, M]$ with the property:

$$\text{span}\,(\varphi_i)_{i\in A_j} = \text{span}\,(\varphi_i)_{i\in A_k}\,,\ \text{for all } 1 \le j, k \le K.$$

The following improvement of the Rado-Horn Theorem shows the surprising fact that this is really the only blockage that can occur.

Theorem 6 (Rado-Horn Theorem III). *Let $\Phi = (\varphi_i)_{i=1}^M$ be a collection of vectors in $\mathcal{H}^N$ and $K \in \mathbb{N}$. Then the following conditions are equivalent.*

(1) There exists a partition $\{A_k : k = 1, \dots, K\}$ of $[1, M]$ such that for each $1 \le k \le K$ the set $\{\varphi_j : j \in A_k\}$ is linearly independent.
(2) For all $J \subset I$,

$$\frac{|J|}{\dim \text{span}\{\varphi_j : j \in J\}} \le K. \tag{1.2}$$

Moreover, in the case that either of the conditions above fails, there exists a partition $\{A_k : k = 1, \dots, K\}$ of $[1, M]$ and a subspace S of $\mathcal{H}^N$ such that the following three conditions hold.

(a) For all $1 \le k \le K$, $S = \text{span}\{\varphi_j : j \in A_k \text{ and } \varphi_j \in S\}$.

(b) For $J = \{i \in I : \varphi_i \in S\}$, $\frac{|J|}{\dim \text{span}(\{\varphi_i : i \in J\})} > K$.

(c) For each $1 \le k \le K$, $\{P_{S^\perp}\varphi_i : i \in A_k, \varphi_i \notin S\}$ is linearly independent, where $P_{S^\perp}$ is the orthogonal projection onto $S^\perp$.

For the purposes of this chapter, we are restricting to $\mathcal{H}^N$, but the result also holds with a slightly different statement for general vector spaces, see [15] for details.

The statement of Theorem 6 is somewhat involved, and the proof even more so, so we delay the proof until Section 1.4. For now, we show how Theorem 6 can be applied in two different cases. For our first application, we will provide a proof of Theorem 4 in the general case.

Theorem 7. *Let $\Phi = (\varphi_i)_{i=1}^M$ be a collection of M vectors in $\mathcal{H}^N$. Let $K, L \in \mathbb{N}$. There exists a set H with $|H| \le L$ such that the set $\Phi \setminus H$ can be partitioned into K linearly independent sets if and only if for every non-empty $J \subset [1, M]$,*

$$\frac{|J| - L}{\dim \text{span}\{\varphi_j : j \in J\}} \le K.$$

Proof. The forward direction is Proposition 7. For the reverse direction, if Φ can be partitioned into K linearly independent sets, then we are done. Otherwise, we can apply the alternative in Theorem 6 to obtain a partition $\{A_k : 1 \le k \le K\}$ and a subspace S satisfying the properties listed.

For $1 \le k \le K$, let $A_k^1 = \{j \in A_k : \varphi_j \in S\}$, and $A_k^2 = A_k \setminus A_k^1 = \{j \in A_k : \varphi_j \notin S\}$. For each $1 \le k \le K$, let $B_k \subset A_k^1$ be defined such that $(\varphi_j : j \in B_k)$ is a basis for S, which is possible by property (a) in Theorem 6. Letting $J = \cup_{k=1}^K A_k^1$ and applying

$$\frac{|J| - L}{\dim \operatorname{span}\{\varphi_j : j \in J\}} \le K$$

yields that there are at most L vectors in J which are not in one of the B_k's. Let $H = J \setminus \cup_{k=1}^K B_k$. Since $|H| \le L$, it suffices to show that letting $C_k = B_k \cup A_k^2$ partitions $[1, M] \setminus H$ into linearly independent sets.

Indeed, fix k and assume that $\sum_{j \in C_k} a_k \varphi_k = 0$. Then

$$\begin{aligned} 0 &= \sum_{j \in C_k} a_k P_{S^\perp} \varphi_j \\ &= \sum_{j \in A_k^2} a_k P_{S^\perp} \varphi_j . \end{aligned}$$

So $a_k = 0$ for all $k \in A_k^2$ by property (c) in Theorem 6. This implies that

$$\begin{aligned} 0 &= \sum_{j \in C_k} a_k \varphi_j \\ &= \sum_{j \in B_k} a_k \varphi_j , \end{aligned}$$

and so $a_k = 0$ for all $k \in B_k$. Therefore, $\{C_k\}$ is a partition of $[1, M] \setminus H$ such that for each $1 \le k \le K$, the set $(\varphi_j : j \in C_k)$ is linearly independent.

We now present an application that is more directly related to frame theory. This theorem will be combined with Theorem 10 to prove Lemma 2.

Theorem 8. *Let $\Phi = (\varphi_i)_{i=1}^M$ be an equal-norm Parseval frame for $\mathcal{H}^N$. Let $K = \lfloor M/N \rfloor$. Then there exists a partition $\{A_k\}_{k=1}^K$ of $[1, M]$ so that*

$$\operatorname{span}\{\varphi_i : i \in A_j\} = \mathcal{H}^N, \quad \textit{for all } j = 1, 2, \ldots, K.$$

Our method of proof of Theorem 8 involves induction on the dimension N. In order to apply the induction step, we will project onto a subspace, which, while it preserves the Parseval frame property, does not preserve equal-norm of the vectors. For this reason, we state a more general theorem that is more amenable to an induction proof.

Theorem 9. *Let $\Phi = (\varphi_i)_{i=1}^M$ be a frame for $\mathcal{H}^N$ with lower frame bound $A \geq 1$, let $\|\varphi_i\|^2 \leq 1$ for all $i \in [1, M]$ and set $K = \lfloor A \rfloor$. Then there exists a partition $\{A_k\}_{k=1}^K$ of $[1, M]$ so that*

$$\text{span } \{\varphi_i : i \in A_k\} = \mathcal{H}^N, \quad \textit{for all } k = 1, 2, \ldots, K.$$

In particular, the number of frame vectors in a unit norm frame with lower frame bound A is greater than or equal to $\lfloor A \rfloor N$.

We will need the following lemma, which we state without proof.

Lemma 1. *Let $\Phi = (\varphi_i)_{i=1}^M$ be a collection of vectors in $\mathcal{H}^N$ and let $I_k \subset [1, M]$, $k = 1, 2, \ldots K$ be a partition of Φ into linearly independent sets. Assume that there is a partition of $[1, M]$ into $\{A_k\}_{k=1}^K$ so that*

$$\text{span } (\varphi_i)_{i \in A_k} = \mathcal{H}^N, \quad \textit{for all } k = 1, 2, \ldots, K.$$

Then,

$$\text{span } \{\varphi_i\}_{i \in I_k} = \mathcal{H}^N, \quad \textit{for all } k = 1, 2, \ldots, K.$$

Proof (of Theorem 10). We replace $(\varphi_i)_{i=1}^M$ by $\left(\frac{1}{\sqrt{K}}\varphi_i\right)_{i=1}^M$ so that our frame has lower frame bound greater than or equal to 1 and $\|\varphi_i\|^2 \leq \frac{1}{K}$, for all $i \in [1, M]$. Assume the frame operator for $(\varphi_i)_{i=1}^M$ has eigenvectors $(e_j)_{j=1}^N$ with respective eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 1$. We proceed by induction on N.

