

STRUCTURE OF THE SET OF DYADIC PFW'S

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ABSTRACT. The purpose of this paper is to reveal the deep and rich structure of the set of Parseval frame wavelets. Two main directions are pursued. First, we study the reproducing properties of the translates of a Parseval frame wavelet within the closed linear span that the translates generate. In particular, we show that the translates need not have good reproducing properties, even though the translates and dilates form a Parseval frame. Second, we describe the effect of a semiorthogonalization procedure on the set of Parseval frame wavelets. Several examples illustrating the various possibilities are given.

1. INTRODUCTION

The study of various reproducing function systems has gained a lot of momentum in recent years. It is motivated on one side by the desire to understand and describe such systems; which is clearly related to some of the fundamental questions of mathematical analysis. On the other side, the tremendous success of orthonormal wavelets (which are particular examples of reproducing systems) in theory and practice has shown that such systems are more and more within our reach. At the same time the development of more general reproducing function systems has been well on its way.

The most basic case, which is the one studied in this paper, is the one of singly generated, one-dimensional systems where the generating groups are integer translations and dyadic dilations. More precisely, we shall analyze the systems of the form

$$\{\psi_{jk}(x)\} := \{2^{j/2}\psi(2^jx - k) : j, k \in \mathbb{Z}\}, \quad (1)$$

where $\psi \in L^2(\mathbb{R})$, and the system given in (1) satisfies the *reproducing property*, that is, for every $f \in L^2(\mathbb{R})$,

$$f = \sum_{j,k \in \mathbb{Z}} \langle f, \psi_{jk} \rangle \psi_{jk} \quad (2)$$

unconditionally in $L^2(\mathbb{R})$. This property is, of course, equivalent to the property that, for every $f \in L^2(\mathbb{R})$,

$$\|f\|_2^2 = \sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{jk} \rangle|^2; \quad (3)$$

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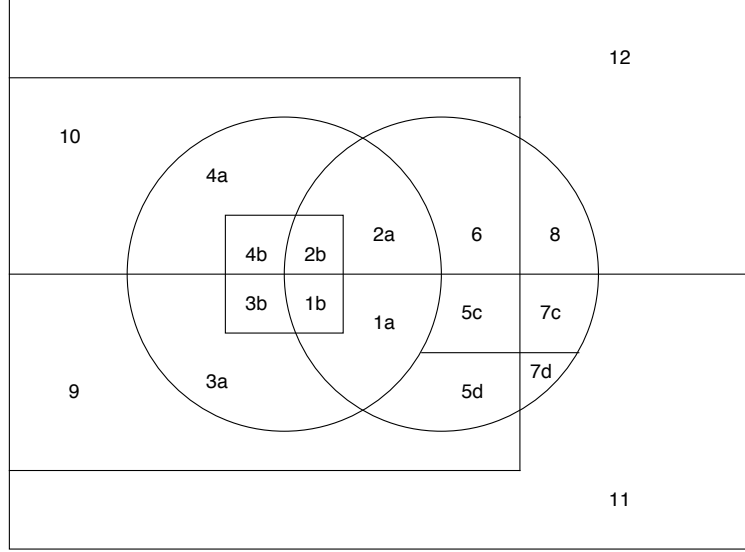
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which means that the system (1) is a normalized tight frame for $L^2(\mathbb{R})$ with upper and lower frame bounds 1. We denote by \mathcal{P} the set of all functions $\psi \in L^2(\mathbb{R})$ such that ψ satisfies (2), and call an element of \mathcal{P} a *Parseval frame wavelet (PFW)*. Many authors have studied various properties of PFW's, or of some subclasses of \mathcal{P} . Furthermore, studies of such systems have been generalized in many directions (multidimensional case, different generating groups, etc.). Despite this, some of the most fundamental questions about \mathcal{P} are still unanswered. In particular, we do not know the full structure of the set \mathcal{P} , and we do not have the systematic classification of the various subclasses of \mathcal{P} . The purpose of this paper is to reveal the deep and rich structure of the set \mathcal{P} . In many ways, this is a continuation of [12, 13].

Already in [13], it was clear that the structure of \mathcal{P} is potentially very rich, but we lacked some very basic answers to confirm our intuition. For example, it was not clear whether the various potential subclasses of \mathcal{P} were empty. As it turned out, the structure is even more complex than we suspected, and in this paper we provide many examples (some of them highly non-trivial) to support this claim. In Section 2, we revisit (and somewhat reinterpret) the necessary basic theory, which we use in Section 3 to describe the rich structure of \mathcal{P} . In Section 4, we emphasize the MRA case even more. By formalizing the process of *semiorthogonalization*, we emphasize the limitations of the known theories and the importance of the filter based approach, which is grounded on ideas that go back to [18] and [11]. In particular, when we study the elements of \mathcal{P} which are not semi-orthogonal, the shift-invariant spaces and MRA structure do not provide fine enough information, while the properties of the filters do.

We end this introduction with a figure that shows the structure of \mathcal{P} . The reader will not *a priori* understand all of the notation, but will need to go into sections 3 and 4. We do believe that the legend, though not being completely understandable to the reader without reading further, does give a description of what this work does. We will present not only results, but many examples. Some of these examples are technically complicated. Nevertheless, there are certain notions that we assume the reader is well acquainted with, such as MRA, non-MRA, semiorthogonality and Riesz bases. With this warning, we believe that the figure does indeed make the reader aware of the contents of the paper. Note that adjacency does not necessarily signify any relationship not already implied by inclusions, and that reflection about the middle line corresponds to toggling MRA and non-MRA.



Legend

Odd numbers = MRA PFW's = \mathcal{P}^{MRA}

Even numbers = $\mathcal{P}^{\text{N}} = \mathcal{P} \setminus \mathcal{P}^{\text{MRA}}$

1 = MRA orthonormal wavelets = $\mathcal{P}_{tf,+}^{\text{MRA}}$

2 = non-MRA orthonormal wavelets = $\mathcal{P}_{tf,+}^{\text{N}}$

3 = MRA semi-orthogonal, non-orthonormal PFW = $\mathcal{P}_{tf,0}^{\text{MRA}}$

4 = semi-orthogonal non-MRA non-orthonormal PFW = $\mathcal{P}_{tf,0}^{\text{N}}$

5 = MRA W_0 Riesz basis, non-orth. PFW = $\mathcal{P}_{f,+}^{\text{MRA}}$

6 = non-MRA W_0 Riesz basis, non-orth. PFW = $\mathcal{P}_{f,+}^{\text{N}}$

7 = MRA non W_0 frame, \mathcal{I} 1-1 = $\mathcal{P}_{0,+}^{\text{MRA}}$

8 = non-MRA non W_0 frame, \mathcal{I} 1-1 = $\mathcal{P}_{0,+}^{\text{N}}$

9 = MRA W_0 frame, non semi-orthogonal = $\mathcal{P}_{f,0}^{\text{MRA}}$

10 = non-MRA W_0 frame, non semi-orthogonal = $\mathcal{P}_{f,0}^{\text{N}}$

11 = MRA non W_0 frame, \mathcal{I} not 1-1 = $\mathcal{P}_{0,0}^{\text{MRA}}$

12 = non-MRA non W_0 frame, \mathcal{I} not 1-1 = $\mathcal{P}_{0,0}^{\text{N}}$

$a \cup b$ = semi-orthogonal PFW's

a = non-MSF semi-orthogonal PFW's

b = MSF PFS's

$1 \cup c \cup d$ = MRA PFW's such that \mathcal{I} is 1-1

$1 \cup c = \mathcal{P}_{-,1}^{\text{MRA}}$ = MRA PFW's whose semiorthogonalization yields o.n. wavelet

2. PRELIMINARIES

In this section, we present the basic notions, known (and needed) facts, and some auxiliary results which are either of independent interest or are necessary for our main study. Our ambient space is $L^2(\mathbb{R})$, and for $\psi \in L^2(\mathbb{R})$, we shall often employ its Fourier transform $\hat{\psi}$, given by

$$\hat{\psi}(\xi) = \int_{\mathbb{R}} \psi(x) e^{-i\xi x} dx,$$

for $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Two basic operators are the translation operator T , defined by $T\psi(x) = \psi(x-1)$, and the dilation operator D , defined by $D\psi(x) = \sqrt{2}\psi(2x)$. They are both unitary operators on $L^2(\mathbb{R})$, and for $k \in \mathbb{Z}$, we denote their respective powers by T_k and D_k . It is often useful to have the translation projection τ on \mathbb{R} , defined by $\tau(\xi) = \eta$, where $\eta \in [-\pi, \pi)$ is such that $\xi - \eta = 2\pi k$ for some $k \in \mathbb{Z}$. The dilation projection d is defined on $\mathbb{R} \setminus \{0\}$ by $d(\xi) = \eta$, where η belongs to the so-called Shannon set, $S := [-2\pi, -\pi) \cup [\pi, 2\pi)$, and is such that $\eta/\xi = 2^k$ for some $k \in \mathbb{Z}$.

2.1. Shift-invariant spaces. A closed subspace $V \subset L^2(\mathbb{R})$ is called *shift-invariant* if for every $f \in V$ and $k \in \mathbb{Z}$, we also have $T_k f \in V$. For a family $\mathcal{F} \subset L^2(\mathbb{R})$,

$$\langle \mathcal{F} \rangle := \overline{\text{span}\{T_k f : f \in \mathcal{F}, k \in \mathbb{Z}\}}$$

is the shift-invariant space generated by \mathcal{F} , i.e. the smallest shift-invariant space that contains \mathcal{F} . If $\mathcal{F} = \{f\}$ is a singleton, then $\langle f \rangle := \langle \{f\} \rangle$ is called a *principal shift-invariant space* (generated by f). The classical study of shift-invariant spaces in the context of separable Hilbert spaces goes back to H. Helson (see [9], for example). The techniques of shift-invariant spaces were used in the context of reproducing systems by C. de Boor, R. A. DeVore, and A. Ron (see, for example, [4]), and further developed in several papers by A. Ron and Z. Shen (for example, [14]). More recently, significant additions to the theory were obtained by M. Bownik [5], and by M. Bownik and Z. Rzeszotnik [6]. The ideas summarized by G. Weiss and E. Wilson [17] are closest to the spirit that we follow here. The results that we are going to present here (and use later in the article) are either in at least one of the articles mentioned here or are easily traceable to the results in these articles (and have become “folklore” by now).

Every shift-invariant space can be decomposed in terms of principal shift-invariant spaces. For a shift-invariant space V , there exists a countable family \mathcal{F} (which is not unique) such that

$$V = \oplus_{f \in \mathcal{F}} \langle f \rangle \tag{4}$$

is the orthogonal sum of principal shift-invariant spaces [5][Theorem 3.3]. Hence, one would like to understand principal shift-invariant spaces. Since for $f \in L^2(\mathbb{R})$, $\langle f \rangle = \overline{\text{span}\{T_k f : k \in \mathbb{Z}\}}$, one obtains significant information on $\langle f \rangle$ by means of the periodization of $|\hat{f}|^2$

$$p_f(\xi) := \sum_{k \in \mathbb{Z}} |\hat{f}(\xi + 2k\pi)|^2, \quad \xi \in \mathbb{R} \tag{5}$$

More precisely, let $L^2(\mathbb{T}, p_f)$ be the L^2 space on the torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ with the measure $p_f(\xi)d\xi$. For $f \in L^2(\mathbb{R})$, there is an isometric isomorphism

$$\mathcal{I} = \mathcal{I}_f : L^2(\mathbb{T}, p_f) \rightarrow \langle f \rangle, \tag{6}$$

given by $\mathcal{I}(t) := (t\hat{f})^\vee$, where we choose the inverse Fourier transform with the factor $\frac{1}{\sqrt{2\pi}}$, so that $(\hat{g})^\vee = g$ for every $g \in L^2(\mathbb{R})$. Consider also the set

$$U_f := \{\xi \in \mathbb{R} : p_f(\xi) > 0\}, \quad (7)$$

which is sometimes referred to as the spectrum of $\langle f \rangle$. Then, for $f, g \in L^2(\mathbb{R})$, we have that $\langle f \rangle = \langle g \rangle$ if and only if $g = \mathcal{I}_f(t)$ for some $t \in L^2(\mathbb{T}, p_f)$ such that $\text{supp}(t) = U_f$ a.e., where by supp we mean the ordinary set support of the function. In particular, $U_g = U_f$ a.e. Furthermore, for $f \in L^2(\mathbb{R})$, we define $h \in L^2(\mathbb{R})$ by

$$\hat{h}(\xi) := \frac{\hat{f}(\xi)}{\sqrt{p_f(\xi)}} \cdot \mathbf{1}_{U_f}(\xi), \quad \xi \in \mathbb{R}, \quad (8)$$

and it follows that $\langle h \rangle = \langle f \rangle$ and that $\{T_k h : k \in \mathbb{Z}\}$ is a normalized (constant 1) tight frame for $\langle h \rangle$. In general, $\{T_k f : k \in \mathbb{Z}\}$ is a normalized (constant 1) tight frame for $\langle f \rangle$ if and only if

$$p_f = \mathbf{1}_{U_f} \text{ a.e.} \quad (9)$$

We emphasize and apply here yet another important function associated with shift-invariant spaces; the *dimension function*. This is a mapping $\dim_V : \mathbb{R} \rightarrow \{0\} \cup \mathbb{N} \cup \{\infty\}$, where V is a shift-invariant space, given by

$$\dim_V(\xi) := \text{dimension of } \overline{\text{span}\{\hat{g}(\xi + 2k\pi) : k \in \mathbb{Z}\} : g \in V}. \quad (10)$$

Observe that $(\hat{g}(\xi + 2k\pi) : k \in \mathbb{Z})$ is an element of $\ell^2(\mathbb{Z})$, so the definition above makes sense, and that actually we do not even have to put the span and the closure in (10), since it is well-known that $\{(\hat{g}(\xi + 2k\pi) : k \in \mathbb{Z}) : g \in V\}$ is a closed subspace of $\ell^2(\mathbb{Z})$. It is also known that \dim_V is a measurable and 2π -periodic function. By (4) and the fact that

$$\dim_{\langle f \rangle} = \mathbf{1}_{U_f}, \quad (11)$$

it is very convenient (and possible) to choose the family \mathcal{F} in (4) so that it forms a finite or infinite sequence $\{f_1, f_2, \dots\}$ with the property that $U_{f_n} \supset U_{f_{n+1}}$ for all $n \in \mathbb{N}$. In this case, the dimension function of V has a particularly nice form. Observe that we also get that

$$Z_V := \{\xi : \dim_V(\xi) = 0\} = U_{f_1}^c, \quad (12)$$

and

$$I_V := \{\xi : \dim_V(\xi) = \infty\} = \bigcap_{n \in \mathbb{N}} U_{f_n}, \quad (13)$$

where in (13), the sets are decreasing.

