

## DYADIC PFW'S AND $W_0$ -BASES

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The rich structure of the set of dyadic PFW's has been presented in [6]. As mentioned in [6, Remark 3.7], the subclass of the so called non- $W_0$ -frames with various levels of  $W_0$ -linear independence (denoted by  $\mathcal{P}_{0,+}$  in [6]) needs extra attention. The recent observation in [4] on  $W_0$ -Schauder bases enables us to properly restructure this subclass ( $\mathcal{P}_{0,+}$ ) and to show that it contains a wealth of examples. This is the purpose of our article. Let us, however, begin by defining some of the notions mentioned above.

We denote by  $\mathcal{P}$  the set of all *Parseval frame wavelets* (PFW's, for short), i.e., the set of functions  $\psi \in L^2(\mathbb{R})$  such that the system

$$\{\psi_{jk}(x)\} := \{2^{j/2}\psi(2^j x - k) : j, k \in \mathbb{Z}\}$$

forms a normalized (frame bounds 1) tight frame for  $L^2(\mathbb{R})$ . This is equivalent to having the reproducing property, i.e., for every  $f \in L^2(\mathbb{R})$

$$f = \sum_{j,k \in \mathbb{Z}} \langle f, \psi_{jk} \rangle \psi_{jk} \quad (1)$$

unconditionally in  $L^2(\mathbb{R})$ . Interestingly enough, the property of being a PFW (on the entire  $L^2(\mathbb{R})$ ) still allows a very varied behaviour on the main (or zeroeth) resolution level

$$W_0 := \overline{\text{span}}\{\psi_{0k} : k \in \mathbb{Z}\}. \quad (2)$$

If the family  $\{\psi_{0k} : k \in \mathbb{Z}\}$  satisfies the property “ $abc$ ” within  $W_0$ , we shall say that  $\psi$  is a “ $W_0 - abc$ ” (for example, if it is a Riesz basis within  $W_0$ , we say that  $\psi$  is a  $W_0$ -Riesz basis). A very useful tool to study such properties is the periodization function  $p_\psi : \mathbb{R} \rightarrow [0, +\infty)$ , defined by

$$p_\psi(\xi) := \sum_{k \in \mathbb{Z}} |\hat{\psi}(\xi + 2k\pi)|^2 \quad , \quad \xi \in \mathbb{R} ,$$

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where the Fourier transform is chosen so that

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x} dx, \text{ for } f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}).$$

Recall that the Hilbert space  $W_0 \subset L^2(\mathbb{R})$  is isometrically isomorphic to the space  $L^2(\mathbb{T}; p_\psi)$  (the  $L^2$  space on the torus  $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$  with the measure  $p_\psi(\xi) \frac{d\xi}{2\pi}$ ; observe that  $p_\psi$  is  $2\pi$ -periodic) via isomorphism

$$\mathcal{I}_\psi : L^2(\mathbb{T}; p_\psi) \rightarrow W_0 ,$$

given by,

$$\mathcal{I}_\psi(t) := (t\hat{\psi})^\vee . \quad (3)$$

If  $\psi \in \mathcal{P}$ , then (see [5])

$$p_\psi \leq 1 \quad \text{a.e.} \quad (4)$$

Let us consider also an operator

$$\tilde{\mathcal{I}}_\psi : L^2(\mathbb{T}; \frac{d\xi}{2\pi}) \rightarrow L^2(\mathbb{T}; p_\psi) ,$$

given by  $\tilde{\mathcal{I}}_\psi(t) = t$ . It follows by (4) that  $\tilde{\mathcal{I}}_\psi$  is a bounded linear operator.

Let us now introduce the main subclasses of  $\mathcal{P}$  which are going to be of interest for us here (for a complete picture see [6]). In the following we shall assume that the reader is familiar with the notions of frame, normalized tight frame, biorthogonal sequence, Schauder basis, Riesz basis and orthonormal basis (see [2] and [7] for basic results and some further references).