We first consider $N = 1$: Since

$$\sum_{i=1}^M \|\varphi_i\|^2 \geq 1, \quad \text{and} \quad \|\varphi_i\|^2 \leq \frac{1}{K}, \tag{1.3}$$

it follows that $|\{i \in I : \varphi_i \neq 0\}| \geq K$ and so we have a partition of the frame into K spanning sets.

Next, we assume the induction hypothesis holds for any Hilbert space of dimension N and let $\mathcal{H}^{N+1}$ be a Hilbert space of dimension $N + 1$. We check two cases:

Case I: Suppose there exists a partition $\{A_k\}_{k=1}^K$ of $[1, M]$ so that $(\varphi_i)_{i \in A_k}$ is linearly independent for all $k = 1, 2, \ldots, K$. In this case,

$$N + 1 \leq (N + 1)\lambda_N \leq \sum_{j=1}^{N+1} \lambda_j = \sum_{i=1}^M \|\varphi_i\|^2 \leq M\frac{1}{K},$$

and hence,

$$M \geq K(N + 1).$$

However, by linear independence, we have

$$M = \sum_{k=1}^{K} |A_k| \leq K(N+1).$$

Thus, $|A_k| = N+1$ for every $k = 1, 2, \ldots, K$ and so $(\varphi_i)_{i \in A_k}$ is spanning for $1 \leq k \leq K$.

Case II: Suppose $(\varphi_i)_{i=1}^M$ cannot be partitioned into K linearly independent sets. In this case, let $\{A_k\}_{k=1}^K$ and a subspace $\emptyset \neq S \subset \mathcal{H}^{N+1}$ be given by Theorem 6. If $S = \mathcal{H}^{N+1}$, we are done. Otherwise, let P be the orthogonal projection onto the subspace S. Let

$$A_k' = \{i \in A_k : \varphi_i \notin S\}, \quad B = \bigcup_{k=1}^{K} A_k'.$$

By Theorem 6(c), $((Id - P)\varphi_i)_{i \in A_k'}$ is linearly independent for all $k = 1, 2, \ldots, K$.

Now, $((Id - P)\varphi_i)_{i \in B}$ has lower frame bound 1 in $(Id - P)(\mathcal{H}^{N+1})$, dim $(Id - P)(\mathcal{H}^{N+1}) \leq N$ and

$$\|(Id - P)\varphi_i\|^2 \leq \|\varphi_i\|^2 \leq \frac{1}{K}$$

for all $i \in B$. Applying the induction hypothesis, we can find a partition $\{B_k\}_{k=1}^K$ of B with span $((Id - P)\varphi_i)_{i \in B_k} = (Id - P)(\mathcal{H}^{N+1})$ for all $k = 1, 2, \ldots, K$. Now, we can apply Lemma 1 together with the partition $\{B_k\}_{k=1}^K$ to conclude span $((Id - P)\varphi_i)_{i \in A_k'} = (Id - P)(\mathcal{H}^{N+1})$, and hence

$$\text{span}\,(\varphi_i)_{i \in A_k} = \text{span}\{S, ((Id - P)\varphi_i)_{i \in A_k'}\} = \mathcal{H}^{N+1}.$$

Up to this point, we have seen that an equal-norm Parseval frame with M vectors in $\mathcal{H}^N$ can be partitioned into $\lfloor M/N \rfloor$ spanning sets and $\lceil M/N \rceil$ linearly independent sets. We now show that there is a single partition which accomplishes both the spanning and linear independence properties.

Theorem 10. *Let $\Phi = (\varphi_i)_{i=1}^M$ be an equal-norm Parseval frame for $\mathcal{H}^N$ and let $K = \lceil M/N \rceil$. There exists a partition $\{A_k\}_{k=1}^K$ of $[1, M]$ such that*

1. *$(\varphi_i : i \in A_k)$ is linearly independent for $1 \leq k \leq K$, and*
2. *$(\varphi_i : i \in A_k)$ spans $\mathcal{H}^N$ for $1 \leq k \leq K - 1$.*

The proof of Theorem 10 is immediate from Corollary 1, Theorem 8 and Lemma 2 below.

Lemma 2. *Let $\Phi = (\varphi_i)_{i=1}^M$ be a finite collection of vectors in $\mathcal{H}^N$ and let $K \in \mathbb{N}$. Assume*

1. *Φ can be partitioned into $K+1$-linearly independent sets, and*
2. *Φ can be partitioned into a set and K spanning sets.*

Then there is a partition $\{A_k\}_{k=1}^{K+1}$ *so that* $(\varphi_j)_{j\in A_k}$ *is a linearly independent spanning set for all* $k = 2, 3, \ldots, K+1$ *and* $(\varphi_i)_{i\in A_1}$ *is a linearly independent set.*

The proof of Lemma 2 requires yet another extension of the Rado-Horn Theorem, which we have not yet discussed and will be proven at the end of Section 1.4.

1.3 The Rado-Horn Theorem I and its proof

In this and the following sections, we discuss the proofs of the Rado-Horn Theorems I and III. While the forward direction is essentially obvious, the reverse direction of the Rado-Horn Theorem I, while elementary, is not simple to prove. Our present goal is a proof of the case $K = 2$, which contains many of the essential ideas of the general proof without some of the bookkeeping difficulties in the general case. The proof of the general case of the Rado-Horn Theorem III will be presented below, which contains in it a proof of the Rado-Horn Theorem I. The main idea for the reverse direction is to take as a candidate partition one that maximizes the sum of the dimensions associated with the partition. Then, if that does not partition the set into linearly independent subsets, one can construct a set of interconnected linearly dependent vectors which directly contradicts the hypotheses of the Rado-Horn Theorem I.

As mentioned above, the forward direction of Rado-Horn Theorem I is essentially obvious, but we provide a formal proof in the following lemma.

Lemma 3. *Let* $\Phi = (\varphi_i)_{i=1}^M \subset \mathcal{H}^N$ *and* $K \in \mathbb{N}$. *If there exists a partition* $\{A_1, \ldots, A_K\}$ *of* $[1, M]$ *such that for each* $1 \le k \le K$, $(\varphi_i : i \in A_k)$ *is linearly independent, then for every non-empty* $J \subset [1, M]$,

$$\frac{|J|}{\dim \operatorname{span}\{\varphi_i : i \in J\}} \le K.$$

Proof. Let $\{A_1, \ldots, A_K\}$ partition Φ into linearly independent sets. Let J be a non-empty subset of $[1, M]$. For each $1 \le k \le K$, let $J_k = J \cap A_k$. Then,

$$|J| = \sum_{k=1}^{K} |J_k| = \sum_{k=1}^{K} \dim \operatorname{span}(\{\varphi_i : i \in J_k\}) \le K \dim \operatorname{span}(\{\varphi_i : i \in J\}),$$

as desired.

The Rado-Horn Theorem I tells us that if we want to partition vectors into K linearly independent subsets, there are no non-trivial obstructions. The only obstruction is that there cannot be a subspace S which contains more than $K \dim(S)$ of the vectors that we wish to partition.