The decomposition (4) has another interesting consequence. Since the dilation operator D is unitary, it follows that for every shift-invariant space V , the space DV is also shift-invariant, and that if V satisfies (4), then

$$DV = \bigoplus_{f \in \mathcal{F}} D\langle f \rangle. \quad (14)$$

Observe that, if we define $h(x) := \sqrt{2}f(2x - 1)$, then for every $k \in \mathbb{Z}$, $T_k Df = DT_{2k}f$, and $T_k h = DT_{2k+1}f$, i.e.,

$$D\langle f \rangle = \langle \{Df, h\} \rangle. \quad (15)$$

One then obtains a useful formula (for a much more general formula, see [6, Corollary 2.5])

$$\dim_{DV}(2\xi) = \dim_V(\xi) + \dim_V(\xi + \pi), \quad (16)$$

for a.e. $\xi \in \mathbb{R}$.

2.2. Singly generated spaces. Let $\psi \in L^2(\mathbb{R})$. It is useful to consider various “resolution levels” of ψ . Hence, for $j \in \mathbb{N} \cup \{0\}$, we define

$$W_j = W_j(\psi) := \overline{\text{span}\{\psi_{jk} : k \in \mathbb{Z}\}}, \quad (17)$$

and it is easy to see that all of them are shift-invariant space. Actually,

$$W_0 = \langle \psi \rangle, \quad \text{and } D_j W_0 = W_j, \quad j \geq 0. \quad (18)$$

Hence,

$$\dim_{W_0}(\xi) = \mathbf{1}_{U_\psi}(\xi) \text{ a.e.},$$

while (16) can be applied to find \dim_{W_j} . Observe that for $j < 0$, (17) does not define a shift-invariant spaces, so we consider

$$V_0 = V_0(\psi) := \langle \psi_{j,0} : j \in \mathbb{Z}, j < 0 \rangle, \quad (19)$$

which is a shift-invariant space. Related to it are $\ell^2(\mathbb{Z})$ vectors

$$\Psi_j(\xi) := \{\hat{\psi}(2^j(\xi + 2k\pi)) : k \in \mathbb{Z}\}, \quad \xi \in \mathbb{R}, \quad (20)$$

where $j \in \mathbb{Z}$ and $j \geq 0$; observe that the 2π -periodic, measurable function

$$\xi \rightarrow \|\Psi_j(\xi)\|_2^2 = \sum_{k \in \mathbb{Z}} |\hat{\psi}(2^j(\xi + 2k\pi))|^2$$

satisfies

$$\int_{-\pi}^{\pi} \|\Psi_j(\xi)\|_2^2 d\xi = \int_{\mathbb{R}} |\hat{\psi}(2^j u)|^2 du = \frac{1}{2^j} \|\hat{\psi}\|_2^2 < \infty. \quad (21)$$

The following useful formula is straightforward: for almost every $\xi \in \mathbb{R}$ and for every $j \in \mathbb{N} \cup \{0\}$,

$$\|\Psi_j(2\xi)\|_2^2 = \|\Psi_{j+1}(\xi)\|_2^2 + \|\Psi_{j+1}(\xi + \pi)\|_2^2. \quad (22)$$

By (10), it follows that

$$\dim_{V_0(\psi)}(\xi) = \dim \overline{\text{span}\{\Psi_j(\xi) : j \geq 1\}}, \quad (23)$$

where on the right hand side, we have the dimension function $\dim_\psi(\xi)$ used in [12, 13]. By \dim_ψ , we will mean $\dim_{V_0(\psi)}$, *not* the dimension function associated with W_0 . Observe also that

$$p_\psi(\xi) = \|\Psi_0(\xi)\|_2^2, \quad \xi \in \mathbb{R}, \quad (24)$$

and we will also use the function D_ψ , given by

$$D_\psi(\xi) := \sum_{j=1}^{\infty} \|\Psi_j(\xi)\|_2^2, \quad \xi \in \mathbb{R}. \quad (25)$$

It is obvious that for a function $\psi \in L^2(\mathbb{R})$, the three functions \dim_ψ , D_ψ and p_ψ are measurable and 2π -periodic, and that they satisfy the following two properties:

$$D_\psi(2\xi) + p_\psi(2\xi) = D_\psi(\xi) + D_\psi(\xi + \pi), \quad \xi \in \mathbb{R} \quad (26)$$

$$\int_{-\pi}^{\pi} D_\psi(\xi) d\xi = \int_{-\pi}^{\pi} p_\psi(\xi) d\xi = \|\hat{\psi}\|_2^2 < \infty. \quad (27)$$

In particular, D_ψ and p_ψ are finite a.e. The function \dim_ψ can have infinite values, but in this case it can happen only with more restrictions than in the case of general shift-invariant spaces. Let us state and prove what we mean. By considering (22) coordinatewise, it is easy to get that for a.e. $\xi \in \mathbb{R}$,

$$\dim_\psi(2\xi) \geq \max(\dim_\psi(\xi) - 1, \dim_\psi(\xi + \pi) - 1). \quad (28)$$

Recall the following lemma (see [7], for example).

Lemma 2.1. *Let $E \subset \mathbb{R}$ be a measurable set such that $E + 2\pi = E$ and $2E \subset E$. Then, either $E = \mathbb{R}$ or $E = \emptyset$ a.e.*

Clearly, the set $I_{V_0(\psi)} = \{\xi : \dim_\psi(x) = \infty\}$ is 2π -periodic, while (28) implies that $2I_{V_0(\psi)} \subset I_{V_0(\psi)}$. By Lemma 2.1, we conclude:

Proposition 2.2. *If $\psi \in L^2(\mathbb{R})$, then either $I_{V_0(\psi)} = \mathbb{R}$ a.e. or $I_{V_0(\psi)} = \emptyset$ a.e.*

Let us also observe that

$$Z_{V_0(\psi)} = \{\xi : D_\psi(\xi) = 0\}. \quad (29)$$

Let us consider $DV_0(\psi)$; it is shift-invariant as we observed in (14). In order to describe it more precisely, we introduce the functions $h_j, j \in \mathbb{Z}, j < 0$ by

$$h_j(x) := \sqrt{2}\psi_{j,0}(2x-1). \quad (30)$$

Since D is a unitary operator, we get $DV_0(\psi) = \overline{\text{span}\{T_k D_j \psi : j < 0, k \in \mathbb{Z}\}} = \overline{\text{span}\{DT_k D_j \psi : j < 0, k \in \mathbb{Z}\}}$. Observe that for $k = 2\ell + 1$ odd, we get

$$DT_k D_j \psi = T_\ell h_j, \quad (31)$$

while for $k = 2\ell$ even, we get

$$DT_k D_j \psi = T_\ell D_{j+1} \psi. \quad (32)$$

Hence,

$$DV_0(\psi) = \langle \{\psi_{j,0} : j \leq 0\} \cup \{h_j : j < 0\} \rangle. \quad (33)$$

Since $\langle \psi_{j,0} : j \leq 0 \rangle$ is equal to the sum of shift-invariant spaces $V_0(\psi) + W_0(\psi)$ (this sum is not necessarily orthogonal!), we get

$$V_0(\psi) \subset V_0(\psi) + W_0(\psi) \subset DV_0(\psi); \quad (34)$$

in particular, $V_0(\psi)$ is a *refinable* shift-invariant space (with respect to D). A consequence of (18) is then that

$$\overline{\text{span}\{\psi_{jk} : j, k \in \mathbb{Z}\}} \subset \overline{\cup_{j \in \mathbb{Z}} D_j V_0(\psi)}, \quad (35)$$

while (16) implies that

$$\dim_\psi(2\xi) \leq \dim_\psi(\xi) + \dim_\psi(\xi + \pi), \quad (36)$$

for a.e. $\xi \in \mathbb{R}$ (for a more general result, see [6, Theorem 3.2]).

2.3. Parseval frame wavelets. Suppose now that $\psi \in \mathcal{P}$, i.e. ψ is a PFW. Recall (see Chapter 7 in [10]) that for $\psi \in L^2(\mathbb{R})$, we have that $\psi \in \mathcal{P}$ if and only if

$$\sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \xi)|^2 = 1 \text{ a.e.} \quad (37)$$

and

$$t_q(\xi) := \sum_{j \geq 0} \hat{\psi}(2^j \xi) \overline{\hat{\psi}(2^j(\xi + 2q\pi))} = 0 \text{ a.e.} \quad (38)$$

whenever q is an odd integer. As a consequence, we have that $\|\psi\|_2 \leq 1$ and $|\hat{\psi}(\xi)| \leq 1$ a.e., for every $\psi \in \mathcal{P}$.

Furthermore, the reproducing property and (35) imply that for $\psi \in \mathcal{P}$, we have

$$\overline{\cup_{j \in \mathbb{Z}} D_j V_0(\psi)} = L^2(\mathbb{R}). \quad (39)$$

Remark 2.3. Since $V_0(\psi)$ is refinable, the question now becomes whether it is possible to have $\psi \in \mathcal{P}$ such that

$$V_0(\psi) = DV_0(\psi).$$

Observe that for such ψ , we have by (39) that $V_0(\psi) = L^2(\mathbb{R})$, and therefore, that $\dim_\psi = \infty$ a.e.

In the case that $\psi \in \mathcal{P}$, there are some additional useful properties. The results from the following proposition are essentially from [13, 15].

Proposition 2.4. *Suppose that $\psi \in \mathcal{P}$. Then*

- (i) $p_\psi \leq 1$ a.e.
- (ii) $D_\psi \leq \dim_\psi$ a.e.
- (iii) $\liminf_{n \rightarrow \infty} D_\psi(2^{-n}\xi) \geq 1$ for a.e. $\xi \in \mathbb{R}$.

Part 1 of Proposition 2.4 enables us to consider another useful operator defined in the context of \mathcal{I}_ψ and (6). Consider the operator $\tilde{\mathcal{I}} = \tilde{\mathcal{I}}_\psi : L^2(\mathbb{T}, dx) \rightarrow L^2(\mathbb{T}, p_\psi)$ given by $\tilde{\mathcal{I}}(f) = f$. Observe that by Proposition 2.4.1, we have that for $\psi \in \mathcal{P}$, the operator $\tilde{\mathcal{I}}_\psi$ is a bounded (norm less than 1) linear operator. The adjoint operator $\tilde{\mathcal{I}}^* : L^2(\mathbb{T}, p_\psi) \rightarrow L^2(\mathbb{T}, dx)$ is given by $\tilde{\mathcal{I}}^*(g) = p_\psi g$. Since both L^2 spaces involved contain bounded, measurable functions as a dense subset, it follows that the range of $\tilde{\mathcal{I}}$ is dense in $L^2(\mathbb{T}, p_\psi)$. Hence, the kernel of $\tilde{\mathcal{I}}^*$ is trivial; that is, $\tilde{\mathcal{I}}^*$ is injective. Consider the standard orthonormal basis $\{e_k(\xi) = \frac{e^{ik\xi}}{\sqrt{2\pi}} : k \in \mathbb{Z}\}$ of the space $L^2(\mathbb{T}, dx)$ and observe that, for every $k \in \mathbb{Z}$,

$$\mathcal{I}_\psi(\tilde{\mathcal{I}}_\psi(e_k)) = \psi_{0k} \in W_0, \quad (40)$$

and $\mathcal{I}_\psi \circ \tilde{\mathcal{I}}_\psi : L^2(\mathbb{T}, dx) \rightarrow W_0$. As we have already seen in [13], the properties of $\{\psi_{0k} : k \in \mathbb{Z}\}$ within W_0 play a major role in the analysis of \mathcal{P} . Recall that we say that $\psi \in \mathcal{P}$ is a W_0 -frame (W_0 -Parseval frame, W_0 -Riesz basis) if $\{\psi_{0k} : k \in \mathbb{Z}\}$ is a frame (Parseval frame, Riesz basis; respectively) for W_0 .

