# THE STABILIZED SET OF $p$'S IN KRIVINE'S THEOREM CAN BE DISCONNECTED

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*In memory of Edward Odell*

ABSTRACT. For any closed subset $F$ of $[1, \infty]$ which is either finite or consists of the elements of an increasing sequence and its limit, a reflexive Banach space $X$ with a 1-unconditional basis is constructed so that in each block subspace $Y$ of $X$, $\ell_p$ is finitely block represented in $Y$ if and only if $p \in F$. In particular, this solves the question as to whether the stabilized Krivine set for a Banach space had to be connected. We also prove that for every infinite dimensional subspace $Y$ of $X$ there is a dense subset $G$ of $F$ such that the spreading models admitted by $Y$ are exactly the $\ell_p$ for $p \in G$.

## 1. INTRODUCTION

In the past, many of the driving questions in the study of Banach spaces concerned the existence of "nice" subspaces of general infinite dimensional Banach spaces. Finding counterexamples to these questions involved developing new ideas for constructing Banach spaces. B. Tsirelson's construction of a reflexive infinite dimensional Banach space which does not contain $\ell_p$ for any $1 < p < \infty$ [T] and W.T. Gowers and B. Maurey's construction of an infinite dimensional Banach space which does not contain an unconditional basic sequence [GM] are two important examples. On the other hand, after Tsirelson's construction, J-L. Krivine proved that every basic sequence contains $\ell_p$ for some $1 \leqslant p \leqslant \infty$ finitely block represented [K] (where the case $p = \infty$ refers to $c_0$), and it is not difficult to show that every normalized weakly null sequence in a Banach space has a subsequence with a 1-suppression unconditional spreading model. Thus, though we cannot always find these properties in infinite dimensional subspaces, they are still always present in certain finite block or asymptotic structure.

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*2010 Mathematics Subject Classification:* Primary 46B03, 46B06, 46B07, 46B25, 46B45
*Key words:* Spreading models, Finite block representability, Krivine's Theorem
The first named author acknowledges support from the Lenfest Summer Grant at Washington and Lee University.
Research of the second author was supported by the National Science Foundation.
The authors would like to acknowledge the support of program ΑΡΙΣΤΕΙΑ-1082.

In his paper on Krivine's Theorem, Rosenthal proved that given any Banach space, the set of $p$'s such that $\ell_p$ is finitely block represented in the Banach space can be stabilized on a subspace [R] (for a simplified proof of the stability result see also [M, page 133]). That is, given any infinite dimensional Banach space $X$, there exists an infinite dimensional subspace $Y \subseteq X$ with a basis and a nonempty closed subset $I \subseteq [1,\infty]$ such that for every block subspace $Z$ of $Y$, $\ell_p$ is finitely block represented in $Z$ if and only if $p \in I$. Rosenthal concluded his paper by asking if this stabilized Krivine set $I$ had to be a singleton. E. Odell and Th. Schlumprecht answered this question by constructing a Banach space $X$ with an unconditional basis which had the property that every unconditional basic sequence is finitely block represented in every block sequence in $X$ [OS1]. Thus, the stabilized Krivine set for this space is the interval $[1,\infty]$. Later, Odell and Schlumprecht constructed a Banach space with a conditional basis which had the property that every monotone basic sequence is finitely block represented in every block sequence in $X$ [OS2]. At this point, the known possible stabilized Krivine sets for a Banach space are singletons and the entire interval $[1,\infty]$. P. Habala and N. Tomczak-Jaegermann proved that if $1 \leqslant p < q \leqslant \infty$ and $X$ is an infinite dimensional Banach space such that $\ell_p$ and $\ell_q$ are finitely block represented in every block subspace of $X$ then $X$ has a quotient $Z$ so that every $r \in [p,q]$ is finitely block represented in $Z$ [HT]. They then asked if the stabilized Krivine set for a Banach space is always connected [HT], which was later included as problem 12 in Odell's presentation of 15 open problems in Banach spaces at the Fields institute in 2002 [O]. We solve the stabilized Krivine set problem with the following theorem.

**Theorem.** Let $F \subseteq [1,\infty]$ be either a finite set or a set consisting of an increasing sequence and its limit. Then there exists a reflexive Banach space $X$ with an unconditional basis such that for every infinite dimensional block subspace $Y$ of $X$:

(i) For all $1 \leqslant p \leqslant \infty$, the space $\ell_p$ is finitely block represented in $Y$ if and only if $p \in F$.
(ii) If $F$ is finite then the spreading models admitted by $Y$ are exactly the spaces $\ell_p$ for $p \in F$.
(iii) If $F$ is an increasing sequence with limit $p_\omega$ then every spreading model admitted by $Y$ is isomorphic to $\ell_p$ for some $p \in F$ and for every $p \in F \setminus \{p_\omega\}$ $\ell_p$ is admitted as a spreading model by $Y$.