The first obstacle to proving the Rado-Horn Theorem I is coming up with a candidate partition which should be linearly independent. There are several ways to do this. The most common, used in [18, 19, 20, 24], is to build the partition while proving the theorem. In [15], it was noticed that any partition which maximizes the sums of dimensions (as explained below) must partition Φ into linearly independent sets, provided any partition can do so. Given a set $\Phi \subset \mathcal{H}^N$ indexed by $[1, M]$ and a natural number K, we say that a partition $\{A_1, \dots, A_K\}$ of $[1, M]$ *maximizes the K-sum of dimensions* of Φ if for any partition $\{B_1, \dots, B_K\}$ of $[1, M]$,

$$\sum_{k=1}^{K} \dim \operatorname{span}\{\varphi_j : j \in A_k\} \geq \sum_{k=1}^{K} \dim \operatorname{span}\{\varphi_j : j \in B_k\}.$$

There are two things to notice about a partition $\{A_1, \dots, A_K\}$ which maximizes the K-sum of dimensions. First, such a partition will always exist since we are dealing with finite sets. Second, such a partition will partition Φ into K linearly independent sets if it is possible for any partition to do so. That is the content of the next two propositions.

Proposition 9. *Let $\Phi = (\varphi_i)_{i=1}^M \subset \mathcal{H}^N, K \in \mathbb{N}$, and $\{A_k\}_{k=1}^K$ be a partition of $[1, M]$. The following conditions are equivalent.*

(1) For every $k \in \{1, \dots, K\}$, $(\varphi_j : j \in A_k)$ is linearly independent.
(2) $\sum_{k=1}^K \dim \operatorname{span}\{\varphi_j : j \in A_k\} = M$.

Proof. (1) ⇒ (2) Clearly,

$$\sum_{k=1}^{K} \dim \operatorname{span}\{\varphi_j : j \in A_k\} = \sum_{k=1}^{K} |A_k| = M.$$

(2) ⇒ (1) Note that

$$M = \sum_{k=1}^{K} \dim \operatorname{span}\{\varphi_j : j \in A_k\} \leq \sum_{k=1}^{K} |A_k| = M.$$

Therefore, $\dim \operatorname{span}\{\varphi_j : j \in A_k\} = |A_k|$ for each $1 \leq k \leq K$ and $(\varphi_j : j \in A_k)$ is linearly independent.

Proposition 10. *Let $\Phi = (\varphi_i)_{i=1}^M \subset \mathcal{H}^N$ and $K \in \mathbb{N}$. If $\{A_k\}_{k=1}^K$ maximizes the K-sum of dimensions of Φ and there exists a partition $\{B_k\}_{k=1}^K$ such that for each $1 \leq k \leq K$, $(\varphi_j : j \in B_k)$ is linearly independent, then $(\varphi_j : j \in A_k)$ is linearly independent for each $1 \leq k \leq K$.*

Proof. We have

$$M = \sum_{k=1}^{K} \dim \operatorname{span}\{\varphi_j : j \in B_k\}$$
$$\leq \sum_{k=1}^{K} \dim \operatorname{span}\{\varphi_j : j \in A_k\} \leq M.$$

Therefore, $(\varphi_j : j \in A_k)$ is linearly independent for each $1 \leq j \leq M$ by Proposition 9.

A third way of partitioning Φ to prove the Rado-Horn Theorem I was given in [5], though not explicitly. Given Φ as above and $K \in \mathbb{N}$, we say a partition $\{A_k\}_{k=1}^K$ *maximizes the K-ordering of dimensions* if the following holds. Given any partition $\{B_k\}_{k=1}^K$ of $[1, M]$, if for every $1 \leq k \leq K$, $\dim \operatorname{span}\{\varphi_j : j \in A_k\} \leq \dim \operatorname{span}\{\varphi_j : j \in B_k\}$, then

$$\dim \operatorname{span}\{\varphi_j : j \in A_k\} = \dim \operatorname{span}\{\varphi_j : j \in B_k\}, \text{ for every } 1 \leq k \leq K.$$

It is easy to see that any partition which maximizes the K-sum of dimensions also maximizes the K-ordering of dimensions. The next proposition shows that the converse holds, at least in the case that one can partition into linearly independent sets. Therefore, when proving the Rado-Horn Theorem, it makes sense to begin with a partition which maximizes the K-ordering of dimensions. We do not present a proof of this proposition, but mention that it follows from Theorem 12.

Proposition 11. *Let $\Phi = (\varphi_i)_{i=1}^M \subset \mathcal{H}^N$ and $K \in \mathbb{N}$. If $\{A_k\}_{k=1}^K$ maximizes the K-ordering of dimensions of Φ and there exists a partition $\{B_k\}_{k=1}^K$ such that for each $1 \leq k \leq K$, the set $(\varphi_j : j \in B_k)$ is linearly independent, then for each $1 \leq k \leq K$, the set $(\varphi_j : j \in A_k)$ is linearly independent.*

A second obstacle to proving the Rado-Horn Theorem I is proving that a candidate partition into linearly independent sets really does partition into linearly independent sets. Our strategy will be to suppose that it does not partition into linearly independent sets, and directly construct a set $J \subset [1, M]$ which violates the hypotheses of the Rado-Horn Theorem I. In order to construct J, we will imagine moving the linearly dependent vectors from one element of the partition to another element of the partition. The first observation is that if a partition maximizes the K-ordering of dimensions, and there is a linearly dependent vector in one of the elements of the partition, then that vector is in the span of each element of the partition.

Proposition 12. *Let $\Phi = (\varphi_i)_{i=1}^M \subset \mathcal{H}^N$, $K \in \mathbb{N}$, and let $\{A_k\}_{k=1}^K$ be a partition of $[1, M]$ which maximizes the K-ordering of dimensions of Φ. Fix $1 \leq m \leq K$. Suppose that there exist scalars $\{a_j\}_{j \in A_m}$, not all of which are zero, such that $\sum_{j \in A_m} a_j \varphi_j = 0$. Let $j_0 \in A_m$ be such that $a_{j_0} \neq 0$. Then for each $1 \leq n \leq K$,*

$$\varphi_{j_0} \in \operatorname{span}\{\varphi_j : j \in A_n\}.$$

Proof. Since removing φ_{j_0} from A_m will not decrease the dimension of the span, adding φ_{j_0} to any of the other A_n's will not increase the dimension of their spans.

A simple, but useful, observation is that if we start with a partition $\{A_k\}_{k=1}^K$ which maximizes the K-ordering of dimensions of Φ, then a new partition obtained by moving one linearly dependent vector out of some A_k into another $A_{k'}$ will also maximize the K-ordering of dimensions.

Proposition 13. *Let $\Phi = (\varphi_i)_{i=1}^M \subset \mathcal{H}^N, K \in \mathbb{N}$, and let $\{A_k\}_{k=1}^K$ be a partition of $[1, M]$ which maximizes the K-ordering of dimensions of Φ. Fix $1 \le m \le K$. Suppose that there exist scalars $\{a_j\}_{j \in A_m}$, not all of which are zero, such that $\sum_{j\in A_m} a_j \varphi_j = 0$. Let $j_0 \in A_m$ be such that $a_{j_0} \neq 0$. For every $1 \le n \le K$, the partition $\{B_k\}_{k=1}^K$ given by*

$$B_k = \begin{cases} A_k & k \neq m, n \\ A_m \setminus \{j_0\} & k = m \\ A_n \cup \{j_0\} & k = n. \end{cases}$$

also maximizes the K-ordering of dimensions of Φ.

Proof. By Proposition 12, the new partition has exactly the same dimension of spans as the old partition.