Let us emphasize yet another point; the basic idea most likely goes back to L. W. Baggett. For $\psi \in \mathcal{P}$, the space

$$\{f \in L^2(\mathbb{R}) : \|f\|_2^2 = \sum_{j \geq 0} \sum_{k \in \mathbb{Z}} |\langle f, \psi_{jk} \rangle|^2\} \quad (41)$$

is shift-invariant (since $j \geq 0$), and is the orthogonal complement of

$$\overline{\text{span}\{\psi_{jk} : k \in \mathbb{Z}, j < 0\}}.$$

It follows now easily, compare with (34), that for $\psi \in \mathcal{P}$,

$$V_0(\psi) = \overline{\text{span}\{\psi_{jk} : k \in \mathbb{Z}, j < 0\}} \quad (42)$$

and that

$$DV_0(\psi) = V_0(\psi) + W_0(\psi). \quad (43)$$

Remark 2.5. One should not conclude too much from this. Observe that the sum in (43) is not necessarily orthogonal and that we do not know immediately that $W_0(\psi)$ is not contained within $V_0(\psi)$. Hence, the question raised in Remark 2.3 remains; with some additional refinements. Namely, it follows that for $\psi \in \mathcal{P}$, the following statements are equivalent:

- (i) $V_0(\psi) = L^2(\mathbb{R})$,
- (ii) $V_0(\psi) = DV_0(\psi)$,

- (iii) $\psi \in V_0(\psi)$,
- (iv) the space in (41) is trivial

2.4. MRA Parseval frame wavelets. Following [12], we add even more structure by assuming that \mathcal{P} has a corresponding filter. A *generalized filter* is a measurable, 2π -periodic function $m : \mathbb{R} \rightarrow \mathbb{C}$ which satisfies

$$|m(\xi)|^2 + |m(\xi + \pi)|^2 = 1, \quad (44)$$

for a.e. $\xi \in \mathbb{R}$. A function $\varphi \in L^2(\mathbb{R})$ will be called a *pseudo-scaling function* if there exists a generalized filter m such that

$$\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi), \quad (45)$$

for a.e. $\xi \in \mathbb{R}$. A PFW ψ is an *MRA PFW* if there exist a pseudo-scaling function φ and an associated generalized filter m such that

$$\hat{\psi}(2\xi) = e^{i\xi} \overline{m(\xi + \pi)} \hat{\varphi}(\xi), \quad (46)$$

for a.e. $\xi \in \mathbb{R}$. We denote the set of all MRA PFW-s by \mathcal{P}^{MRA} . As it was proven in [12], if $\psi \in \mathcal{P}^{MRA}$ and m is its associated filter, then m has to be a *generalized low pass filter*, i.e. for a.e. $\xi \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \prod_{j=1}^{\infty} |m(\frac{2^{-n}\xi}{2^j})| = 1. \quad (47)$$

It was proven in [11] that for a generalized filter, the limit in (47) always exists and is either 0 or 1. Furthermore, starting with any generalized low pass filter, we can build (using the multiplier techniques explained in [12]) an MRA PFW whose associated filter is the starting one.

One of the key results in [13] is the characterization of MRA PFW's; for $\psi \in \mathcal{P}$, we have

$$\psi \in \mathcal{P}^{MRA} \iff \dim_{\psi} \in \{0, 1\}. \quad (48)$$

Let us observe some consequences of this results. Obviously, for $\psi \in \mathcal{P}^{MRA}$, we have

$$0 \leq D_{\psi}(\xi) \leq \dim_{\psi}(\xi) = \mathbf{1}_{Z_{V_0(\psi)}^c}(\xi), \quad (49)$$

for a.e. $\xi \in \mathbb{R}$. It is also immediate that

$$I_{V_0(\psi)} = \emptyset \text{ a.e. and } \psi \notin V_0(\psi). \quad (50)$$

Furthermore, using the expression for the Fourier transform of ψ_{jk} , (46) and (6), it is clear that

$$V_0(\psi) \subset \langle \varphi \rangle,$$

while (48) shows that $V_0(\psi) = \langle f \rangle$, for some f . Observe that

$$|\hat{\varphi}(\xi)|^2 = \sum_{j=1}^{\infty} |\hat{\psi}(2^j \xi)|^2, \quad (51)$$

for a.e. $\xi \in \mathbb{R}$, which also shows that

$$D_{\psi} = p_{\varphi}. \quad (52)$$

It follows that

$$U_{\varphi}^c = Z_{V_0(\psi)}, \quad (53)$$

which implies the following result.

Proposition 2.6. *Suppose that $\psi \in \mathcal{P}$. Then $\psi \in \mathcal{P}^{\text{MRA}}$ if and only if $V_0(\psi)$ is a principal shift-invariant space. If $\psi \in \mathcal{P}^{\text{MRA}}$ and φ is the associated pseudo-scaling function, then*

$$V_0(\psi) = \langle \varphi \rangle.$$

Remark 2.7. One has to be somewhat careful in applying the above result. For example, we could have that $V_0(\psi) = \langle \varphi_1 \rangle$, but that does not necessarily mean that φ_1 is the pseudo-scaling function associated with ψ in the sense of (46). We will return to this issue in more detail in Section 4.

We complete this section with a few more simple and useful technical details. Observe that (49) and Proposition 2.4, part (3), imply that for every $\psi \in \mathcal{P}^{\text{MRA}}$, we have

$$\lim_{n \rightarrow \infty} D_\psi\left(\frac{\xi}{2^n}\right) = \lim_{n \rightarrow \infty} \dim_\psi\left(\frac{\xi}{2^n}\right) = 1, \quad (54)$$

for a.e. $\xi \in \mathbb{R}$.

Lemma 2.8. *Suppose $\psi \in \mathcal{P}^{\text{MRA}}$ and φ and m are associated pseudo-scaling function and filter, respectively. Then, for a.e. $\xi \in \mathbb{R}$,*

- (i) $D_\psi(2\xi) = |m(\xi)|^2 D_\psi(\xi) + |m(\xi + \pi)|^2 D_\psi(\xi + \pi)$,
- (ii) $p_\psi(2\xi) = |m(\xi + \pi)|^2 p_\psi(\xi) + |m(\xi)|^2 p_\psi(\xi + \pi)$,
- (iii) $D_\psi(2\xi), p_\psi(2\xi) \in [\min\{D_\psi(\xi), D_\psi(\xi + \pi)\}, \max\{D_\psi(\xi), D_\psi(\xi + \pi)\}]$.

Proof The proof of (ii) goes along the same line as the proof of (i), while (iii) is an obvious consequence of (i) and (ii). Hence, we prove (i) using (45) and (46). Indeed,

$$\begin{aligned} D_\psi(2\xi) &= \sum_{k \text{ even}} |m(\xi + k\pi)|^2 |\hat{\varphi}(\xi + k\pi)|^2 + \sum_{k \text{ odd}} |m(\xi + k\pi)|^2 |\hat{\varphi}(\xi + k\pi)|^2 \\ &= |m(\xi)|^2 p_\varphi(\xi) + |m(\xi + \pi)|^2 p_\varphi(\xi + \pi). \end{aligned}$$

□

3. STRUCTURE OF \mathcal{P}

We shall introduce some additional notation in order to be able to go through the various subclasses of \mathcal{P} in a systematic way. We have already seen the first natural breaking point, a PFW ψ can be MRA, i.e., $\psi \in \mathcal{P}^{\text{MRA}}$ or non-MRA. We denote the collection of non-MRA PFW's by $\mathcal{P}^{\text{N}} := \mathcal{P} \setminus \mathcal{P}^{\text{MRA}}$. We will also use two subscripts. The first, say x , will indicate which properties (of frames) the family $\{\psi_{0k} : k \in \mathbb{Z}\}$ has with respect to W_0 , while the second, say y , will indicate which property $\tilde{\mathcal{I}}_\psi$ has, given in terms of p_ψ . Hence, we will always have

$$\mathcal{P}_{x,y} = \mathcal{P}_{x,y}^{\text{N}} \cup \mathcal{P}_{x,y}^{\text{MRA}}, \quad (55)$$

with the union being disjoint.

3.1. Non W_0 -frames such that $\tilde{\mathcal{I}}_\psi$ is not injective. This is the class of PFW's with the least amount of structure, and its precise definition and our notation for it is as follows:

$$\mathcal{P}_{0,0} := \{\psi \in \mathcal{P} : \psi \text{ is not a } W_0\text{-frame and the kernel of } \tilde{\mathcal{I}}_\psi \text{ is not trivial}\}. \quad (56)$$

If we denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}$ by $|A|$ and we use the known fact (see [13]) that for $\psi \in \mathcal{P}$, ψ is a W_0 -frame if and only if

$$(\exists 0 < c \leq 1) (p_\psi \geq c \mathbf{1}_{U_\psi} \text{ a.e.}), \quad (57)$$

then we have the following characterization of the set $\mathcal{P}_{0,0}$ and its subsets $\mathcal{P}_{0,0}^N$ and $\mathcal{P}_{0,0}^{\text{MRA}}$.

Proposition 3.1. *Suppose $\psi \in \mathcal{P}$. Then, $\psi \in \mathcal{P}_{0,0}$ if and only if*

- (i) $|U_\psi^c| > 0$, and
- (ii) $|\{\xi : 0 < p_\psi(\xi) < \epsilon\}| > 0$, for every $\epsilon > 0$.

Furthermore, if $\psi \in \mathcal{P}_{0,0}$, then $\psi \in \mathcal{P}_{0,0}^{\text{MRA}}$ if and only if $\dim_\psi = \mathbf{1}_{Z_{V_0(\psi)}^c}$, which is equivalent to \dim_ψ taking only the values of 0 and 1.

Observe that none of the conditions in Proposition 3.1 can be removed. However, it is not *a priori* clear that the classes $\mathcal{P}_{0,0}^N$ and $\mathcal{P}_{0,0}^{\text{MRA}}$ (and even $\mathcal{P}_{0,0}$ itself) are non-empty. The following examples show that they are.

Example 3.2. *We show $\mathcal{P}_{0,0}^N \neq \emptyset$.*

Consider the set $F := [\frac{3}{2}\pi, \frac{5}{3}\pi)$. It is easy to see that $d(F + 4\pi) = [\frac{11}{8}\pi, \frac{17}{12}\pi)$ and that

$$d|_{F \cup (F+4\pi)} \text{ is injective} \quad (58)$$

Observe also that $\tau(F) = [-\frac{1}{2}\pi, -\frac{1}{3}\pi)$, and $\tau(2F) = [-\pi, -\frac{2}{3}\pi)$. Hence,

$$0 \notin \overline{\tau(F \cup 2F)} \quad (59)$$

and

$$\tau|_{F \cup 2F} \text{ is injective.} \quad (60)$$

By (59), there exists $J \in \mathbb{N} \cup \{0\}$ such that for each $j \geq J$,

$$[-\frac{2\pi}{2^j}, \frac{2\pi}{2^j}] \cap \overline{\tau(F \cup 2F)} = \emptyset. \quad (61)$$

We define the set E by

$$E := (2^{-J}d(F \cup (F + 4\pi)))^c \cap ([-\frac{2\pi}{2^J}, -\frac{\pi}{2^J}) \cup [\frac{\pi}{2^J}, \frac{2\pi}{2^J})).$$

Observe that E and F are two measurable sets of positive Lebesgue measure such that

$$d|_{E \cup F \cup (F+4\pi)} \text{ is a bijection,} \quad (62)$$

$$\tau|_{F \cup 2F \cup \tilde{E}} \text{ is injective, where } \tilde{E} := \cup_{j=0}^{\infty} 2^{-j}E, \text{ and} \quad (63)$$

$$2^{-j}E, j \geq 0; F; F + 4\pi; 2F; 2F + 8\pi \text{ are pairwise disjoint.} \quad (64)$$

It is easy to check (64) directly, (62) follows from (58) and the definition of E , while (63) follows from (60) and (61).

Consider now any $a \in (0, \frac{1}{2})$ and define $\psi \in L^2(\mathbb{R})$ such that

$$\hat{\psi}(\xi) = \begin{cases} -\sqrt{\frac{1}{2} + a}, & \xi \in 2F \\ \sqrt{\frac{1}{2} - a}, & \xi \in F \cup (2F + 8\pi) \\ \sqrt{\frac{1}{2} + a}, & \xi \in F + 4\pi \\ \sqrt{2^{-(j+1)}}, & \xi \in 2^{-j}E, j \geq 0 \\ 0, & \text{otherwise.} \end{cases} \quad (65)$$

We show that $\psi \in \mathcal{P}$. We need to show that $\hat{\psi}$ satisfies (37) and (38). Observe that (62) implies that it is enough to check (37) for $\xi \in E \cup F \cup (F + 4\pi)$. This, then, follows directly from (65).