This theorem is somewhat surprising in that the corresponding question for finite representability instead of finite block representability is very different. Indeed, if $\ell_p$ is finitely representable in a Banach space $X$ for some $1 \leqslant p < 2$ then $\ell_r$ is finitely representable in $X$ for all $r \in [p,2]$. However, for $2 < p < \infty$ the Banach space $\ell_r$ is finitely representable in $\ell_p$ if and only if $r = 2$ or $r = p$. Thus the position in $[1,\infty]$ of the set $F$ of $p$'s that are finitely represented in a space $X$ determines whether $F$ is an interval.

Our results show that in the case of block finitely represented, the position of $F$ in the interval $[1,\infty]$ does not matter.

Theorem 1 also solves several open questions on spreading models raised by G. Androulakis, E. Odell, Th. Schlumprecht, and Tomczak-Jaegermann [AOST]. They asked in particular the following three questions: Does there exist a Banach space so that every subspace has exactly $n$ many different spreading models? Does there exist a Banach space so that every subspace has exactly countably infinitely many different spreading models? If a Banach space admits $\ell_1$ and $\ell_2$ spreading models in every subspace must it admit uncountably many spreading models? In [AM2], S.A. Arygros and the third author have constructed a space so that every subspace admits every unconditional basis as a spreading model. In [ABM], S.A. Argyros with the first and third named authors created a Banach space such that every infinite dimensional subspace admits exactly two spreading models up to isomorphism, namely $\ell_1$ and $c_0$. Theorem 1 includes the case that $F$ is an increasing sequence and its limit, and so it is natural to question if the case of a decreasing sequence is possible. However, B. Sari proved that if a Banach space admits a countable collection of spreading models which form a strictly increasing sequence in terms of domination, then the Banach space admits uncountably many spreading models [S]. Thus, Theorem 1 (iii) would be impossible in the case that $F$ is a decreasing sequence and its limit as the spaces $\{\ell_p\}_{p\in F}$ would include an increasing sequence in terms of domination.

Given a Banach space $X$ with a basis, one may consider the set of all $p$'s, such that $\ell_p$ is admitted as a spreading model by $X$. Although this set may fail to coincide with the Krivine set of the space, or may even be empty [OS1], it is always contained in the Krivine set. Therefore, for a given subset $F$ of $[1,\infty]$ when constructing a space $X$, one way to ensure that $F$ is contained in any stabilized Krivine set of $X$, is to have $\ell_p$ admitted as a spreading model by every subspace of the space for every $p\in F$. For any single $1\leqslant p<\infty$, Tsirelson's method allows one to build a reflexive space not containing $\ell_p$ that is asymptotic $\ell_p$. Every spreading model admitted by this space is isomorphic to $\ell_p$ and the Krivine set of every infinite dimensional subspace is the singleton $\{p\}$. In this paper, for any finite set of $p$'s, we build a space with exactly these $\ell_p$'s hereditarily as spreading models and exactly these $p$'s hereditary as Krivine $p$'s. Moreover, for any increasing sequence of $p$'s we get almost the same result. In this case, the only caveat is that, although the basis of the space $X$ admits only the limit $p$ as a spreading model, we did not prove that the limit $p$ is admitted in every subspace.

In the case of two distinct $p$'s, our construction is rooted in the convexified Tsirelson's spaces in the sense of T. Figiel and W.B. Johnson's description [FJ] and the work of Odell and Schlumprecht [OS1], [OS2]. The methods we follow are based on the ones from [ABM]. In particular, for the simplest case of $F=\{1,\infty\}$, our construction reduces to a small modification of the space $\mathfrak{X}^1_{0,1}$, which is the simplest case of the construction defined in that paper, and in which it is shown that $\mathfrak{X}^1_{0,1}$ admits only $c_0$ and $\ell_1$ spreading models in every subspace. In recent literature, the spaces in [OS1], [OS2], [ABM], [AM1] and [AM2] are referred to as Tsirelson spaces with constraints or multi-layer Tsirelson spaces. For the sake of understanding our construction in the simplest case $F = \{1, \infty\}$, the norm satisfies the following implicit equation for $x \in c_{00}$:

$$\|x\| = \|x\|_0 \vee \sup \frac{1}{4} \sum_{i=1}^{n} \|E_i x\|_{m_i},$$

where the supremum is over successive intervals $(E_i)_{i=1}^n$ and $(m_i)_{i=1}^n$ with

$$(\min E_i)_{i=1}^n \in \mathcal{S}_1, \min E_i > (\max E_{i-1})^2 \text{ and } m_i > \max E_{i-1},$$

and for each $m \in \mathbb{N}$,

$$\|x\|_m = \frac{1}{4m} \sup \sum_{i=1}^{m} \|F_i x\|,$$

where the supremum is over successive intervals $(F_i)_{i=1}^m$. These $m$-norms and the way they are combined above are the previously mentioned contraints. Since the constraints are based on averages, local and asymptotic $c_0$ structure occurs in every subsapce. Furthermore, in contrast to Tsirelson space, which has homogeneous asymptotic $\ell_1$ structure, the above construction hereditarily provides both $\ell_1$ and $c_0$ local and asymptotic structure.