The idea for constructing the set J which will contradict the hypotheses of the Rado-Horn Theorem I is to suppose that a partition which maximizes the K-ordering of dimensions does not partition into linearly independent sets. We will take a vector which is linearly dependent, and then see that it is in the span of each of the other elements of the partition. We create new partitions, which again maximize the K-ordering of dimensions, by moving the linearly dependent vector into other sets of the partition. The partition element to which we moved the vector will also be linearly dependent. We then repeat and take the index of all vectors which can be reached in such a way as our set J. It is easy to imagine that the bookkeeping aspect of this proof will get involved relatively quickly. For that reason, we will restrict to the case $K = 2$ and prove the Rado-Horn Theorem I in that case, using the same idea that will work in the general case. The bookkeeping in this case is somewhat easier, yet all of the ideas are already there.

A key concept in our proof of the Rado-Horn Theorem I is that of a *chain of dependencies* of length P. Given two collections of vectors $(\varphi_j : j \in A_1)$ and $(\varphi_j : j \in A_2)$, where $A_1 \cap A_2 = \emptyset$, we define a chain of dependencies of length P to be a finite sequence of distinct indices $\{i_1, i_2, \dots, i_P\} \subset A_1 \cup A_2$ with the following properties:

1. i_k will be an element of A_1 for odd indices k, and an element of A_2 for even indices k,
2. $\varphi_{i_1} \in \operatorname{span}\{\varphi_j : j \in A_1 \setminus \{i_1\}\}$, and $\varphi_{i_1} \in \operatorname{span}\{\varphi_j : j \in A_2\}$,

3. for odd k, $1 < k \leq P$, $\varphi_{i_k} \in \operatorname{span}\{\varphi_j : j \in \big(A_1 \cup \{i_2, i_4, \ldots, i_{k-1}\}\big) \setminus \{i_1, i_3, \ldots, i_{k-2}\}\}$ and $\varphi_{i_k} \in \operatorname{span}\{\varphi_j : j \in \big(A_2 \cup \{i_1, i_3, \ldots, i_{k-2}\}\big) \setminus \{i_2, i_4, \ldots, i_{k-1}\}\}$,
4. for even k, $1 < k \leq P$, $\varphi_{i_k} \in \operatorname{span}\{\varphi_j : j \in \big(A_2 \cup \{i_1, i_3, \ldots, i_{k-1}\}\big) \setminus \{i_2, i_4, \ldots, i_k\}\}$, and $\varphi_{i_k} \in \operatorname{span}\{\varphi_j : j \in \big(A_1 \cup \{i_2, i_4, \ldots, i_{k-2}\}\big) \setminus \{i_1, i_3, \ldots, i_{k-1}\}\}$.

A chain of dependencies is constructed as follows. Start with a linearly dependent vector. Moving that vector to another set in the partition cannot increase the sum of the dimensions of the spans, so that vector is also in the span of the vectors in the set to which it has been moved. Now, that makes the new set linearly dependent, so take a second vector, which is linearly dependent in the second set, and move it to a third set. Again, the second vector is in the span of the vectors in the third set. Continuing in this fashion gives a chain of dependencies.

With this new definition, it is easier to describe the technique of proof of the Rado-Horn Theorem I. Suppose that a partition which maximizes the 2-ordering of dimensions does not partition into linearly independent sets. Let J be the union of all of the chains of dependencies. We will show that J satisfies

$$\frac{|J|}{\dim \operatorname{span}\{\varphi_i : i \in J\}} > 2.$$

Example 2. We give an example of chains of dependencies in $\mathcal{H}^3$. Let $\varphi_1 = \varphi_5 = (1,0,0)^T$, $\varphi_2 = \varphi_6 = (0,1,0)^T$, $\varphi_3 = \varphi_7 = (0,0,1)^T$ and $\varphi_4 = (1,1,1)^T$. Suppose also that $A_1 = \{1,2,3,4\}$ and $A_2 = \{5,6,7\}$. Then, the set $\{4,5,1,6,2,7,3\}$ is a chain of dependencies of length 7. Note also that $\{4,5,1\}$ is a chain of dependencies of length 3.

Note that if we let J be the union of all of the sets of dependencies based on the partition $\{A_1, A_2\}$, then

$$\frac{|J|}{\dim \operatorname{span}\{\varphi_i : i \in J\}} = \frac{7}{3} > 2.$$

The following example illustrates what can happen if we do not start with a partition which maximizes the K-ordering of dimensions.

Example 3. Let $\varphi_1 = (1,0,0)^T, \varphi_2 = (0,1,0)^T, \varphi_3 = (1,1,0)^T, \varphi_4 = (1,0,0)^T$, $\varphi_5 = (0,0,1)^T$, and $\varphi_6 = (0,1,1)^T$. Imagine starting with our partition consisting of $A_1 = \{1,2,3\}$ and $A_2 = \{4,5,6\}$. We can make a chain of dependencies $\{3,6\}$, but notice that $\{\varphi_6, \varphi_1, \varphi_2\}$ is linearly independent. This indicates that we have removed one linear dependence, and in fact, the new partition $B_1 = \{1,2,6\}$, $B_2 = \{3,4,5\}$ is linearly independent.

Note that the new partition does maximize the K-ordering of dimensions.

A slight generalization of Proposition 13 is given below.

Lemma 4. *Let $\Phi = (\varphi_i)_{i=1}^M \subset \mathcal{H}^N$, and suppose that Φ cannot be partitioned into two linearly independent sets. Let $\{A_1, A_2\}$ be a partition of $[1, M]$ which maximizes the 2-ordering of dimensions. Let $\{i_1, \dots, i_P\}$ be a chain of dependencies of length P based on the partition $\{A_1, A_2\}$. For each $1 \le k \le P$, the partition $\{B_1(k), B_2(k)\}$ given by*

$$B_1(k) = \big(A_1 \cup \bigcup_{1\le j\le k/2} \{i_{2j}\}\big) \setminus \bigcup_{1\le j\le (k+1)/2} \{i_{2j-1}\},$$
$$B_2(k) = \big(A_2 \cup \bigcup_{1\le j\le (k+1)/2} \{i_{2j-1}\}\big) \setminus \bigcup_{1\le j\le k/2} \{i_{2j}\}$$

also maximizes the 2-ordering of dimensions.

We introduce one notational convenience at this point. Given a set $A \subset [1, M]$, a finite sequence of elements $\{i_1, \dots, i_P\}$ and disjoint sets $Q, R \subset [1, P]$, we define

$$A(Q; R) = \big(A \cup \bigcup_{j\in Q} \{i_j\}\big) \setminus \bigcup_{j\in R} \{i_j\}.$$

Lemma 5. *Let $\Phi = \Phi = (\varphi_i)_{i=1}^M \subset \mathcal{H}^N$, and suppose that Φ cannot be partitioned into two linearly independent sets. Let $\{A_1, A_2\}$ be a partition of $[1, M]$ which maximizes the 2-ordering of dimensions. Let J be the union of all chains of dependencies of Φ based on the partition $\{A_1, A_2\}$. Let $J_1 = J \cap A_1$ and $J_2 = J \cap A_2$, and $S = \operatorname{span}\{\varphi_i : i \in J\}$. Then,*

$$S = \operatorname{span}\{\varphi_i : i \in J_k\}$$

for $k = 1, 2$.