In order to prove that (38) is satisfied, it is not difficult to see that the interesting cases are when $\xi \in \mathbb{R}$ and $q \in 2\mathbb{Z}+1$ such that there exists $j \geq 0$ with $2^j \xi \in \text{supp}(\hat{\psi})$ and $2^j(\xi + 2q\pi) \in \text{supp}(\hat{\psi})$. Without loss of generality, we consider the case $q > 0$. Since $\tau(2^j \xi) = \tau(2^j(\xi + 2q\pi))$, it follows by (63) that these conditions on j and q are satisfied only in the case that either $(2^j \xi \in F$ and $2^j(\xi + 2q\pi) \in F + 4\pi)$ or $(2^j \xi \in 2F$ and $2^j(\xi + 2q\pi) \in 2F + 8\pi)$. In the first case, we get $2^j q = 4$, which implies $q = 1$ and $j = 1$. In the second case, we get $2^j q = 8$, which implies that $q = 1$ and $j = 2$. Hence, in both cases, $\xi \in F/2$, so we get

$$t_q(\xi) = \sqrt{\frac{1}{2} - a} \sqrt{\frac{1}{2} + a} + (-\sqrt{\frac{1}{2} + a}) \sqrt{\frac{1}{2} - a} = 0.$$

It remains to prove that $\psi \in \mathcal{P}_{0,0}^N$; that is, to check that 1 and 2 from Proposition 3.1 hold, and that \dim_ψ attains values bigger than 2 on a set of positive measure. Notice that

$$\tau(\text{supp}\hat{\psi}) = \tau(F \cup 2F \cup \tilde{E}) \subset [-\pi, -\frac{2}{3}\pi) \cup [-\frac{1}{2}\pi, -\frac{1}{3}\pi) \cup [-\frac{2\pi}{2^J}, \frac{2\pi}{2^J}).$$

Since $p_\psi = 0$ outside the set above (which does not include all of $[-\pi, \pi)$), condition i) is clearly fulfilled.

For $\xi \in 2^{-j}E$, we get $p_\psi(\xi) = |\hat{\psi}(\xi)|^2 = 2^{-(j+1)}$. Since $|2^{-j}E| > 0$, for every $j \geq 0$, and $\lim_{j \rightarrow \infty} 2^{-(j+1)} = 0$, we obtain ii).

For the last requirement, consider $\xi \in F/2$; recall that $|F/2| > 0$. Observe that among the vectors $\Psi_j(\xi)$ we have

$$(\dots, 0, 0, \sqrt{\frac{1}{2} - a}, \sqrt{\frac{1}{2} + a}, 0, 0, \dots)$$

and

$$(\dots, 0, 0, -\sqrt{\frac{1}{2} + a}, \sqrt{\frac{1}{2} - a}, 0, 0, \dots)$$

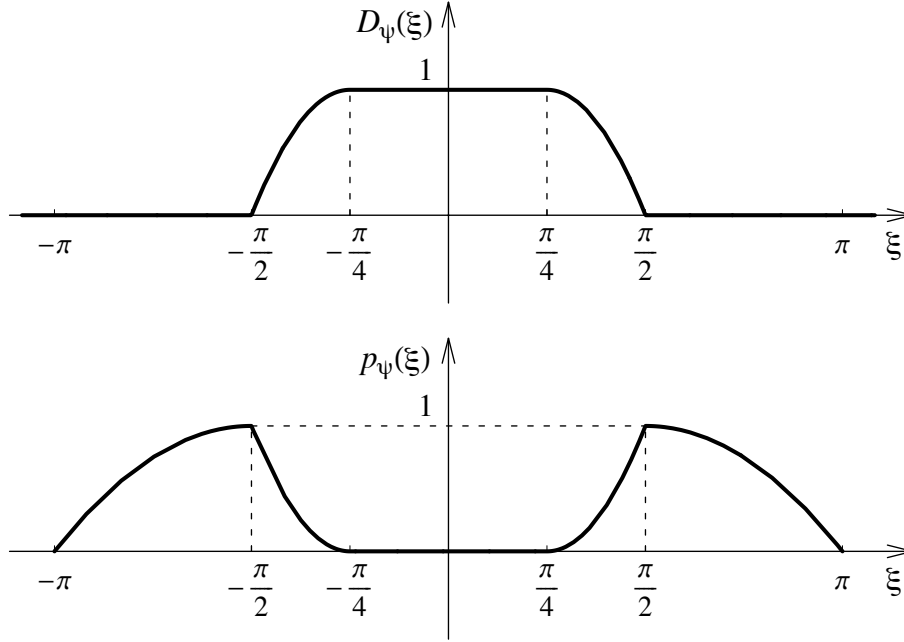
which are linearly independent. Hence, $\dim_\psi(\xi) \geq 2$ for $\xi \in F/2$. \square

Example 3.3. We show $\mathcal{P}_{0,0}^{MRA} \neq \emptyset$.

Consider [13, Example 2.6] which give $\psi \in \mathcal{P}^{MRA}$ such that the associated scaling pseudo-scaling function φ is given by

$$\hat{\varphi}(\xi) = \begin{cases} 1, & |\xi| \leq \frac{\pi}{4} \\ -\frac{4}{\pi}|\xi| + 2, & \frac{\pi}{4} \leq |\xi| \leq \frac{\pi}{2} \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to check then that the graphs of D_ψ and p_ψ are such that $\psi \in \mathcal{P}_{0,0}^{MRA}$.



3.2. **Non W_0 -frames such that $\tilde{\mathcal{I}}_\psi$ is injective.** The notation for this collection of functions is

$$\mathcal{P}_{0,+} := \{\psi \in \mathcal{P} : \psi \text{ is not a } W_0\text{-frame and } \ker(\tilde{\mathcal{I}}_\psi) = \{0\}\} \quad (66)$$

Observe that $\mathbf{1}_{U_\psi^c \cap [-\pi, \pi]}$ is equal to zero when considered in $L^2(\mathbb{T}, p_\psi)$. Then, it is straightforward to get

$$\ker(\tilde{\mathcal{I}}_\psi) = \{0\} \iff |U_\psi^c| = 0. \quad (67)$$

Hence, we get the following characterization of $\mathcal{P}_{0,+}$ in which none of the conditions are redundant.

Proposition 3.4. *Suppose $\psi \in \mathcal{P}$. Then, $\psi \in \mathcal{P}_{0,+}$ if and only if*

- (i) $|U_\psi^c| = 0$ and
- (ii) $|\{\xi : 0 < p_\psi(\xi) < \epsilon\}| > 0$ for every $\epsilon > 0$.

The only construction of examples in $\mathcal{P}_{0,+}$ that we know of is somewhat delicate. We recall the following theorem, which is a special case of [8, Theorem 1] along the lines of [16, Theorem 1.1].

Theorem 3.5 (DLS). *Let $E \subset [-\pi, \pi]$ and $F \subset [-2\pi, -\pi] \cup [\pi, 2\pi]$ be measurable sets such that $0 \in E^\circ$ and $E^\circ \neq \emptyset$. Then, there exists measurable $G \subset \mathbb{R}$ such that $\tau|_G$ and $d|_G$ are injective functions with $\tau(G) = E$ and $d(G) = F$.*

Example 3.6. *We show $\mathcal{P}_{0,+} \neq \emptyset$.*

Consider sets $E := [-\frac{2\pi}{3}, \frac{\pi}{3}]$ and $F := [-2\pi, -\pi]$. Apply Theorem 3.5 to obtain the set $G \subset \mathbb{R}$ such that $\tau : G \rightarrow E$ and $d : G \rightarrow F$ are measurable bijections. Observe that G is necessarily a subset of $(-\infty, 0)$. Consider $0 < \alpha < \frac{1}{2}$ and sets $H_1 := [\frac{\pi}{3}, \frac{2\pi}{3}]$, $H_2 := 2H_1 = [\frac{2\pi}{3}, \frac{4\pi}{3}]$. Observe that

$$\tau|_{H_1 \cup H_2 \cup G} : H_1 \cup H_2 \cup G \rightarrow [-\pi, \pi] \text{ is a bijection.} \quad (68)$$

We define $\psi \in L^2(\mathbb{R})$ by

$$\hat{\psi}(\xi) := \begin{cases} \min(|\xi - \pi|^\alpha, \sqrt{\frac{1}{2}}) & \xi \in H_2 \\ \sqrt{1 - |\min(|2\xi - \pi|^\alpha, \sqrt{\frac{1}{2}})|^2} & \xi \in H_1 \\ 1 & \xi \in G \\ 0 & \text{otherwise.} \end{cases} \quad (69)$$

Note that on H_1 , $\hat{\psi}(\xi) = \sqrt{1 - |\hat{\psi}(2\xi)|^2}$. See that $\hat{\psi}$ satisfies (37), since for $\xi > 0$ the dyadic orbit of ξ hits H_1 and H_2 exactly once, and for $\xi < 0$, it hits G only once. It is also easy to check (38), since $\hat{\psi}(\xi) \neq 0$ implies that $\hat{\psi}(\xi + 2k\pi) = 0$ for every $0 \neq k \in \mathbb{Z}$. Therefore, $\psi \in \mathcal{P}$.

Property (ii) from Proposition 3.4 follows from the definition of $\hat{\psi}$ on H_2 , since $\hat{\psi}(\pi) = 0$ and $\hat{\psi}$ continuously approaches 0 around π (observe that $|\hat{\psi}(\xi)|^2$ equals $p_\psi(\xi)$ for $\xi \in H_1 \cup H_2 \cup G$).

We also have an even stronger condition than (i) from Proposition 3.4. It is easy to check that

$$\frac{1}{p_\psi} \Big|_{H_1 \cup H_2 \cup G} \text{ is in } L^1. \quad (70)$$

Remark 3.7. An obvious question is if ψ from Example 3.6 belongs to $\mathcal{P}_{0,+}^{\mathbb{N}}$ or $\mathcal{P}_{0,+}^{\text{MRA}}$. Although we have simple characterizations of these subclasses of $\mathcal{P}_{0,+}$, it is not easy to check the properties of \dim_ψ , since we do not have a detailed description of the set G .

Remark 3.8. By taking $\alpha \geq \frac{1}{2}$ in Example 3.6, we can adjust ψ so that $|U_\psi^c| = 0$, but $\frac{1}{p_\psi} \notin L^1$, which shows that the class $\mathcal{P}_{0,+}$ is even more interesting than one would expect. For example, when $\frac{1}{p_\psi} \in L^1$, as in our Example 3.6, then, for every h bounded in $L^2(\mathbb{T}, dx)$, we have that $\frac{h}{p_\psi} \in L^2(\mathbb{T}, p_\psi)$ and

$$\tilde{\mathcal{I}}_\psi^* \left(\frac{h}{p_\psi} \right) = h. \quad (71)$$

In particular, the sequence $\{y_k\}$, where $y_k := \frac{e_k}{p_\psi} \in L^2(\mathbb{T}, p_\psi)$, is biorthogonal to the sequence $\{x_k\}$, where $x_k := \tilde{\mathcal{I}}_\psi(e_k)$. Using \mathcal{I} , we get that $\{\psi_{0k}\}$ has a biorthogonal sequence in W_0 . It is not difficult to prove that this fact is actually equivalent to $\frac{1}{p_\psi} \in L^1$. However, since such questions go beyond dimension one and the case of PFW's, we hope to address them in a separate article.

3.3. W_0 -frames, but not more. As we have seen, when a PFW ψ has the property that the family $\{\psi(\cdot - k) : k \in \mathbb{Z}\}$ is a frame for W_0 , we say that ψ is a W_0 -frame. Recall that for a $\psi \in \mathcal{P}$, this is equivalent to the following property:

$$(\exists 0 < c \leq 1) \text{ such that } p_\psi \geq c \mathbf{1}_{U_\psi} \text{ a.e.} \quad (72)$$

This property can be improved in at least two important ways; we could require that $c = 1$ and that $U_\psi = \mathbb{R}$ a.e. The first improvement would lead to semi-orthogonality and the second to W_0 -Riesz bases. Here, we are interested in a subclass which does not have either of these improvements. Hence,

$$\mathcal{P}_{f,0} := \{ \psi \in \mathcal{P} : \psi \text{ is a } W_0\text{-frame, } \psi \text{ is not semi-orthogonal,} \quad (73) \\ \text{and } \psi \text{ is not a } W_0\text{-Riesz basis} \}.$$

In the following characterization, none of the conditions are redundant.

Proposition 3.9. *Suppose that $\psi \in \mathcal{P}$ ($\psi \in \mathcal{P}^N, \psi \in \mathcal{P}^{MRA}$, respectively). Then, $\psi \in \mathcal{P}_{f,0}$ ($\psi \in \mathcal{P}_{f,0}^N, \psi \in \mathcal{P}_{f,0}^{MRA}$, respectively) if and only if (72) holds and*

- (i) $|U_\psi^c| > 0$,
- (ii) $|\{\xi : 0 < p_\psi(\xi) < 1\}| > 0$.

Furthermore, in the equivalence above, the condition (ii) can be replaced by either of the following conditions:

- (iii) p_ψ is not integer valued,
- (iv) D_ψ is not integer valued.

Proof: This is also more or less a straightforward application of the ideas from [12] and [13]. Being a W_0 -frame is equivalent to (72). Adding $|U_\psi^c| = 0$ gives us a W_0 -Riesz basis, while allowing c in (72) to be 1 would give us a semi-orthogonal PFW. Conditions (ii) and (iii) are clearly equivalent, while (iii) implies (iv) by (26). Finally, if (iv) holds, then D_ψ can not be equal to \dim_ψ ; hence, ψ is not semi-orthogonal. Therefore, (ii) holds.