It is shown in [6] that the class  $\mathcal{P}_{\cdot,+}$ , defined by

$$\mathcal{P}_{\cdot,+} := \left\{ \psi \in \mathcal{P} : \ker(\tilde{\mathcal{I}}_\psi) = \{0\} \right\} , \quad (5)$$

is the disjoint union

$$\mathcal{P}_{\cdot,+} = \mathcal{P}_{0,+} \cup \mathcal{P}_{f,+} \cup \mathcal{P}_{tf,+} , \quad (6)$$

where

$$\begin{aligned} \mathcal{P}_{0,+} &= \{ \psi \in \mathcal{P}_{\cdot,+} : \psi \text{ is not a } W_0\text{-frame} \} \\ \mathcal{P}_{f,+} &= \{ \psi \in \mathcal{P}_{\cdot,+} \setminus \mathcal{P}_{0,+} : \psi \text{ is not a } W_0\text{-normalized tight frame} \} \\ \mathcal{P}_{tf,+} &= \{ \psi \in \mathcal{P}_{\cdot,+} : \psi \text{ is a } W_0\text{-normalized tight frame} \} \end{aligned}$$

Recall that  $\mathcal{P}_{tf,+}$  is actually the set of orthonormal wavelets, while  $\mathcal{P}_{f,+}$  is the set of non-semiorthogonal  $W_0$ -Riesz basis. However, the class  $\mathcal{P}_{0,+}$  has been touched only briefly in [6], where it was shown that  $\mathcal{P}_{0,+}$  is non-empty. Following the most recent advances in [4], we are able to show that  $\mathcal{P}_{0,+}$  has a much more rich structure with various

levels of linear independence (or basis) type conditions which naturally fit with other subclasses of  $\mathcal{P}_{\cdot,+}$ .

**Remark.** Parseval frames are orthogonal projections of orthonormal bases, and as such, are not necessarily linearly independent. However, in the special case of PFW's, there is more structure. First of all, for every  $\psi \in \mathcal{P}$  the family  $\{\psi_{0k} : k \in \mathbb{Z}\}$  is linearly independent. Suppose contrary that the family  $\{e^{i\xi k} : k \in \mathbb{Z}\}$  (here we are using (3)) is linearly dependent in  $L^2(\mathbb{T}; p_\psi)$ . Since  $\|\psi\|_2 > 0$ , we have that the Lebesgue measure  $|\{\xi \in \mathbb{T} : p_\psi(\xi) > 0\}| > 0$ . Hence, without loss of generality, we would have that a constant function 1 is a.e. on a set of positive Lebesgue measure equal to a finite linear combination of non-constant exponentials  $\{e^{i\xi k}\}$ , which is not possible. Additionally, every frame (even Bessel sequence) can be partitioned into linearly independent sets [?]. Combining these two results, it is natural to conjecture that PFW's must already be linearly independent. This is also true, and will be appear in future work. ■

We can not expect more than linear independence of  $\{\psi_{0k} : k \in \mathbb{Z}\}$  outside of  $\mathcal{P}_{\cdot,+}$ . However, inside  $\mathcal{P}_{\cdot,+}$  we have a rich hierarchy of subclasses. Let us remind the reader (see [7] for more details) that a sequence  $\{x_n : n \in \mathbb{N}\}$  in a Banach space is said to be  $\ell^2$ -linearly independent if

$$(\{\alpha_n\} \in \ell^2, \sum_{n=1}^{\infty} \alpha_n x_n = 0 \Rightarrow \alpha_n = 0, \forall n \in \mathbb{N}) . \quad (7)$$

**Theorem 1.** *Suppose that  $\psi \in \mathcal{P}$ . Then, the following are equivalent:*

- (a)  $\psi \in \mathcal{P}_{\cdot,+}$  ;
- (b)  $p_\psi > 0$  a.e. ;
- (c)  $\{\psi_{0k} : k \in \mathbb{Z}\}$  is  $\ell^2$ -linearly independent in  $W_0$ .

Observe that in (7), the tacit assumption is that  $\lim_{N \rightarrow \infty} \sum_{n=1}^N \alpha_n x_n$  exists in the Banach space, so the particular ordering of vectors matters. This will be the case in several other statements in this article. Here (Theorem 1.(c)), as throughout the article, we assume that  $\mathbb{Z}$  is ordered as

$$\{0, 1, -1, 2, -2, \dots\} . \quad (8)$$

*Proof.* The equivalence (a)  $\Leftrightarrow$  (b) is more or less obvious (and is given already in [6]). Let us prove (b)  $\Leftrightarrow$  (c).