Proof. We will prove the lemma in the case $k = 1$, the other case being similar. It suffices to show that for every chain of dependencies $\{i_1, \dots, i_P\}$, all of the even indexed vectors φ_k are in the span of J_1, which we will do by induction.

Note that $\varphi_{i_2} \in \operatorname{span}\{\varphi_i : i \in A_1 \setminus \{i_1\}\}$. Therefore, there exist scalars $\{a_i : i \in A_1 \setminus \{i_1\}\}$ such that

$$\varphi_{i_2} = \sum_{i\in A_1\setminus\{i_1\}} a_i \varphi_i.$$

Let $i \in A_1 \setminus \{i_1\}$ be such that $a_i \ne 0$. We show that $\{i_1, i_2, i\}$ is a chain of dependencies of length 3. First, note that $\varphi_i \in \operatorname{span}\{\varphi_j : j \in (A_1(\{2\}; \{1\})\}$. By Lemma 4, the partition $\{A_1(\{2\}; \{1\}), A_2(\{1\}; \{2\})\}$ maximizes 2-ordering of dimensions. Since φ_i is a dependent vector in $(\varphi_j : j \in A_1(\{2\}; \{1\}))$, the partition $\{A_1(\{2\}; \{1,i\}), A_2(\{1,i\}; \{2\})\}$ has the same dimensions as the partition $\{A_1(\{2\}; \{1\}), A_2(\{1\}; \{2\})\}$. In particular, $\varphi_i \in \operatorname{span}\{\varphi_j : j \in$

$A_2(\{1\};\{2\})\}$. Therefore $\{i_1, i_2, i\}$ is a chain of dependencies of length 3, and $\varphi_{i_2} \in \text{span}\{\varphi_j : j \in J_1\}$.

Now, suppose that $\varphi_{i_2}, \ldots, \varphi_{i_{2m-2}} \in \text{span}\{\varphi_j : j \in J_1\}$. We show that $\varphi_{i_{2m}} \in \text{span}\{\varphi_j : j \in J_1\}$. Note that $\varphi_{i_{2m}} \in \text{span}\{\varphi_j : j \in A_1(\{2,4,\ldots,2m-2\};\{1,3,\ldots,2m-1\})\}$. Therefore, there exist scalars $\{a_i : i \in A_1(\{2,4,\ldots,2m-2\};\{1,3,\ldots,2m-1\})\}$ such that

$$\varphi_{i_{2m}} = \sum_{i \in A_1(\{2,4,\ldots,2m-2\};\{1,3,\ldots,2m-1\})} a_i \varphi_i. \tag{1.4}$$

By the induction hypothesis, for the even indices $j < 2m$, $\varphi_j \in \text{span}\{\varphi_i : i \in J_1\}$, so it suffices to show that for all $i \in A_1(\emptyset;\{1,3,\ldots,2m-1\})$ such that $a_i \neq 0$, the set $\{i_1, \ldots, i_{2m}, i\}$ is a chain of dependencies. (Note that there may not be any i in this set.) By (1.4), $\varphi_i \in \text{span}\{\varphi_j : j \in A_1(\{2,4,\ldots,2m\};\{1,3\ldots,2m-1\})\}$. By Lemma 4, the partition $\{A_1(\{2,4,\ldots,2m\};\{1,3,\ldots,2m-1\}), A_2(\{1,3\ldots,2m-1\};\{2,4,\ldots,2m\})\}$ maximizes 2-ordering of dimensions. Therefore, since φ_i is a dependent vector in $(\varphi_j : j \in A_1(\{2,4,\ldots,2m\};\{1,3,\ldots,2m-1\}))$, moving i into the second partition by forming the new partition $\{A_1(\{2,4,\ldots,2m\};\{1,3,\ldots,2m-1,\text{i}\}), A_2(\{1,3,\ldots,2m-1,\text{i}\};\{2,4,\ldots,2m\})\}$ does not change the dimensions. In particular,

$$\varphi_i \in \text{span}\{\varphi_j : j \in A_2(\{1,3,\ldots,2m-1\};\{2,4,\ldots,2m\}\}.$$

Therefore $\{i_1, i_2, \ldots, i_{2m}, i\}$ is a chain of dependencies of length $2m+1$, and $\varphi_{i_{2m}} \in \text{span}\{\varphi_j : j \in J_1\}$.

Theorem 11. *Let $\Phi = (\varphi_i)_{i=1}^M \subset \mathcal{H}^N$. If for every non-empty $J \subset [1, M]$,*

$$\frac{|J|}{\dim \text{span}\{\varphi_i : i \in J\}} \le 2,$$

then Φ can be partitioned into two linearly independent sets.

Proof. Suppose that Φ cannot be partitioned into two linearly independent sets. We will construct a set J such that

$$\frac{|J|}{\dim \text{span}\{\varphi_i : i \in J\}} > 2.$$

Let $\{A_1, A_2\}$ be a partition of $[1, M]$ which maximizes the 2-ordering of dimensions. By hypothesis, this partition of $[1, M]$ does not partition Φ into linearly independent sets, so at least one of the collections $(\varphi_j : j \in A_k)$, $k = 1, 2$ must be linearly dependent. Without loss of generality, we assume that $(\varphi_j : j \in A_1)$ is linearly dependent.

Let J be the union of all chains of dependencies based on the partition $\{A_1, A_2\}$. We claim that J satisfies

$$\frac{|J|}{\dim \operatorname{span}\{\varphi_i : i \in J\}} > 2.$$

Indeed, let $J_1 = J \cap A_1$ and $J_2 = J \cap A_2$. By Lemma 5, $(\varphi_j : j \in J_k)$, $k = 1, 2$ span the same subspace $S = (\varphi_j : j \in J)$. Since $(\varphi_j : j \in J_1)$ is not linearly independent, $|J_1| > \dim S$. Therefore,

$$\begin{aligned} |J| &= |J_1| + |J_2| \\ &> \dim S + \dim S = 2 \dim\{\varphi_j : j \in J\}, \end{aligned}$$

and the theorem is proved.

A careful reading of the proof of Theorem 11 yields that we have proven more than what has been advertised. In fact, we have essentially proven the more general Theorem 12 in the special case of partitioning into two sets.

1.4 The Rado-Horn Theorem III and its proof

The final section of this chapter is devoted to the proof of the third version of the Rado-Horn Theorem, which we recall below (see Theorem 6). We did not include all elements of the theorem, as a discussion of partitions maximizing the K-ordering of dimensions would have taken us too far astray at that time, and we only needed the full version of the theorem in the proof of Lemma 2, whose proof we have delayed until the end of this section.

Theorem 12 (Rado-Horn Theorem III). *Let $\Phi = (\varphi_i)_{i=1}^M$ be a collection of vectors in $\mathcal{H}^N$ and $K \in \mathbb{N}$. Then the following conditions are equivalent.*

(1) There exists a partition $\{A_k : k = 1, \ldots, K\}$ of $[1, M]$ such that for each $1 \leq k \leq K$ the set $(\varphi_j : j \in A_k)$ is linearly independent.
(2) For all $J \subset I$,

$$\frac{|J|}{\dim \operatorname{span}\{\varphi_j : j \in J\}} \leq K. \tag{1.5}$$

Moreover, in the case that both of the conditions above are true, any partition which maximizes the K-ordering of dimensions will partition the vectors into linearly independent sets. In the case that either of the conditions above fails, there exists a partition $\{A_k : k = 1, \ldots, K\}$ of $[1, M]$ and a subspace S of $\mathcal{H}^N$ such that the following three conditions hold.