Observe that the fact that none of the conditions can be removed is actually proven by examples in this article. \square

Remark 3.10. As for the other classes, we know that for a $\psi \in \mathcal{P}_{f,0}$, we have that $\psi \in \mathcal{P}_{f,0}^{MRA}$ if and only if \dim_ψ attains only the values 0 and 1. In particular, for $\psi \in \mathcal{P}_{f,0}^{MRA}$, we have $0 \leq D_\psi \leq 1$. It is natural to ask if within $\mathcal{P}_{f,0}$, the condition $0 \leq D_\psi \leq 1$ is also sufficient for ψ being in $\mathcal{P}_{f,0}^{MRA}$. The following example, which also shows that $\mathcal{P}_{f,0}^N \neq \emptyset$, shows that the answer is negative. Let us also mention that this example was announced in [13, Remark 3.7(b)].

Example 3.11. $\mathcal{P}_{f,0}^N \neq \emptyset$.

Proof: Let $\epsilon > 0$ be a small number to be specified later. Let $E := [\frac{\pi}{16} - \epsilon, \frac{\pi}{16} + \epsilon]$ so that $d(E) = [\pi, \pi + 16\epsilon) \cup [2\pi - 32\epsilon, 2\pi)$. For $\epsilon > 0$ small enough, we have that $d|_{E \cup (E + \pi)}$ is injective and

$$d(E \cup (E + \pi)) = [\pi, \pi + 16\epsilon) \cup [\frac{17}{16}\pi - \epsilon, \frac{17}{16}\pi + \epsilon) \cup [2\pi - 32\epsilon, 2\pi).$$

We define $\tilde{F} := S \setminus d(E \cup (E + \pi))$, where S is the Shannon set, and $F := 2^{-6}\tilde{F}$. Hence, $F \subset [-\frac{\pi}{32}, -\frac{\pi}{64}) \cup [\frac{\pi}{64}, \frac{\pi}{32})$.

We define $\psi \in L^2(\mathbb{R})$ by

$$\hat{\psi}(\xi) := \begin{cases} \frac{1}{\sqrt{2}} & \xi \in E \cup (E + \pi) \\ \frac{1}{\sqrt{6}} & \xi \in 2E \\ -\frac{1}{\sqrt{6}} & \xi \in 2E + 2\pi \\ \frac{1}{\sqrt{8}} - \frac{1}{\sqrt{24}} & \xi \in 4E \cup (8E + 8\pi) \\ \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{24}} & \xi \in 8E \cup (4E + 4\pi) \\ 1 & \xi \in F \\ 0 & \text{otherwise.} \end{cases} \quad (74)$$

Let us show that ψ is a PFW. To check (37), observe that it is enough to do it for $\xi \in F \cup E \cup (E + \pi)$. For $\xi \in F$, the result is immediate, while for E and $E + \pi$, the calculations are essentially the same. For $\xi \in E$, the only elements in

the dyadic orbit which are also in $\text{supp}(\hat{\psi})$ are ξ , 2ξ , 4ξ and 8ξ , and one checks directly that the corresponding sum is 1.

As usual, showing (38) is somewhat more delicate. Observe first that without loss of generality, we can assume that $q > 0$ in the q from t_q . Observe also that

$$\tau(F) = F \subset \left[-\frac{\pi}{32}, -\frac{\pi}{64}\right) \cup \left[\frac{\pi}{64}, \frac{\pi}{32}\right) \quad (75)$$

$$\tau(E) = E = \left[\frac{\pi}{16} - \epsilon, \frac{\pi}{16} + \epsilon\right) \quad (76)$$

$$\tau(E + \pi) = E - \pi = \left[\frac{15}{16}\pi - \epsilon, \frac{15}{16}\pi + \epsilon\right) \quad (77)$$

$$\tau(2E + 2\pi) = \tau(2E) = \left[\frac{\pi}{8} - 2\epsilon, \frac{\pi}{8} + 2\epsilon\right) \quad (78)$$

$$\tau(4E + 4\pi) = \tau(4E) = \left[\frac{\pi}{4} - 4\epsilon, \frac{\pi}{4} + 4\epsilon\right) \quad (79)$$

$$\tau(8E + 8\pi) = \tau(8E) = \left[\frac{\pi}{2} - 8\epsilon, \frac{\pi}{2} + 8\epsilon\right). \quad (80)$$

In particular, assuming $\epsilon > 0$ is small enough, we have that if for some $0 \neq k \in \mathbb{Z}$ that $\xi \in \text{supp}(\hat{\psi}) \cap (\text{supp}(\hat{\psi}) + 2k\pi)$, then $\xi \in 2E \cup (2E + 2\pi) \cup 4E \cup (4E + 4\pi) \cup 8E \cup (8E + 8\pi)$.

Take $\xi \in \mathbb{R}$. If for every $j \in \mathbb{N} \cup \{0\}$, we have

$$2^j \xi \notin \text{supp}(\hat{\psi}) \text{ or } 2^j(\xi + 2q\pi) \notin \text{supp}(\hat{\psi}),$$

then $t_q(\xi) = 0$. Otherwise, there exists $j \in \mathbb{N} \cup \{0\}$ such that $2^j \xi$ and $2^j(\xi + 2q\pi)$ are in $\text{supp}(\hat{\psi})$. It follows then by (75)-(80) and the assumption that $q > 0$ that we have three options:

$$\begin{aligned} \xi \in 2E & \quad j = 0, q = 1 \\ \xi \in 2E & \quad j = 1, q = 1 \\ \xi \in 2E & \quad j = 2, q = 1. \end{aligned}$$

Hence, we get that

$$t_1(\xi) = \frac{1}{\sqrt{6}}\left(-\frac{1}{\sqrt{6}}\right) + 2\left(\frac{1}{\sqrt{8}} - \frac{1}{\sqrt{24}}\right)\left(\frac{1}{\sqrt{8}} + \frac{1}{\sqrt{24}}\right) = 0.$$

We have proven that $\psi \in \mathcal{P}$. It is straightforward from (75)-(80) to conclude that for $\epsilon > 0$ small enough, we have (72) and (i) and (ii) from Proposition 3.9, i.e. $\psi \in \mathcal{P}_{f,0}$. It remains to show that ψ is not an MRA PFW and that $0 \leq D_\psi \leq 1$. The first claim follows from the fact that for $\xi \in 2E$ we will have among the vectors $\Psi_j(\xi), j \geq 0$, at least the vectors

$$\begin{aligned} & \left(\dots, 0, \frac{1}{\sqrt{8}} - \frac{1}{\sqrt{24}}, \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{24}}, 0, \dots\right) \\ & \left(\dots, 0, \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{24}}, \frac{1}{\sqrt{8}} - \frac{1}{\sqrt{24}}, 0, \dots\right), \end{aligned}$$

which are linearly independent. Hence, $\dim_\psi(\xi) \geq 2$ and $\psi \in \mathcal{P}_{f,0}^N$.

In order to calculate D_ψ , we first consider the function

$$S_\psi(\xi) := \sum_{j=1}^{\infty} |\hat{\psi}(2^j \xi)|^2. \quad (81)$$

Observe that D_ψ is the periodization of S_ψ and that for every choice of j and k we have

$$2^j E \cap 2^k (E + \pi) = 2^j E \cap 2^k F = 2^j F \cap 2^k (E + \pi) = \emptyset.$$

Hence, we obtain

$$S_\psi(\xi) = \begin{cases} 1 & \xi \in \cup_{j \geq 1} 2^{-j} (E \cup (E + \pi) \cup F) \\ \frac{1}{2} & \xi \in E \cup (E + \pi) \\ \frac{1}{3} & \xi \in 2E \cup 2(E + \pi) \\ (\frac{1}{\sqrt{8}} + \frac{1}{\sqrt{24}})^2 & \xi \in 4E \\ (\frac{1}{\sqrt{8}} - \frac{1}{\sqrt{24}})^2 & \xi \in 4(E + \pi) \\ 0 & \text{otherwise.} \end{cases} \quad (82)$$

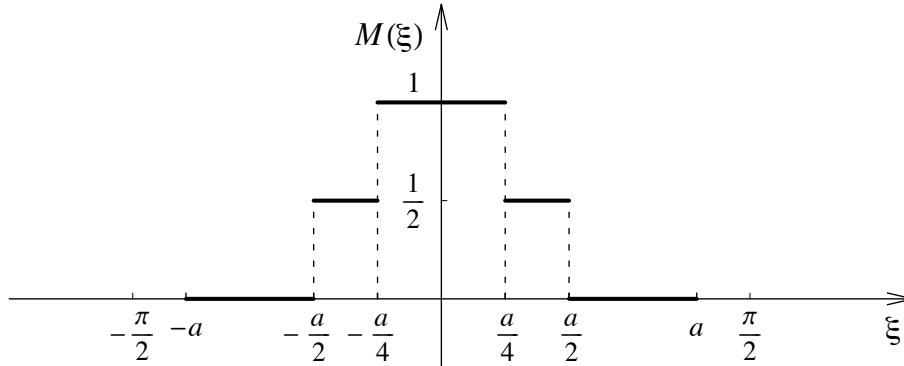
Recall that D_ψ is 2π -periodic, so it is enough to compute it on $[-\pi, \pi)$. It follows from (82) that for $\xi \in [-\pi, \pi)$, we obtain

$$D_\psi(\xi) = \begin{cases} 1 & \xi \in \cup_{j \geq 1} 2^{-j} (E \cup (E + \pi) \cup F) \\ \frac{1}{2} & \xi \in E \cup (E - \pi) \\ \frac{2}{3} & \xi \in 2E \\ \frac{1}{3} & \xi \in 4E \\ 0 & \text{otherwise,} \end{cases}$$

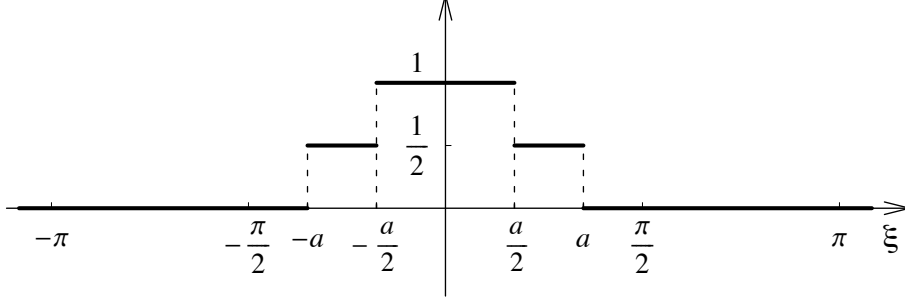
where we used the fact that $E - \pi$ is disjoint from all the other sets appearing above. Thus, in particular, $0 \leq D_\psi(\xi) \leq 1$ for every ξ , as desired. \square

Example 3.12. $\mathcal{P}_{f,0}^{MRA} \neq \emptyset$.

Proof: Consider $0 < a < \pi/2$. Since every generalized filter can be extended, through (44), from $[-\pi/2, \pi/2)$ to $[-\pi, \pi)$, it is obvious that there exists a generalized filter m such that the graph of the function $M(\xi) := |m(\xi)|^2$ on $[-a, a)$ is given by



Since $m(\xi) = 0$ for $\xi \in [-a, -a/2) \cup [a/2, a)$, it follows by (45) that $\hat{\varphi}(\xi) = 0$ for $|\xi| > a$, where φ is the corresponding pseudo-scaling function. Furthermore, it follows that the graph of the function $\xi \rightarrow |\hat{\varphi}(\xi)|^2$ is as follows:



Observe that $M(\xi) = 1$ on $(-a/4, a/4)$, which ensures that (47) is valid, i.e., m is a generalized low pass filter, and by the well-known construction from [12], we know that it generates an MRA PFW ψ .

Since $M(\xi)$ and $|\hat{\varphi}(\xi)|^2$ attain only the values 0, 1/2 and 1, by (46), the same is true of $|\hat{\psi}(\xi)|^2$. Hence, (72) is fulfilled. Again, by (46) and (3.3), we see that $\text{supp}(\hat{\psi}) \subset [-2a, 2a]$, which is a proper subset of $[-\pi, \pi]$. This shows (i) from Proposition 3.9. The second consequence is that, by (52), we get (iv) from Proposition 3.9. Hence, $\psi \in \mathcal{P}_{f,0}^{\text{MRA}}$.

3.4. W_0 -Riesz bases, but not more. The next natural step is to consider W_0 -frames for which $\tilde{\mathcal{I}}$ is injective, i.e.

$$\mathcal{P}_{f,+} := \{\psi \in \mathcal{P} : \psi \text{ is a } W_0\text{-frame, } \psi \text{ is not semi-orthogonal, } \tilde{\mathcal{I}}_\psi \text{ is injective}\}. \quad (83)$$

Recall that two of the conditions in (83), namely that ψ is a W_0 -frame and $\tilde{\mathcal{I}}$ is injective, are actually equivalent to the single condition:

$$(\exists 0 < c \leq 1) \text{ such that } p_\psi \geq c, \text{ a.e.}, \quad (84)$$

which as we know from [13], is equivalent to ψ being a W_0 -Riesz basis. Observe also that (84) is equivalent to

$$\frac{1}{p_\psi} \text{ is bounded}, \quad (85)$$

which implies that $\tilde{\mathcal{I}}_\psi$ is invertible and its inverse is a bounded operator. Hence, $\{\psi_{0k}\}$ is the image of the orthonormal basis via the regular operator. This is to be expected. Observe, however, that in this case both the orthonormal basis $\{e_k\}$ and the regular operator are explicitly given.

It is fairly straightforward to get the following characterization of $\mathcal{P}_{f,+}$ where none of the conditions are redundant.