Suppose first that  $p_\psi > 0$  a.e., i.e., that  $\ker(\tilde{\mathcal{I}}_\psi) = \{0\}$ . Let us denote by  $\{x_n : n \in \mathbb{N}\}$  the family  $\{\psi_{0k} : k \in \mathbb{Z}\}$  ordered by (8). Suppose

that  $\{\alpha_n\} \in \ell^2$  and  $\sum_{n=1}^{\infty} \alpha_n x_n = 0$ . Let us denote by  $\{e_n : n \in \mathbb{N}\}$  the family of exponentials  $\{e^{i\xi k} : k \in \mathbb{Z}\}$  ordered by (8). Since  $\tilde{\mathcal{I}}_\psi$  is a bounded linear operator, and  $\tilde{\mathcal{I}}_\psi(e_n) = x_n$ , we get  $\tilde{\mathcal{I}}_\psi(\sum_{n=1}^{\infty} \alpha_n e_n) = 0$ . Since the kernel of  $\tilde{\mathcal{I}}_\psi$  is trivial, we get  $\sum_{n=1}^{\infty} \alpha_n e_n = 0$  in  $L^2(\mathbb{T}; \frac{d\xi}{2\pi})$ . However,  $\{e_n : n \in \mathbb{N}\}$  forms an orthonormal basis in  $L^2(\mathbb{T}; \frac{d\xi}{2\pi})$ , which implies  $\alpha_n = 0, \forall n \in \mathbb{N}$ .

Suppose now that the set  $\{\xi : p_\psi(\xi) = 0\}$  has a positive Lebesgue measure. Observe that  $h := 1_{\{p_\psi=0\}}$  is a non-zero function in  $L^2(\mathbb{T}; \frac{d\xi}{2\pi})$ , but is a zero function in  $L^2(\mathbb{T}; p_\psi)$ . Hence, the Fourier coefficients of  $h$  are not trivial, i.e., there exists an  $\ell^2$ -sequence  $\{\alpha_n\}$ , whose elements are not all equal to zero, such that  $h = \sum_{n=1}^{\infty} \alpha_n e_n$  in  $L^2(\mathbb{T}; \frac{d\xi}{2\pi})$ . Since  $\tilde{\mathcal{I}}_\psi$  is a bounded linear operator, we get that

$$0 = h = \tilde{\mathcal{I}}_\psi(h) = \sum_{n=1}^{\infty} \alpha_n x_n$$

in  $L^2(\mathbb{T}; p_\psi)$ . Hence,  $\{x_n\}$  is not  $\ell^2$ -linearly independent. Q.E.D.

If we combine this result with the recent note [4], we get a nice description of the structure of  $\mathcal{P}_{+,+}$ . Let us remind the reader that a measurable,  $2\pi$ -periodic function  $w : \mathbb{R} \rightarrow (0, \infty)$  is an  $\mathcal{A}_2$ -weight (see [3] for more details) if there exists a constant  $M > 0$  such that for every interval  $I \subseteq \mathbb{R}$ ,

$$\left( \frac{1}{|I|} \int_I w(\xi) d\xi \right) \left( \frac{1}{|I|} \int_I \frac{1}{w(\xi)} d\xi \right) \leq M,$$

where  $|I|$  is the Lebesgue measure of  $I$ .

**Corollary 2.** *Let  $\psi \in \mathcal{P}$ . Then*

- (a)  $\psi \in \mathcal{P}_{+,+}$  if and only if  $\{\psi_{0k} : k \in \mathbb{Z}\}$  is  $\ell^2$ -linearly independent;
- (b)  $\psi \in \mathcal{P}_{+,+}$  if and only if  $p_\psi > 0$  a.e. ;
- (c)  $\{\psi_{0k} : k \in \mathbb{Z}\}$  belongs to a biorthogonal sequence in  $W_0$  if and only if  $\frac{1}{p_\psi} \in L^1(\mathbb{T}; \frac{d\xi}{2\pi})$
- (d)  $\psi$  is a  $W_0$ -Schauder basis if and only if  $p_\psi$  is an  $\mathcal{A}_2(\mathbb{T})$ -weight;
- (e)  $\psi$  is a  $W_0$ -Riesz basis if and only if  $\frac{1}{p_\psi}$  is bounded;
- (f)  $\psi$  is a  $W_0$ -orthonormal basis (and, therefore, an orthonormal wavelet) if and only if  $p_\psi \equiv 1$  a.e.

Furthermore, any  $\psi$  from either (b), (c), or (d), which is also a  $W_0$ -frame, belongs to (e). For any  $W_0$ -Schauder basis, which is not a  $W_0$ -frame, the Schauder basis  $\{\psi_{0k}\}$  in  $W_0$  is a conditional basis.