(a) For all $1 \leq k \leq K$, $S = \operatorname{span}\{\varphi_j : j \in A_k \text{ and } \varphi_j \in S\}$.

(b) For $J = \{i \in I : \varphi_i \in S\}$, $\frac{|J|}{\dim \operatorname{span}\{\varphi_i : i \in J\}} > K$.

(c) For each $1 \leq k \leq K$, $(P_{S^\perp}\varphi_i : i \in A_k, \varphi_i \notin S)$ is linearly independent, where $P_{S^\perp}$ is the orthogonal projection onto $S^\perp$.

We saw in the previous section how to prove the more elementary version of the Rado-Horn Theorem in the case of partitioning into 2 subsets. The details of the proof in the general setting are similar, and where the proofs follow the same outline we will omit them. The interested reader can refer to [15, 5] for full details.

As before, our general plan is to start with a partition which maximizes the K-ordering of dimensions. We will show that if that partition does not partition into linearly independent sets, then we can construct a set J which directly contradicts the hypotheses of the Rado-Horn Theorem. The set J constructed will span the subspace S in the conclusion of the theorem.

Let $\{A_1, \ldots, A_K\}$ be a partition of $[1, M]$ and let $\{i_1, \ldots, i_P\} \subset [1, M]$. We say $\{a_1, \ldots, a_P\}$ are the associated partition indices if for all $1 \le p \le P$, $i_p \in A_{a_p}$. We define the chain of partitions $\{\mathcal{A}^j\}_{j=1}^P$ associated with $\mathcal{A} = \{A_1, \ldots, A_K\}$ and $\{i_1, \ldots, i_P\}$ as follows. Let $\mathcal{A}^1 = \mathcal{A}$, and given that the partitions $\mathcal{A}^j = \{A_k^j\}_{k=1}^K$ have been defined for $1 \le j \le p$ and $p \le P$, we define $\mathcal{A}^{p+1} = \{A_1^{p+1}, \ldots, A_K^{p+1}\}$ by

$$A_k^{p+1} = \begin{cases} A_k^p & k \neq a_p, a_{p+1} \\ A_{a_p}^p \setminus \{i_p\} & k = a_p \\ A_{a_{p+1}}^p \cup \{i_p\} & k = a_{p+1}. \end{cases}$$

A chain of dependencies of length P based on the partition $\{A_1, \ldots, A_K\}$ is a set of distinct indices $\{i_1, \ldots, i_P\} \subset [1, M]$ with associated partition indices $\{a_1, \ldots, a_P\}$ and the $P+1$ associated partitions $\{A_k^p\}_{k=1}^K$, $1 \le p \le P+1$ such that the following conditions are met.

1. $a_p \neq a_{p+1}$ for all $1 \le p < P$.
2. $a_1 = 1$.
3. $\varphi_{i_1} \in \operatorname{span}\{\varphi_j : j \in A_1^2\}$, and $\varphi_{i_1} \in \operatorname{span}\{\varphi_j : j \in A_{a_2}^1\}$.
4. $\varphi_{i_p} \in \operatorname{span}\{\varphi_j : j \in A_{a_p}^p \setminus \{i_p\}\}$ for all $1 < p \le P$.
5. $\varphi_{i_p} \in \operatorname{span}\{\varphi_j : j \in A_{a_{p+1}}^p\}$ for all $1 < p < P$.

Lemma 6. *With the notation above, for each $1 \le p \le P+1$, the partition $\{A_k^p\}_{k=1}^K$ maximizes the K-ordering of dimensions.*

Proof. As in Lemma 4, when we are constructing the pth partition, we are taking a vector that is dependent in the $(p-1)$st partition, and moving it to a new partition element. Since removing the dependent vector does not reduce the dimension, all of the dimensions in the pth partition must remain the same. Hence, it maximizes the K-ordering of dimensions.

Lemma 7. *Let $\Phi = (\varphi_i)_{i=1}^M \subset \mathcal{H}^N$, and suppose that Φ cannot be partitioned into K linearly independent sets. Let $\{A_1, \ldots, A_K\}$ be a partition of $[1, M]$ which maximizes the K-ordering of dimensions. Let J be the union of all chains of dependencies of Φ based on the partition $\{A_1, \ldots, A_K\}$. For $1 \le k \le K$, let $J_k = J \cap A_k$, and let $S = \operatorname{span}\{\varphi_i : i \in J\}$. Then,*

$$S = \operatorname{span}\{\varphi_i : i \in J_k\}$$

for $k = 1, \ldots, K$.

Proof. We sketch the proof for $k = 1$. The details are similar to Lemma 5. Clearly, it suffices to show that if $\{i_1, \ldots, i_P\}$ is a chain of dependencies based on $\{A_1, \ldots, A_K\}$, then each $\varphi_{i_p} \in \operatorname{span}\{\varphi_i : i \in J_1\}$ for each $1 \le p \le P$. For $p = 1$, this is true since $a_1 = 1$. (For $k \neq 1$, it is true since moving a dependent vector from A_1 to A_k cannot increase the dimension of $(\varphi_i : i \in A_k)$.)

Proceeding by induction on p, assume that $\varphi_{i_1}, \ldots, \varphi_{i_{p-1}} \in \operatorname{span}\{\varphi_i : i \in J_1\}$. Let $\{a_1, \ldots, a_P\}$ be the associated partition indices and $\mathcal{A}^p = \{A_k^p\}_{k=1}^K$ the associated partitions. If $a_p = 1$, then we are done. Otherwise, we know that $\varphi_{i_p} \in \operatorname{span}\{\varphi_j : j \in A_{a_p}^{p+1}\}$. Note that $i_p \in A_{a_p}^p$ and $i_p \notin A_{a_p}^{p+1}$. Therefore, removing i_p from $A_{a_p}^p$ does not change the span of the vectors indexed by $A_{a_p}^p$, and by Lemma 6,

$$\varphi_{i_p} \in \operatorname{span}\{\varphi_j : j \in A_1^p\}.$$

Write

$$\varphi_{i_p} = \sum_{j \in A_1^p} \alpha_j \varphi_j$$

for some scalars α_j. We claim that for each j such that $\alpha_j \neq 0$, $\varphi_j \in \operatorname{span}\{\varphi_i : i \in J_1\}$. Since $A_1^p \subset A_1 \cup \{i_1, \ldots, i_{p-1}\}$, by the induction hypothesis it suffices to show that whenever $j_0 \in A_1^p \backslash \{i_1, \ldots, i_p\}$, $\varphi_{j_0} \in \operatorname{span}\{\varphi_i : i \in J_1\}$. To do so, we claim that $\{i_1, \ldots, i_p, j_0\}$ is a chain with associated indices $\{a_1, \ldots, a_p, 1\}$. Indeed, noting that $A_1^{p+1} = \big(A_1^p \cup \{i_p\}\big)$, property 4 of a chain of dependencies ensures

$$\varphi_{j_0} \in \operatorname{span}\{\varphi_i : i \in \big(A_1^p \cup \{i_p\}\big) \setminus \{j_0\}\}.$$

Proof (of Theorem 12). Suppose that Φ cannot be partitioned into K linearly independent sets. Let $\mathcal{A}$ be a partition of $[1, M]$ which maximizes the K-ordering of subspaces. By hypothesis, this partition does not partition Φ into linearly independent sets, so without loss of generality, we assume that $(\varphi_i : i \in A_1)$ is linearly dependent.