Proposition 3.13. *Suppose that $\psi \in \mathcal{P}$ (resp. $\psi \in \mathcal{P}^N, \psi \in \mathcal{P}^{\text{MRA}}$). Then, $\psi \in \mathcal{P}_{f,+}$ (resp. $\psi \in \mathcal{P}_{f,+}^N, \psi \in \mathcal{P}_{f,+}^{\text{MRA}}$) if and only if (84) is valid and*

$$(i) |\{\xi : p_\psi(\xi) \neq 1\}| > 0.$$

Furthermore, in the equivalence above, the condition (i) can be replaced by either of the following conditions:

- (ii) p_ψ is not integer valued,
- (iii) D_ψ is not integer valued.

Remark 3.14. The elements of $\mathcal{P}_{f,+}$ are W_0 -Riesz bases which are not semi-orthogonal. Example 2.5 in [13], which corresponds to the generalized low pass

filter $m(\xi) = \frac{1}{2}(1 + e^{3i\xi})$, shows that $\mathcal{P}_{f,+}^{\text{MRA}} \neq \emptyset$. In particular, $\mathcal{P}_{f,+} \neq \emptyset$. We will revisit the class $\mathcal{P}_{f,+}^{\text{MRA}}$ in more detail in Section 4. The question whether $\mathcal{P}_{f,+}^{\text{N}} = \emptyset$ remains open at this point.

3.5. Semiorthogonal PFW's, which are not orthonormal. Let us consider all “resolution levels” $W_j(\psi)$, see (17). Recall that we say that a PFW ψ is *semi-orthogonal* if $W_j(\psi) \perp W_k(\psi)$ whenever $j \neq k$, $j, k \in \mathbb{Z}$. Observe that for such a ψ , we have $D_\psi = \dim_\psi$, so it is not possible that $\dim_\psi = \infty$. In particular, for a semi-orthogonal PFW ψ , we always have

$$\psi \notin V_0(\psi), \quad (86)$$

in particular, $V_0(\psi) \neq L^2(\mathbb{R})$.

It is useful to recall the various characterizations of semi-orthogonality within \mathcal{P} . Using (43) and [13, Theorem 2.7, Theorem 3.1, Corollary 3.2], we obtain directly the following theorem.

Theorem 3.15. *Suppose ψ is a PFW. The following are equivalent.*

- (i) ψ is semi-orthogonal,
- (ii) ψ is a W_0 -Parseval frame,
- (iii) p_ψ is integer valued,
- (iv) $p_\psi = \mathbf{1}_{U_\psi}$,
- (v) D_ψ is integer valued,
- (vi) $D_\psi = \dim_\psi$,
- (vii) $\|\psi\|_2^2 = \sum_{k \in \mathbb{Z}} |\langle \psi, \psi_{0k} \rangle|^2$,
- (viii) $DV_0(\psi) = V_0(\psi) \oplus W_0(\psi)$

Remark 3.16. Condition (viii) in the previous theorem shows also that there is a close connection between the GMRA structure and semi-orthogonal PFW's. (See, for example, [1] for the definition and basics of GMRA's.) Indeed, if ψ is a semi-orthogonal PFW, then (viii) implies that

$$V_0(\psi) = \bigoplus_{j < 0} W_j(\psi), \quad (87)$$

and $V_0(\psi)$ is the core space for a GMRA, assuming that ψ generated a GMRA to begin with. Observe also that in this case, the function S_ψ given in (81) is the spectral function of the shift-invariant space $V_0(\psi)$, see [6] for details.

Conversely, if $\{V_j\}$ is a GMRA and ψ is a Parseval frame for $V_1 \cap V_0^\perp$, then ψ is a semi-orthogonal PFW. Since these facts are part of folklore even in higher dimensions and for more general dilations, we do not say more here. Rather, we will emphasize that semi-orthogonal PFW's are most closely related to the theory of GMRA's, while when we go beyond semi-orthogonality, we obtain only very limited information through the GMRA approach. For example, if $\psi \in \mathcal{P}$ and $\psi \notin V_0(\psi)$, then we could consider

$$\psi_1 = \psi - \psi_0,$$

where ψ_0 is the orthogonal projection of ψ onto $V_0(\psi)$, and we can associate with ψ the corresponding semi-orthogonal PFW, which is going to have the same GMRA as ψ . However, since at this level of generality, we have a much more basic question given in Remark 2.5, we will address this issue in detail only in the MRA case in Section 4.

Let us also emphasize that a straightforward consequence of Theorem 3.15 is the following set of two analogous properties.

Corollary 3.17. *Suppose $\psi \in \mathcal{P}$ (resp. $\psi \in \mathcal{P}^{MRA}$). Then, ψ is a semi-orthogonal MRA PFW if and only if D_ψ (resp. p_ψ) has only values 0 and 1.*

Observe also that for a semi-orthogonal $\psi \in \mathcal{P}^N$, we have $\dim_\psi = D_\psi$, and hence, $|\{\xi : D_\psi(\xi) \geq 2\}| > 0$. The integrability condition (27) then implies that the Lebesgue measure of the set of zeroes of D_ψ must be positive, too. Hence, we have proven the following proposition.

Proposition 3.18. *If $\psi \in \mathcal{P}^N$ is semi-orthogonal, then $|Z_{V_0(\psi)}| > 0$.*

It is now easy to characterize the class that interests us here, i.e.

$$\mathcal{P}_{tf,0} := \{\psi \in \mathcal{P} : \psi \text{ is a } W_0\text{-Parseval frame and } \tilde{\mathcal{I}}_\psi \text{ is not injective}\}. \quad (88)$$

Observe that for $\psi \in \mathcal{P}_{tf,0}$, we have that $p_\psi = \mathbf{1}_{U_\psi}$ and $|U_\psi^c| > 0$, so $L^2(\mathbb{T}, p_\psi)$ really becomes $L^2(\mathbb{T} \cap U_\psi, dx)$ and $\tilde{\mathcal{I}}_\psi$ is the orthogonal projection from $L^2(\mathbb{T}, dx)$ onto $L^2(\mathbb{T} \cap U_\psi, dx)$. Observe also that $\tilde{\mathcal{I}}_\psi^*$ then becomes the inclusion of $L^2(\mathbb{T} \cap U_\psi, dx)$ into $L^2(\mathbb{T}, dx)$.

Proposition 3.19. (i) *Suppose $\psi \in \mathcal{P}$. Then, $\psi \in \mathcal{P}_{tf,0}$ if and only if ψ is semi-orthogonal and $|U_\psi^c| > 0$. Furthermore, the last condition can be replaced by $\|\psi\|_2 < 1$. The same is valid for the class \mathcal{P}^N .*

(ii) *Suppose $\psi \in \mathcal{P}$. Then, $\psi \in \mathcal{P}_{tf,0}^{MRA}$ if and only if $D_\psi = \mathbf{1}_{Z_{V_0(\psi)}^c}$ and $|Z_{V_0(\psi)}| > 0$.*

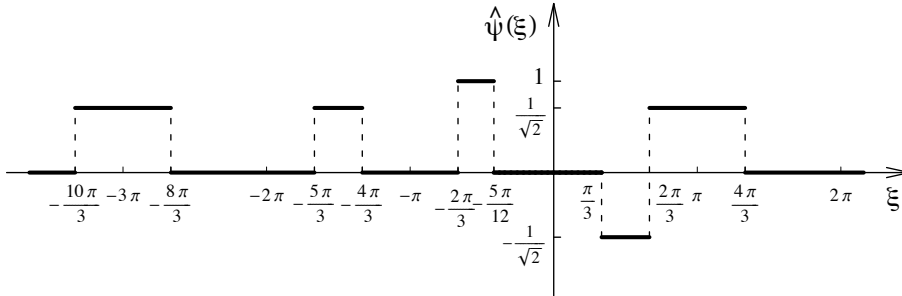
(iii) *Suppose $\psi \in \mathcal{P}^{MRA}$. Then, $\psi \in \mathcal{P}_{tf,0}^{MRA}$ if and only if $p_\psi = \mathbf{1}_{U_\psi}$ and $|U_\psi^c| > 0$.*

Observe that whenever $\psi \in \mathcal{P}_{tf,0}$, we always have $|Z_{V_0(\psi)}| > 0$. However, in Proposition 3.19 (i), the condition $|U_\psi^c| > 0$ can not be replaced by $|Z_{V_0(\psi)}| > 0$ (consider, for example, the Journé wavelet). In Proposition 3.19 (iii), it is possible to replace the condition $|U_\psi^c| > 0$ with $|Z_{V_0(\psi)}| > 0$. We omit the details.

In terms of the examples, we will revisit $\mathcal{P}_{tf,0}^{MRA}$ later, while at this point, let us consider $\mathcal{P}_{tf,0}^N$.

Example 3.20. $\mathcal{P}_{tf,0}^N \neq \emptyset$.

Proof: Consider the following graph of $\hat{\psi}$. Here, we have set a which will appear below equal to $\pi/3$. Any $0 < a \leq \pi/3$ will work, but for the remainder of this example we use a and $\frac{\pi}{3}$ interchangeably.



Observe that $\text{supp}(\hat{\psi})$ consists of intervals of the form $[2a - 4\pi, 4a - 4\pi]$, $[a - 2\pi, 2a - 2\pi]$, $[a - \pi, a/4 - \pi/2]$, $[a, 2a]$ and $[2a, 4a]$. It is easy to see that (37) is satisfied.

In order to check (38), observe that if $2^j\xi$ is in $\text{supp}(\hat{\psi})$ for $j \geq 2$, then $\hat{\psi}(2^j(\xi + 2q\pi)) = 0$. Hence, if $\xi \in \text{supp}(\hat{\psi})$, then $[2a - 4\pi, 4a - 4\pi) \cup [2a, 4a)$ is resolved trivially, while the other possibilities are resolved in an analogous way. For example, $\xi \in [a - 2\pi, 2a - 2\pi)$ leads to

$$t_q(\xi) = \hat{\psi}(\xi)\hat{\psi}(\xi + 2\pi) + \hat{\psi}(2\xi)\hat{\psi}(2\xi + 4\pi) = \frac{1}{\sqrt{2}}\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = 0.$$

Hence, $\psi \in \mathcal{P}$.

It is easy to check directly that for $\xi \in [-\pi, \pi)$ we obtain

$$p_\psi(\xi) = \begin{cases} 1 & \xi \in [-\pi, -\frac{5\pi}{12}) \cup [\frac{\pi}{3}, \pi) \\ 0 & \xi \in [-\frac{5\pi}{12}, \frac{\pi}{3}). \end{cases}$$

Hence, ψ is semi-orthogonal and $|U_\psi^c| > 0$.

If $\xi \in [\frac{\pi}{6}, \frac{\pi}{3})$, then $2\xi \in [\frac{\pi}{3}, \frac{2\pi}{3})$ and $4\xi \in [\frac{2\pi}{3}, \frac{4\pi}{3})$. Therefore, we obtain

$$\begin{aligned} \Psi_1(\xi) &= (\dots, 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, \dots) \\ \Psi_2(\xi) &= (\dots, 0, 0, \frac{1}{\sqrt{2}}, 0, \dots). \end{aligned}$$

This implies that $\dim_\psi(\xi) \geq 2$ for $\xi \in [\frac{\pi}{6}, \frac{\pi}{3})$. Therefore, $\psi \notin \mathcal{P}^{\text{MRA}}$, and $\psi \in \mathcal{P}_{tf,0}^{\text{N}}$. \square

3.6. Orthonormal wavelets. The next natural class of interest is

$$\mathcal{P}_{tf,+} := \{\psi \in \mathcal{P} : \psi \text{ is a } W_0\text{-Parseval frame and } \tilde{\mathcal{I}}_\psi \text{ is injective}\}. \quad (89)$$

For this class, we have

$$p_\psi \equiv 1, \quad (90)$$

so $\tilde{\mathcal{I}}_\psi$ is the identity operator and ψ_{0k} is the orthonormal basis for W_0 . Hence,

$$\psi \in \mathcal{P}_{tf,+} \iff \psi \text{ is an orthonormal wavelet.} \quad (91)$$

Observe also that $\psi \in \mathcal{P}_{tf,+}^{\text{MRA}}$ if and only if ψ is an MRA orthonormal wavelet in the standard sense, as shown in, for example, [12]. Various examples of orthonormal wavelets are well known, and we know that both classes $\mathcal{P}_{tf,+}^{\text{MRA}}$ and $\mathcal{P}_{tf,+}^{\text{N}}$ are very rich.

For the sake of completeness, we recall various characterizations of orthonormal wavelets within \mathcal{P} , see [13], for example.

Proposition 3.21. *Suppose $\psi \in \mathcal{P}$. Then, the following are equivalent.*

- (i) ψ is an orthonormal wavelet,
- (ii) $\|\psi\|_2 = 1$,
- (iii) $p_\psi \equiv 1$,
- (iv) ψ is a semi-orthogonal W_0 -Riesz basis,
- (v) ψ is a W_0 -Parseval frame and $\tilde{\mathcal{I}}_\psi$ is injective.

Furthermore, for $\psi \in \mathcal{P}$, the following are equivalent:

- (i) ψ is an MRA orthonormal wavelet,
- (ii) $D_\psi \equiv 1$.

Remark 3.22. (i) Observe that, for $\psi \in \mathcal{P}$, $D_\psi \equiv 1$ already implies that ψ is an orthonormal wavelet, so the characterization of PFW's that are MRA orthonormal wavelets is the same as the characterization of orthonormal wavelets that come from an MRA.