Following [6] we can expect that some of these classes are not easy to understand; partially for the lack of examples. It is not a priori clear that any of those subclasses is non-empty. However, as we shall see here, the approach via the function  $p_\psi$  is not just suitable for the characterization of these subclasses, but is also helpful for the construction of examples. We shall construct a continuum of examples of PFW's whose periodization function can be arbitrarily chosen on an interval, so as to satisfy any of the desired conditions from Corollary 2. Hence, every subclass is far from being empty.

We shall construct sets with the properties that  $[-\pi, \pi)$  will be divided into three disjoint sets, say  $A, B, C$  (each of them is a union of intervals). The periodization function will be chosen arbitrarily on  $A$  (and, hence, can be adjusted to satisfy any of the requirements given in Corollary 2), will be equal to 1 on  $B$  (and, hence, will not affect any of the properties from Corollary 2). On  $C$  the periodization function will be of the form  $\frac{1}{2}a + \frac{1}{2}b$ , where  $a$  and  $b$  will be bounded from below. Observe that in this fashion, we can construct examples that satisfy any condition from Corollary 2, parts (a) – (e). The only condition which is not included is Corollary 2, part (f), i.e.,  $p_\psi \equiv 1$ . But this is the case of orthonormal wavelets, which has been studied thoroughly and many examples are known.

Let us denote by  $\tau$  the translation projection on  $\mathbb{R}$ , defined by  $\tau(\xi) = \eta$ , where  $\eta \in [-\pi, \pi)$  is such that  $\xi - \eta = 2\pi k$  for some  $k \in \mathbb{Z}$ . The dilation projection  $d$  is defined on  $\mathbb{R} \setminus \{0\}$  by  $d(\xi) = \eta$ , where  $\eta$  belongs to  $[-2\pi, -\pi) \cup [\pi, 2\pi)$  and  $\eta/\xi = 2^k$  for some  $k \in \mathbb{Z}$ . We recall the following theorem, which plays a crucial role in our construction (see [1] and [8]).

**Theorem 3.** [1]. *Let  $E \subseteq [-\pi, \pi)$  and  $F \subseteq [-2\pi, -\pi) \cup [\pi, 2\pi)$  be measurable sets such that  $0 \in E^\circ$  and  $E^\circ \neq \emptyset$ . Then, there exists measurable  $G \subseteq \mathbb{R}$  such that  $\tau|_G$  and  $d|_G$  are injective functions with  $\tau(G) = E$  and  $d(G) = F$ .*

**Example 4.** Let  $I$  be a small interval around  $\frac{9\pi}{4}$  so that  $d(I) \cap d(I + \pi) = \emptyset$ ,  $\tau(I) \cap \tau(I + \pi) = \emptyset$ , and  $\tau(I) \cap \tau(2^j I + 2k\pi) = \emptyset$  for  $j = 1, 2$  and  $k = 1, 2$ . Moreover, we require  $I$  to be small enough to ensure that  $E$ , defined by  $E := [-\pi, \pi) \setminus (\tau(I) \cup \tau(I + \pi) \cup \tau(2I) \cup \tau(4I))$ , satisfies the requirements of Theorem 3. It is easy to see that such a choice of  $I$  is possible. We define  $F$  by  $F := [[-2\pi, -\pi) \cup [\pi, 2\pi)] \setminus [d(I) \cup d(I + \pi)]$ . Using Theorem 3, we obtain  $G$  such that  $\tau|_G$  and  $d|_G$  are 1 – 1 and  $\tau(G) = E$ ,  $d(G) = F$ .

Take any function  $f : I \rightarrow \mathbb{C}$ , such that  $0 < |f(\xi)| < 1$ , for every  $\xi \in I$ . We claim that  $\psi$  is a PFW, when defined by

$$\hat{\psi}(\xi) := \begin{cases} 1 & \xi \in G \\ f(\xi) & \xi \in I \\ f(\xi - \pi) & \xi \in I + \pi \\ (1/\sqrt{2})\sqrt{1 - |f(\xi/2)|^2} & \xi \in 2I \\ -(1/\sqrt{2})\sqrt{1 - |f((\xi - 2\pi)/2)|^2} & \xi \in 2I + 2\pi \\ (1/\sqrt{2})\sqrt{1 - |f(\xi/4)|^2} & \xi \in 4I \\ (1/\sqrt{2})\sqrt{1 - |f((\xi - 4\pi)/4)|^2} & \xi \in 4I + 4\pi \\ 0 & \text{otherwise .} \end{cases}$$