Let J be the union of all chains of dependencies based on the partition $\mathcal{A}$ and $S = \operatorname{span}\{\varphi_i : i \in J\}$. By Lemma 7, J satisfies

$$J = \{i \in [1, M] : \varphi_i \in S\}.$$

We show that J and S satisfy the conclusions of Theorem 12.

First, let $J_k = A_k \cap J$ for $1 \le k \le K$. We have that $\operatorname{span}\{\varphi_i : i \in J_k\} = S$ for $1 \le k \le K$ be Lemma 7, and $|J_1| > \dim S$ by the assumption that $\mathcal{A}$ does not partition into linearly independent sets. Therefore,

$$|J| = \sum_{k=1}^{K} |J_k| > K \dim S = K \dim \operatorname{span}\{\varphi_i : i \in J\}.$$

In particular, if it were possible to partition into linearly independent sets, $\mathcal{A}$ would do it.

To see (a) in the list of conclusions in Theorem 12, note that $S \supset \text{span}\{\varphi_i : i \in A_k, \varphi_i \in S\}$ is obvious, and $S \subset \text{span}\{\varphi_i : i \in A_k, \varphi_i \in S\}$ follows from Lemma 7. Part (b) follows from Lemma 7 and the computations above.

It remains to prove (c). Suppose there exist $\{\alpha_j\}_{j\in A_k\setminus J}$ not all zero such that $\sum_{j\in A_k\setminus J} \alpha_j\varphi_j \in S$. Since J is the union of the set of all chains of dependencies, $\sum_{j\in A_k\setminus J} \alpha_j\varphi_j \neq 0$. Let $\{\beta_j\}_{j\in J_k}$ be scalars such that

$$\sum_{j\in A_k\setminus J} \alpha_j\varphi_j = \sum_{j\in J_k} \beta_j\varphi_j. \tag{1.6}$$

Choose j_0 and a chain of dependencies $\{i_1, \ldots, i_{P-1}, j_0\}$ such that $\beta_{j_0} \neq 0$ and such that P is the minimum length of all chains of dependencies whose final element is in $\{\beta_j : j \neq 0\}$. Let $m \in A_k \setminus J$ such that $\alpha_m \neq 0$. We claim that $\{i_1, \ldots, i_{P-1}, m\}$ is a chain of dependencies, which contradicts $m \notin J$ and finishes the proof.

The key observation to proving the claim is to observe that the minimality of the length of the chain $\{i_1, \ldots, i_{P-1}, j_0\}$ forces

$$\{j : \beta_j \neq 0\} \cup \{j : \alpha_j \neq 0\} \subset A^P_{a_P}. \tag{1.7}$$

To verify property 5 of a chain of dependencies, since $\varphi_{i_{P-1}} \in \text{span}\{\varphi_j : j \in A^P_{a_P} \setminus \{j_0\}\}$, equations (1.6) and (1.7) imply that $\varphi_{i_{P-1}} \in \text{span}\{\varphi_j : j \in A^P_{a_P} \setminus \{m\}\}$. To see property 4 of a chain of dependencies, write

$$\varphi_{j_0} = \sum_{j\in A^P_{a_P}\setminus\{j_0\}} \gamma_j\varphi_j.$$

If $\gamma_m \neq 0$, then $\varphi_m \in \text{span}\{\varphi_i : i \in A^P_{a_P} \setminus \{m\}\}$ directly from the above equation. If $\gamma_m = 0$, then replacing φ_{j_0} in equation (1.6) with $\sum_{j\in A^P_{a_P}\setminus\{j_0\}} \gamma_j\varphi_j$ shows that $\varphi_m \in \text{span}\{\varphi_i : i \in A^P_{a_P} \setminus \{m\}\}$.

We end with a proof of Lemma 2, which we restate now for the reader's convenience.

Theorem 13. *Let $\Phi = (\varphi_i)_{i=1}^M$ be a finite collection of vectors in $\mathcal{H}^N$, and let $K \in \mathbb{N}$. Assume*

1. Φ can be partitioned into $K+1$-linearly independent sets, and
2. Φ can be partitioned into a set and K spanning sets.

Then there is a partition $\{A_k\}_{k=1}^{K+1}$ so that $(\varphi_j)_{j\in A_k}$ is a linearly independent spanning set for all $k = 2, 3, \ldots, K+1$ and $(\varphi_i)_{i\in A_1}$ is a linearly independent set.

Proof. We choose a partition $\{A_k\}_{k=1}^{K+1}$ of $[1,M]$ that maximizes $\dim \operatorname{span}\{\varphi_j\}_{j\in A_1}$ taken over all partitions so that the last K sets span $\mathcal{H}^N$. If $\{B_k\}_{k=1}^{K+1}$ is a partition of $[1,M]$ such that for all $1 \le k \le K+1$,

$$\dim \operatorname{span}\{\varphi_j\}_{j\in B_i} \ge \dim \operatorname{span}\{\varphi_j\}_{j\in A_i},$$

then

$$\dim \operatorname{span}\{\varphi_j\}_{j\in A_i} = \dim \operatorname{span}\{\varphi_j\}_{j\in B_i}$$

for all $i = 2, \ldots, K+1$ since $\dim \operatorname{span}\{\varphi_j\}_{j\in A_i} = N$, and $\dim \operatorname{span}\{\varphi_j\}_{j\in A_1} \ge \dim \operatorname{span}\{\varphi_j\}_{j\in B_1}$ by construction. This means that the partition $\{A_k\}_{k=1}^{K+1}$ maximizes the $(K+1)$-ordering of dimensions. By Theorem 12, since there is a partition of Φ into $K+1$ linearly independent sets, $\{A_k\}_{k=1}^{K+1}$ partitions Φ into linearly independent sets, as desired.

1.5 The Maximal Number of Spanning Sets in a Frame

In this section, we determine the maximal number of spanning sets contained in a frame. Partitioning into spanning sets has not had been studied as much as partitioning into linearly independent sets, and several of the results in this section are, as far as we know, new.

In one sense, the difficulties associated with choosing spanning sets contained in a frame is very similar to the difficulties associated with choosing linearly independent sets. Namely, choosing spanning sets at random will not necessarily provide the maximum number of spanning sets. A trivial example is given in $\mathbb{R}^2$ by the frame (e_1, e_1, e_2, e_1+e_2) where $e_1 = (1,0)^T, e_2 = (0,1)^T$. If we choose (e_2, e_1+e_2), then we can only get one spanning set while if we choose $(e_1, e_2), (e_1, e_1+e_2)$ we get two spanning sets. Recently [4], the problem of determining the maximal number of spanning sets was resolved. We begin with some preliminary results.