- (ii) It is not possible to add a third equivalence of $\dim_\psi \equiv 1$ to the characterization of MRA orthonormal wavelets given above. See Section 4 for details.

3.7. MSF Parseval frame wavelets. An important class for the theory of PFW's is the one consisting of MSF PFW's. These are PFW's such that $|\hat{\psi}(\xi)| \in \{0, 1\}$. By [12, Corollary 3.5], every MSF PFW is semi-orthogonal. If ψ is an MSF PFW and we denote by K the set where $|\hat{\psi}(\xi)| = 1$, then

$$p_\psi = \mathbf{1}_{\tau(K)}. \quad (92)$$

Obviously, MSF PFW's can be either within the class of orthonormal wavelets or outside of it. Clearly, an MSF PFW ψ is an orthonormal wavelet if and only if

$$\tau(K) = [-\pi, \pi]. \quad (93)$$

The examples of various kinds of orthonormal wavelets which are or are not MSF are by this time well known. Hence, let us see what happens outside of orthonormal wavelets. If ψ is an MSF PFW and is not an orthonormal wavelet, then

$$\psi \in \mathcal{P}_{tf,0}^N \cup \mathcal{P}_{tf,0}^{\text{MRA}} = \mathcal{P}_{tf,0}.$$

We shall see what happens within $\mathcal{P}_{tf,0}^{\text{MRA}}$ in Section 4. Let us comment on $\mathcal{P}_{tf,0}^N$. Observe that the class of elements in $\mathcal{P}_{tf,0}^N$ which are not MSF is non-empty, since the PFW from Example 3.20 is not an MSF PFW.

Example 3.23. $\text{MSF} \cap \mathcal{P}_{tf,0}^N \neq \emptyset$.

Proof: Let $K = [-\frac{\pi}{2}, -\frac{\pi}{4}) \cup [\frac{\pi}{8}, \frac{7\pi}{32}) \cup [\frac{15\pi}{64}, \frac{\pi}{4}) \cup [7\pi, \frac{15\pi}{2})$. One checks that (37) holds, and that $\tau|_{\text{supp}(\hat{\psi})}$ is injective, but not onto. It follows that (38) holds, so ψ is an MSF PFW and ψ is not an orthonormal wavelet. It remains to show that ψ is not an MRA PFW. Since ψ is MSF, we have $D_\psi = \dim_\psi$. Consider $\xi \in [-\frac{\pi}{4}, -\frac{\pi}{8})$. In this case, $2\xi \in [-\frac{\pi}{2}, -\frac{\pi}{8})$ and $4(\xi + 2\pi) \in [7\pi, \frac{15\pi}{2})$, so $D_\psi(\xi) \geq 2$, so ψ is not MRA. \square

4. SEMIORTHOGONALIZATION

Results in this section are very much related to the ideas presented in Remark 3.16, but here we work on MRA PFW's. There are at least two strong facts that help us within \mathcal{P}^{MRA} . The first is that for $\psi \in \mathcal{P}^{\text{MRA}}$, we have $\dim_\psi \leq 1$, and therefore

$$\psi \notin V_0(\psi),$$

and $V_0(\psi)$ is the core space of a GMRA with respect to the dilation D . Second, within \mathcal{P}^{MRA} we can work on filters, which is very pleasing if we have in mind the construction from [12] that enables us to construct MRA PFW's from generalized low pass filters. Recall that this construction builds, from a given generalized low pass filter m , the associated pseudo-scaling function, which we denote φ_m , and the associated MRA PFW, which we denote ψ_m . Furthermore, every $\psi \in \mathcal{P}^{\text{MRA}}$ can be constructed in this way. One needs to observe that, given m , its ψ_m is uniquely determined (as well as φ_m), but given ψ we can have several filters which are going to provide the same ψ .

The idea is now to modify the filter m in a minimal way, so as to obtain the new filter which corresponds to the semi-orthogonal MRA PFW. We define a map ζ ,

which we call the *semiorthogonalization map*, from the set of generalized low-pass filters into the set of generalized low-pass filters, by

$$\zeta(m)(\xi) := \begin{cases} \sqrt{\frac{D_{\psi_m}(\xi)}{D_{\psi_m}(2\xi)}} m(\xi) & D_{\psi_m}(2\xi) \neq 0 \\ m(\xi) & D_{\psi_m}(2\xi) = 0, \end{cases} \quad (94)$$

where m is the generalized low-pass filter. This semiorthogonalization procedure is similar in spirit to one outlined in the proof of Theorem 3.3 in [6], but here we get explicit formulas rather than expressing the procedure in terms of projections.

Theorem 4.1. *Suppose m is a generalized low-pass filter. Then $\zeta(m)$ is a generalized low-pass filter such that $\psi_{\zeta(m)}$ is a semi-orthogonal MRA PFW and*

$$V_0(\psi_{\zeta(m)}) = V_0(\psi_m); \quad (95)$$

in particular, $\dim_{\psi_{\zeta(m)}} = \dim_{\psi_m}$. Furthermore, there are direct formulas for $\varphi_{\zeta(m)}$ and $\psi_{\zeta(m)}$, i.e.,

$$\hat{\varphi}_{\zeta(m)} = \begin{cases} \frac{1}{\sqrt{D_{\psi_m}(\xi)}} \hat{\varphi}_m(\xi) & D_{\psi_m}(\xi) \neq 0 \\ 0 & D_{\psi_m}(\xi) = 0, \end{cases} \quad (96)$$

$$\hat{\psi}_{\zeta(m)}(\xi) = \begin{cases} \sqrt{\frac{D_{\psi_m}(\xi/2+\pi)}{D_{\psi_m}(\xi)D_{\psi_m}(\xi/2)}} \hat{\psi}_m(\xi) & D_{\psi_m}(\xi/2) \cdot D_{\psi_m}(\xi) \neq 0 \\ \frac{1}{\sqrt{D_{\psi_m}(\xi)}} \hat{\psi}_m(\xi) & D_{\psi_m}(\xi/2) \neq 0 \text{ and } D_{\psi_m}(\xi) = 0 \\ 0 & \text{otherwise.} \end{cases} \quad (97)$$

Proof. Let us first show that m is a generalized filter. Observe that $D_{\psi_m}(2(\xi + \pi)) = D_{\psi_m}(2\xi)$. Hence, if $D_{\psi_m}(2\xi) \neq 0$, then

$$\begin{aligned} |\zeta(m)(\xi)|^2 + |\zeta(m)(\xi + \pi)|^2 &= \frac{D_{\psi_m}(\xi)}{D_{\psi_m}(2\xi)} |m(\xi)|^2 + \frac{D_{\psi_m}(\xi + \pi)}{D_{\psi_m}(2(\xi + \pi))} |m(\xi + \pi)|^2 \\ &= \frac{1}{D_{\psi_m}(2\xi)} (|m(\xi)|^2 D_{\psi_m}(\xi) + |m(\xi + \pi)|^2 D_{\psi_m}(\xi + \pi)) \\ &= \frac{D_{\psi_m}(2\xi)}{D_{\psi_m}(2\xi)} = 1, \end{aligned}$$

where in the second to last equality, we have used Lemma 2.8(i). If $D_{\psi_m}(2\xi) = D_{\psi_m}(2(\xi + \pi)) = 0$, then

$$|\zeta(m)(\xi)|^2 + |\zeta(m)(\xi + \pi)|^2 = |m(\xi)|^2 + |m(\xi + \pi)|^2 = 1.$$

The fact that $\zeta(m)$ is a generalized filter immediately shows that there is a corresponding pseudo-scaling function $\varphi_{\zeta(m)}$, as given by the multiplier construction from [12]. Hence, in order to prove that $\zeta(m)$ is a generalized low-pass filter, we need to prove that for a.e. $\xi \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} |\hat{\varphi}_{\zeta(m)}(2^{-n}\xi)|^2 = 1. \quad (98)$$

We also know from [11] that this limit is either 0 or 1, and it is 0 if and only if all the members of the sequence are 0. Observe that by (54) applied on D_{ψ_m} , we get that there exists $n_0 = n_0(\xi) \in \mathbb{N}$ such that for every $n \geq n_0$, $n \in \mathbb{N}$, we have

$$|\zeta(m)(2^{-n}\xi)|^2 = \frac{D_{\psi_m}(2^{-n}\xi)}{D_{\psi_m}(2^{-n+1}\xi)} |m(2^{-n}\xi)|^2.$$

It follows that, for every $n \geq n_0$, we have

$$\begin{aligned} |\hat{\varphi}_{\zeta(m)}(2^{-n}\xi)|^2 &= \frac{1}{D_{\psi_m}(2^{-n}\xi)} \cdot \lim_{k \rightarrow \infty} D_{\psi_m}(2^{-n-k}\xi) \cdot \prod_{j=1}^k |m(2^{-n-j}\xi)|^2 \\ &= \frac{1}{D_{\psi_m}(2^{-n}\xi)} |\hat{\varphi}_m(2^{-n}\xi)|^2 \geq |\hat{\varphi}_m(2^{-n}\xi)|^2, \end{aligned}$$

where the second equality is from (54).

Since $\lim_{n \rightarrow \infty} |\hat{\varphi}_m(2^{-n}\xi)|^2 = 1$, by the assumption that m is a generalized low-pass filter, it follows that $\zeta(m)$ is a generalized low-pass filter, too. The consequence is that $\psi_{\zeta(m)} \in \mathcal{P}^{\text{MRA}}$.

In order to check that $\varphi_{\zeta(m)}$ satisfies (96), it is enough to check that the function given by (96) is the pseudo-scaling function of $\zeta(m)$; recall the multiplier approach from [12] and the fact that $\varphi_{\zeta(m)}$ satisfies (98). Indeed, if $D_{\psi_m}(2\xi) \neq 0$, then by (94) and (96), we get

$$\frac{1}{\sqrt{D_{\psi_m}(2\xi)}} \hat{\varphi}_m(2\xi) = \frac{1}{\sqrt{D_{\psi_m}(2\xi)}} m(\xi) \hat{\varphi}_m(\xi) = \frac{1}{\sqrt{D_{\psi_m}(2\xi)}} m(\xi) \sqrt{D_{\psi_m}(\xi)} h(\xi),$$

where h is the function defined by (96) (observe that $\sqrt{D_{\psi_m}(\xi)} h(\xi) = \hat{\varphi}_m(\xi)$ irrespective of $D_{\psi_m}(\xi)$ being zero or not, by (52)). Hence, $h(2\xi) = \zeta(m)(\xi) h(\xi)$ if $D_{\psi_m}(2\xi) \neq 0$. If $D_{\psi_m}(2\xi) = 0$, then $h(2\xi) = 0$ and $\zeta(m)(\xi) = m(\xi)$. Furthermore, by (52), we also have $\hat{\varphi}_m(2\xi) = 0$, which implies that either $m(\xi) = 0$ or $\hat{\varphi}_m(\xi) = 0$. In any case, we get $\zeta(m)(\xi) h(\xi) = 0$. This proves (96), which by Proposition 2.6 and (8), shows that

$$V_0(\psi_{\zeta(m)}) = \langle \varphi_{\zeta(m)} \rangle = \langle \varphi_m \rangle = V_0(\psi_m).$$

Obviously, then $\dim_{\psi_{\zeta(m)}} = \dim_{\psi_m}$. Moreover, (8) and (9) applied on (96) show that

$$p_{\varphi_{\zeta(m)}} = \mathbf{1}_{Z_{V_0(\psi_m)}^c} = \mathbf{1}_{Z_{V_0(\psi_{\zeta(m)})}^c},$$

which by (52) and Corollary 3.17 proves that $\psi_{\zeta(m)}$ is semi-orthogonal. Finally, we leave it to the reader to check that (97) follows from (46), (94) and (96). \square

Let us clarify some aspects of the semiorthogonalization.

- Remark 4.2.**
- (i) It is easy to see that if m is a generalized low-pass filter such that ψ_m is semi-orthogonal, then $\zeta(m) = m$, $\varphi_{\zeta(m)} = \varphi_m$ and $\psi_{\zeta(m)} = \psi_m$.
 - (ii) It is possible to consider (97) directly as the definition of the semiorthogonalization given on MRA PFW's, and we obtain the same results. The reader can check the details. Whichever way one considers it, the semiorthogonalization always maps MRA PFW's into semi-orthogonal MRA PFW's.
 - (iii) One way to be tempted by (ii) is to extend (97) to arbitrary PFW's. However, this does not work, as one can check on various examples. Hence, to extend semiorthogonalization, one needs to use ideas from Remark 3.16. The problem is that they work only if $\psi \notin V_0(\psi)$, and we do not know when this is the case.
 - (iv) One consequence of $\dim_{\psi_{\zeta(m)}} = \dim_{\psi_m}$ is that the sets of zeroes of $D_{\psi_{\zeta(m)}}$ and D_{ψ_m} are equal. Observe that (96) gives us even more, i.e.,

$$\text{supp}(\hat{\varphi}_{\zeta(m)}) = \text{supp}(\hat{\varphi}_m). \quad (99)$$

One also easily obtains from (97) that

$$\text{supp}(p_{\psi_{\zeta(m)}}) = \text{supp}(p_{\psi_m}). \quad (100)$$

Since $\dim_{\psi_{\zeta(m)}} = D_{\psi_{\zeta(m)}}$, and (26) is valid, we obtain that for every $\psi \in \mathcal{P}^{\text{MRA}}$,

$$\dim_{\psi}(\xi) + \dim_{\psi}(\xi + \pi) = \dim_{\psi}(2\xi) + \mathbf{1}_{U_{\psi}}(2\xi), \quad (101)$$

for a.e. $\xi \in \mathbb{R}$.