We first need to show that  $\sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \xi)|^2 = 1$  a.e., first. Clearly, it is enough to check this for  $\xi \in [-2\pi, -\pi) \cup [\pi, 2\pi)$ . This means that  $\xi$  is either in  $G$ , in  $I$  or in  $(I + \pi)$  (modulo dilations by 2). If  $\xi \in G$ , then  $\hat{\psi}(2^j \xi) = 0$ , for every  $j \in \mathbb{Z} \setminus \{0\}$ . Hence,

$$\sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \xi)|^2 = |\hat{\psi}(\xi)|^2 = 1.$$

If  $\xi \in I$ , then  $\hat{\psi}(2^j \xi) = 0$ , for every  $j \in \mathbb{Z} \setminus \{0, 1, 2\}$ . Hence

$$\begin{aligned} \sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \xi)|^2 |\hat{\psi}(\xi)|^2 + |\hat{\psi}(2\xi)|^2 + |\hat{\psi}(4\xi)|^2 = \\ |f(\xi)|^2 + \frac{1}{2}(1 - |f(\xi)|^2) + \frac{1}{2}(1 - |f(\xi)|^2) = 1. \end{aligned}$$

The case  $\xi \in (I + \pi)$  goes along the same line.

Secondly, we need to show that for every odd integer  $q$  and for almost every  $\xi$  we have

$$t_q(\xi) := \sum_{j \geq 0} \hat{\psi}(2^j \xi) \overline{\hat{\psi}(2^j(\xi + 2q\pi))} = 0.$$

Let us observe terms of the form

$$\hat{\psi}(2^j \xi) \cdot \overline{\hat{\psi}(2^j(\xi + 2q\pi))}.$$

Such a product is zero except in the following cases

$$\begin{aligned} \xi \in 2I, & & j = 0, q = 1 \\ \xi \in 2I + 2\pi, & & j = 0, q = -1 \\ \xi \in 2I, & & j = 1, q = 1 \\ \xi \in 2I + 2\pi, & & j = 1, q = -1. \end{aligned}$$

In the first case ( $\xi \in 2I, j = 0, q = 1$ ), we get

$$\begin{aligned} t_q(\xi) &= \hat{\psi}(\xi) \overline{\hat{\psi}(\xi + 2\pi)} + \hat{\psi}(2\xi) \overline{\hat{\psi}(2\xi + 4\pi)} = \\ &= -\frac{1}{2} \sqrt{1 - |f(\xi/2)|^2} \sqrt{1 - |f(\xi/2)|^2} + \\ &\quad + \frac{1}{2} \sqrt{1 - |f(\xi/2)|^2} \sqrt{1 - |f(\xi/2)|^2} = 0. \end{aligned}$$

Other cases are checked in a similar way.

Observe that we have a PFW  $\psi$  and the function  $p_\psi$  has the following properties on  $[-\pi, \pi)$  (since  $p_\psi$  is  $2\pi$ -periodic this is all we need)

$$p_\psi(\xi) = \begin{cases} 1 & \xi \in G \\ |f(\xi)|^2 & \xi \in I \\ |f(\xi - \pi)|^2 & \xi \in I + \pi \\ \frac{1}{2} [(1 - |f(\frac{\xi}{2})|^2) + (1 - |f(\frac{\xi}{2})|^2)] & \xi \in 2I \\ \frac{1}{2} [(1 - |f(\frac{\xi}{4})|^2) + (1 - |f(\frac{\xi}{4})|^2)] & \xi \in 4I. \end{cases}$$

In particular,  $p_\psi > 0$  a.e. Hence, for each choice of  $f, f : I \rightarrow \mathbb{C}$ ,  $0 < |f(\xi)| < 1$ , we get a PFW  $\psi$  which is in  $\mathcal{P}_{\cdot,+}$ . Observe that if we require  $|f|$  to be bounded away from 1, then  $p_\psi|_{G \cup 2I \cup 4I}$  is bounded below and away from zero. Since we can adjust  $p_\psi$  freely on  $I$  (through the choice of  $f$ ) it is clear that we get a continuum of examples for any subclass described in Corollary 2. (except the subclass described in Corollary 2.(f), i.e., orthonormal wavelets).

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