Theorem 14. *Let P be a projection on $\mathcal{H}^M$ and let $(e_i)_{i=1}^M$ be an orthonormal basis for $\mathcal{H}^M$. If $I \subset \{1,2,\ldots,M\}$, the following are equivalent:*

(1) $(Pe_i)_{i\in I}$ *spans* $P(\mathcal{H}^M)$.
(2) $((Id-P)e_i)_{i\in I^c}$ *is linearly independent.*

Proof. (1) $\Rightarrow$ (2): Assume that $((Id-P)e_i)_{i\in I^c}$ is not linearly independent. Then there exist scalars $\{b_i\}_{i\in I^c}$, not all zero, so that

$$\sum_{i\in I^c} b_i (Id-P)e_i = 0.$$

It follows that

$$x = \sum_{i\in I^c} b_i e_i = \sum_{i\in I^c} b_i P e_i \in P(\mathcal{H}^M).$$

Thus,

$$\langle x, Pe_j\rangle = \langle Px, e_j\rangle = \sum_{i\in I^c} b_i\langle e_i, e_j\rangle = 0, \text{ if } j \in I.$$

So $x \perp \text{span}\{Pe_i\}_{i\in I}$ and hence this family is not spanning for $P(\mathcal{H}^M)$.

(2) $\Rightarrow$ (1): We assume that $\text{span}\{Pe_i\}_{i\in I} \neq P(\mathcal{H}^M)$. That is, there is a $0 \neq x \in P(\mathcal{H}^M)$ so that $x \perp \text{span}\{Pe_i\}_{i\in I}$. Also, $x = \sum_{i=1}^M \langle x, e_i\rangle Pe_i$. Then

$$\langle x, Pe_i\rangle = \langle Px, e_i\rangle = \langle x, e_i\rangle = 0, \text{ for all } i \in I.$$

Hence, $x = \sum_{i\in I^c}\langle x, e_i\rangle e_i$. That is,

$$\sum_{i\in I^c}\langle x, e_i\rangle e_i = x = Px = \sum_{i\in I^c}\langle x, e_i\rangle Pe_i.$$

That is,

$$\sum_{i\in I^c}\langle x, e_i\rangle (I-P)e_i = 0,$$

i.e., $((Id-P)e_i)_{i\in I^c}$ is not linearly independent.

We state an immediate consequence.

Corollary 2. *Let P be a projection on $\mathcal{H}^M$. The following are equivalent:*

(1) There is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \ldots, M\}$ so that $(Pe_i)_{i\in A_j}$ spans $P(\mathcal{H}^M)$ for all $j = 1, 2, \ldots, r$.
(2) There is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \ldots, M\}$ so that $((Id-P)e_i)_{i\in A_j^c}$ is linearly independent for every $j = 1, 2, \ldots, r$.

Now we can prove the main result, which gives the maximal number of spanning sets contained in a frame. Recall that this problem is independent of applying an invertible operator to the frame and hence we only need to prove the result for Parseval frames.

Theorem 15. *[4] Let $(\varphi_i)_{i=1}^M$ be a Parseval frame for $\mathcal{H}^N$, let P be a projection on $\mathcal{H}^M$ with $(\varphi_i)_{i=1}^M = (Pe_i)_{i=1}^M$ where $(e_i)_{i=1}^M$ is an orthonormal basis for $\mathcal{H}^M$, and let $(\psi_i)_{i=1}^{(r-1)M}$ be the multiset*

$$\{(Id-P)e_1, \ldots, (Id-P)e_1, (Id-P)e_2, \ldots, (Id-P)_2, \ldots, \tag{1.8}$$
$$(Id-P)e_M, \ldots, (Id-P)e_M\}.$$

The following are equivalent:

(1) $(\varphi_i)_{i=1}^M$ can be partitioned into r spanning sets.
(2) $(\psi_i)_{i=1}^{(r-1)M}$ can be partitioned into r linearly independent sets.

(3) For all $I \subset \{1, 2, \ldots, (r-1)M\}$,

$$\frac{|I|}{\dim \operatorname{span}\{\psi_i\}_{i \in I}} \leq r. \tag{1.9}$$

Proof. (1) $\Rightarrow$ (2): Let $\{A_j\}_{j=1}^r$ be a partition of $\{1, 2, \ldots, M\}$ so that $(Pe_i)_{i \in A_j}$ is spanning for every $j = 1, 2, \ldots, r$. Then $((Id - P)e_i)_{i \in A_j^c}$ is linearly independent for every $j = 1, 2, \ldots, r$. Since $\{A_j\}_{j=1}^r$ is a partition, each $(Id - P)e_i$ appears in exactly $r-1$ of the collections $((Id - P)e_i)_{i \in A_j^c}$. So the multiset $(\psi_i)_{i=1}^{(r-1)M}$ has a partition into r linearly independent sets.

(2) $\Rightarrow$ (1): Let $\{A_j\}_{j=1}^r$ be a partition of $\{1, 2, \ldots, (r-1)M\}$ so that $((Id - P)e_i)_{i \in A_j}$ is linearly independent for all $j = 1, 2, \ldots, r$. Since the collection $((Id - P)e_i)_{i \in A_j}$ is linearly independent, it contains at most one of the r copies of $(Id - P)e_i$ for each $i = 1, 2, \ldots, M$. Hence, each $(Id - P)e_i$ is in exactly $r-1$ of the collections $((Id - P)e_i)_{i \in A_j}$. That is, each i is in all but one of the these sets A_j. For each $j = 1, 2, \ldots, r$, let B_j be the complement of A_j in $\{1, 2, \ldots, M\}$. Since $((Id - P)e_i)_{i \in A_j}$ is linearly independent, $(Pe_i)_{i \in B_j}$ is spanning. Also, for all $i, j = 1, \ldots, r$ with $i \neq j$, we have $B_i \cap B_j = \emptyset$, since if $k \in B_i \cap B_j$ then $k \notin A_i$, and $k \notin A_j$ which is a contradiction.

(2) $\iff$ (3): This is the Rado-Horn Theorem I.

1.6 Problems

We end with the problems which are still left open in this theory. The Rado-Horn theorem and its variants tell us the minimal number of linearly independent sets we can partition a frame into. But this is unusable in practice since it requires doing a calculation for every subset of the frame. What we have done in this chapter is to try to use the Rado-Horn Theorem to identify, *in terms of frame properties,* the minimal number of linearly independent sets into which we can partition a frame. We have shown that there are many cases where we can do this, but the general problem is still open.

Problem 2. Identify, in terms of frame properties, the minimal number of linearly independent sets into which we can partition a frame.

By frame properties we mean using the eigenvalues of the frame operator of a frame $(\varphi_i)_{i=1}^M$, the norms of the frame vectors, or the norms of the vectors of the associated Parseval frame or perhaps the norms of the frame vectors of the canonical Parseval frame associated to $\{\frac{\varphi_i}{\|\varphi_i\|}\}_{i=1}^M$.

The main problem concerning spanning and independence properties of frames is the following.

Problem 3. Given a frame Φ for $\mathcal{H}^N$, find integers $r_0, r_1, \ldots, r_{N-1}$ so that Φ can be partitioned into r_0 sets of co-dimension 0 (i.e. r_0 spanning sets), r_1 sets of codimension 1, and in general, r_i sets of codimension i for $i = 0, 1, 2, \ldots, N-1$. Moreover, do this in a maximal way in the sense that r_0 is the maximal number of spanning sets and whenever we take r_0 spanning sets out of the frame, r_1 is the maximal number of hyperplanes we can obtain from the remaining vectors and whenever r_0, r_1 are known, r_2 is the maximal number of subsets of codimension 2 which can be obtained from the remaining vectors, etc.

Finally, we need to know how to answer the above problems in practice.

Problem 4. Find real time algorithms for answering the problems above.

Problem 4 is particularly difficult since it requires finding an algorithm for proving the Rado-Horn Theorem just to get started.

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