- (v) Sometimes it may not be trivial to see what we get by the semiorthogonalization procedure, despite having explicit formulas (94), (96) and (97). For example, apply them to the filter $m(\xi) = \frac{1}{2}(1 + e^{3i\xi})$, and you may not immediately see the result. E. Hernandez and F. Soria proved in unpublished notes that for every nonnegative integer n , the semiorthogonalization procedure applied to the filter $m_n(\xi) := \frac{1}{2}(1 + e^{(2n+1)i\xi})$ gives that $\psi_{\zeta(m_n)}$ is the Haar wavelet. Recall that ψ_{m_1} is an example of a W_0 -Riesz basis MRA PFW which is not semi-orthogonal.

Remark 4.3. We think that the semiorthogonalization procedure, even considered only on \mathcal{P}^{MRA} , raises an important issue with respect to the GMRA approach to the analysis of \mathcal{P} . More precisely, one can (as several authors do) say that a $\psi \in \mathcal{P}$ is associated with a GMRA if $V_0(\psi)$ is the core space for the GMRA. This notion can be somewhat misleading. Namely, if a semi-orthogonal PFW is associated with its GMRA in this way, everything is fine in the sense that all the crucial information about the PFW is given in the GMRA. However, things are different when we go outside of semi-orthogonal PFW's. If $\psi \in \mathcal{P}^{\text{MRA}}$, then it is always associated with a GMRA (in the above sense). What are all the possible GMRA's that we can get from \mathcal{P}^{MRA} ? By Theorem 4.1, these are exactly those that we can get from semi-orthogonal MRA PFW's, i.e., the ones studied by J. J. Benedetto and S. Li [2] and by J. J. Benedetto and O. M. Treiber [3]. But, the associated GMRA's are not going to tell us anything about some important features of PFW's, as soon as it is not semi-orthogonal. Take the filters $m_0(\xi) = \frac{1}{2}(1 + e^{i\xi})$ and $m_1(\xi) = \frac{1}{2}(1 + e^{3i\xi})$ from Remark 4.2 (v). They will both generate exactly the same GMRA, but one of them, ψ_{m_0} is the orthonormal Haar wavelet, while the other, ψ_{m_1} , is a PFW which is a W_0 -Riesz basis but is not even semi-orthogonal.

The semiorthogonalization procedure points to yet another interesting class of MRA PFW's. Through the semiorthogonalization procedure, an MRA PFW will end either in $\mathcal{P}_{tf,0}^{\text{MRA}}$ or in $\mathcal{P}_{tf,+}^{\text{MRA}}$ (the orthonormal MRA wavelets). Clearly, from Theorem 4.1, MRA PFW's which are going to end in $\mathcal{P}_{tf,+}^{\text{MRA}}$ are precisely the elements of the class

$$\mathcal{P}_{-,1}^{\text{MRA}} := \{\psi \in \mathcal{P} : \dim_{\psi} = 1\}. \quad (102)$$

Remark 4.4. Other authors have encountered this class in related situations. In particular, in [6], the authors consider Parseval frame wavelets (in higher dimensions with an expansive dilation) for which the space $V_0(\psi)$ is the core space of a GMRA. Then, they ask which ones come from a classical MRA, and they prove that these are precisely the ones for which $D_{\psi} > 0$ a.e. Observe that in the case of dyadic, one-dimensional wavelets, this is exactly our class $\mathcal{P}_{-,1}^{\text{MRA}}$. Indeed, for $\psi \in \mathcal{P}^{\text{MRA}}$, this is obvious, and for $\psi \in \mathcal{P}^{\text{N}}$ and such that it is associated with a GMRA, we will have $\psi \notin V_0(\psi)$, so we can apply Remark 3.16, and then using the fact that for semi-orthogonal elements in \mathcal{P}^{N} , $|Z_{V_0(\psi)}| > 0$, conclude that $|\{\xi : D_{\psi}(\xi) = 0\}| > 0$. Hence, the class $\mathcal{P}_{-,1}^{\text{MRA}}$ is clearly of importance, and we shall analyze it further.

Using Corollary 3.7 from [6] and the previous Remark 4.4, we obtain the following consequence.

Corollary 4.5. *If $\psi \in \mathcal{P}$ is compactly supported and $\dim_{V_0(\psi)}$ is not identically infinity, then $\psi \in \mathcal{P}_{-,1}^{MRA}$.*

The following result is both useful and of independent interest.

Lemma 4.6. *Suppose $\psi \in \mathcal{P}^{MRA}$. If $|U_\psi^c| > 0$, then $|Z_{V_0(\psi)}| > 0$.*

Proof. By Lemma 2.8 (iii), whenever $p_\psi(2\xi) = 0$, then either $D_\psi(\xi) = 0$ or $D_\psi(\xi + \pi) = 0$. The result now follows easily. \square

Proposition 4.7. *The following holds:*

$$\mathcal{P}_{if,+}^{MRA} \subset \mathcal{P}_{-,1}^{MRA} \subset \mathcal{P}_{if,+}^{MRA} \cup \mathcal{P}_{f,+}^{MRA} \cup \mathcal{P}_{0,+}^{MRA}.$$

Proof. The first inclusion is obvious, while for the second inclusion, we apply Lemma 4.6 together with the obvious fact that for $\psi \in \mathcal{P}_{-,1}^{MRA}$, we must have $|Z_{V_0(\psi)}| = 0$. Indeed, if $\psi \in \mathcal{P}_{if,0}^{MRA} \cup \mathcal{P}_{f,0}^{MRA} \cup \mathcal{P}_{0,0}^{MRA}$, then $|U_\psi^c| > 0$, that is $\psi \notin \mathcal{P}_{-,1}^{MRA}$. \square

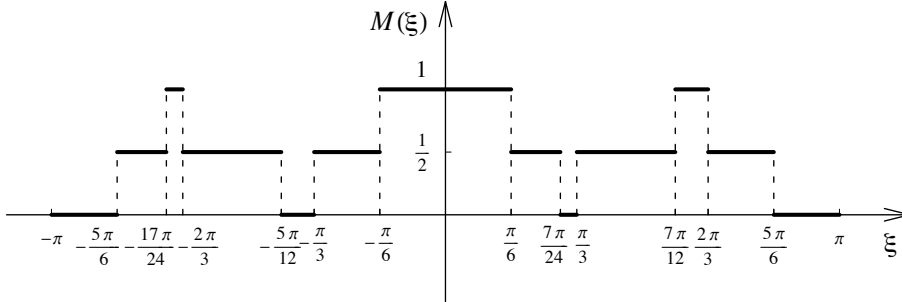
We have

$$\mathcal{P}_{-,1}^{MRA} \cap \mathcal{P}_{f,+}^{MRA} \neq \emptyset, \quad (103)$$

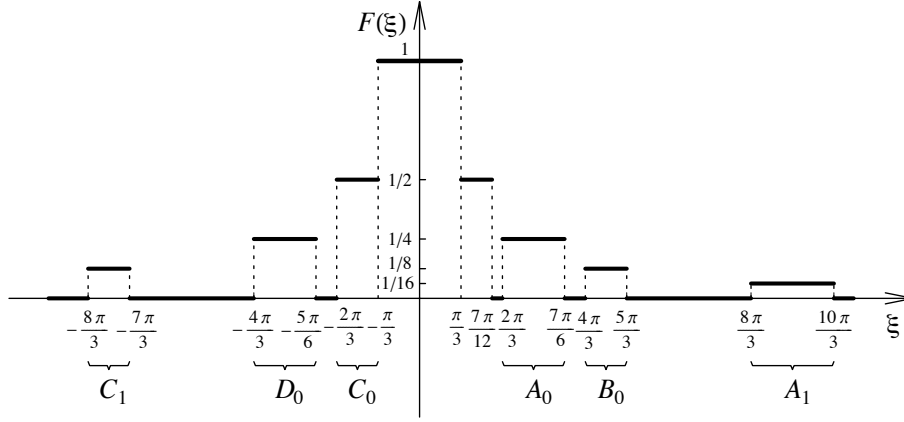
since the MRA PFW generated by the filter $m(\xi) = \frac{1}{2}(1 + e^{3i\xi})$ is in both classes. One would expect that the semiorthogonalization of a W_0 -Riesz basis is always going to produce an orthonormal wavelet, i.e. that $\mathcal{P}_{f,+}^{MRA} \subset \mathcal{P}_{-,1}^{MRA}$. However, this is not true, as the following example shows. Observe that it also shows that the converse of Lemma 4.6 is not true.

Example 4.8. $\mathcal{P}_{f,+}^{MRA} \not\subset \mathcal{P}_{-,1}^{MRA}$.

Proof. Our ψ is an MRA PFW which is generated by the filter m , whose function $M(\xi) = |m(\xi)|^2$ has the following graph (clearly, such an m exists and it is a generalized low-pass filter).



In order to analyze the key properties of this ψ , it is enough to look into the function $F(\xi) = |\hat{\varphi}_{|m|}(\xi)|^2 = \prod_{j=1}^{\infty} M(2^{-j}\xi)$. We claim that the graph of F is as follows.



We divide $\text{supp}(F) \setminus [-\frac{\pi}{3}, \frac{7\pi}{12})$ into four classes of intervals, two on a positive side and two on the negative side. Consider the positive side first. We define, for $j \geq 0$,

$$A_j := \text{supp}(F) \cap 4^j \left[\frac{2\pi}{3}, \frac{4\pi}{3} \right)$$

$$B_j := \text{supp}(F) \cap 4^j \left[\frac{4\pi}{3}, \frac{8\pi}{3} \right).$$

We can prove by induction that

$$A_j = \left[\frac{4^j 2\pi}{3}, \frac{(4^{j+1} + 3)\pi}{6} \right), \tau(A_j) = \tau(A_0)$$

$$\tau(A_0) = \left[\frac{2\pi}{3}, \pi \right) \cup \left[-\pi, -\frac{5\pi}{6} \right), F|_{A_j} = \left(\frac{1}{4} \right)^{j+1}$$

$$B_j = \left[\frac{4^j 4\pi}{3}, \frac{(4^{j+1} + 1)\pi}{3} \right), \tau(B_j) = \tau(B_0)$$

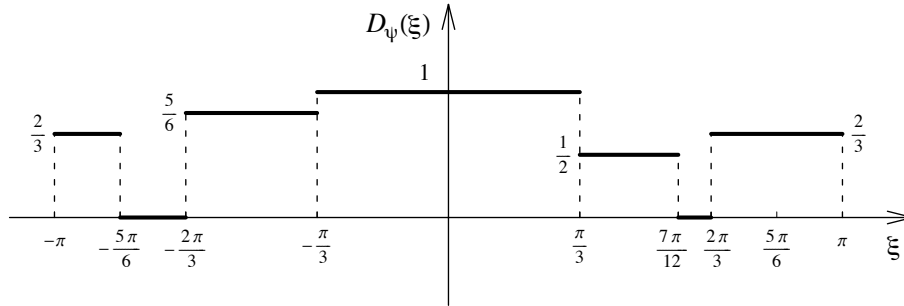
$$\tau(B_0) = \left[-\frac{2\pi}{3}, -\frac{\pi}{3} \right), F|_{B_j} = \frac{1}{8} \left(\frac{1}{4} \right)^j.$$

Similarly, we get two classes on the negative side, i.e. for $j \geq 0$,

$$C_j = \left[-\frac{4^j 2\pi}{3}, -\frac{(4^j \cdot 2 - 1)\pi}{3} \right), \tau(C_j) = \left[-\frac{2\pi}{3}, -\frac{\pi}{3} \right)$$

$$D_j = \left[-\frac{4^j 4\pi}{3}, -\frac{(4^{j+1} \cdot 2 - 3)\pi}{6} \right), \tau(D_j) = \left[\frac{2\pi}{3}, \pi \right) \cup \left[-\pi, -\frac{5\pi}{6} \right).$$

Observe that $D_\psi(\xi) = \sum_{k \in \mathbb{Z}} F(\xi + 2k\pi)$, i.e. we get the graph of D_ψ :



Since $p_\psi(\xi) = D_\psi(\xi/2) + D_\psi(\xi/2 + \pi) - D_\psi(\xi)$, it is easy to check that $p_\psi(\xi) \geq 1/2$ a.e. Hence, ψ is an MRA PFW, it is a W_0 -Riesz basis, but $|Z_{V_0(\psi)}| = |\{\xi : D_\psi(\xi) = 0\}| > 0$. In other words, $\psi \in \mathcal{P}_{f,+}^{\text{MRA}} \setminus \mathcal{P}_{-,1}^{\text{MRA}}$.

Observe also that the semiorthogonalization of ψ provides us with an example in $\mathcal{P}_{tf,0}^{\text{MRA}}$ which is not MSF. \square

In order to complete the picture about MSF PFW's, let us mention that it is indeed not difficult to construct MRA PFW's which are MSF, but are not orthonormal. Let us also mention that if the generalized low-pass filter m has the property that $|m(\xi)|$ attains only values 0 and 1, then the associated ψ must be an MSF.

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