In the case that $F = \{p_1 < \cdots < p_n\} \subset [1, \infty]$ we present a space $X$, admitting hereditarily $\ell_{p_1}, \ldots, \ell_{p_n}$ asymptotic structure and nothing more. For this purpose we define a new norm, which has $n$-many layers, each one corresponding to an $\ell_{p_k}$ structure, for $k = 1, \ldots, n$. The base layer corresponds to the $\ell_{p_n}$ norm, while for $k = 1, \ldots, n-1$ the $k^{th}$ layer corresponds to norm $\ell_{p_k}$ and it is defined using the previous layers. To avoid the domination of some of these layers over the rest, to each of these layers, except for the basic one, some constraints have to be applied. The constraints are based on $p_n'$-averages, where $p_n'$ is the conjugate exponent of $p_n$.

When $F$ consists of an increasing sequence $(p_k)_k$ and its limit $p_\omega$, countably many layers of the norm are used. In this case, the norm $\|\cdot\|_*$ of the space is defined through the following formulas. We state here the implicit equations for the norms for the sake of giving insight into our construction, but we will actually use a different definition in Section 3 in terms of norming functionals. For $0 < \theta \leqslant 1/4$ and $x \in c_{00}(\mathbb{N})$ we define:

$$\|x\|_\omega = \theta \sup \left( \sum_{q=1}^{d} \|E_q x\|_*^{p_\omega} \right)^{1/p_\omega} \quad \text{and} \quad \|x\|_{0,m} = \theta \sup \frac{1}{m^{1/p_\omega'}} \sum_{q=1}^{m} \|E_q x\|_*$$

where both suprema are are taken over all $d \in \mathbb{N}$ and successive intervals $(E_q)_{q=1}^d$ of the natural numbers. These $\|\cdot\|_{0,m}$ norms are the constraints applied to the norm of the space. If for some $k > 0$ the norms $\|\cdot\|_{i,m}$ have been defined for every $0 \leqslant i < k$ and $m \in \mathbb{N}$, for $x \in c_{00}(\mathbb{N})$ and $m \in \mathbb{N}$ we define:

$$\|x\|_{k,m} = \theta \sup \left( \sum_{q=1}^{d} \|E_q x\|_{i_q, m_q}^{p_k} \right)^{1/p_k}$$

where the supremum is taken over all $d \in \mathbb{N}$, $0 \leqslant i_q < k$ for $q = 1, \dots, d$ and $(E_q)_{q=1}^d$, $(m_q)_{q=1}^d$ which satisfy certain growth conditions depending on $m$. The norm of the space satisfies the following implicit equation:

$$\|x\|_* = \max \left\{ \|x\|_\infty,\ \|x\|_\omega, \sup\{\|x\|_{k,m} :\ k, m \in \mathbb{N}\} \right\}.$$

Using the above description of the norm, it is easy to see that any block sequence in our space satisfies a lower $\ell_{p_\omega}$ estimate with constant $\theta$. Likewise, in section 4 we prove that any block sequence satisfies an upper $\ell_{p_1}$ estimate with constant 2. In the case $F$ is finite with $F = \{p_1 < \cdots < p_n\}$ then the norm of the space satisfies the same formula, where $p_\omega$ is replaced with $p_n$.

The paper is organized as follows. In section 2 we give a few preliminary definitions. Section 3 contains the definition of the spaces. In section 4 we set notation that we will use in our subsequent evaluations and prove upper and lower estimates on normalized block sequences. In sections 5 and 6 we prove the spaces have the desired spreading model structure. Finally, in section 7 we show that in every block subspace the only Krivine $p$'s are those admitted as spreading models.

The majority of the research included in this paper was conducted while the first two authors where visiting the National Technical University of Athens. We sincerely thank Spiros Argyros for his hospitality and enlightening conversations.

## 2. Preliminaries

We begin with some preliminary definitions. Two basic sequences $(x_i)$ and $(y_i)$ are *$C$-equivalent* for some $C \geqslant 1$ if $\sqrt{C}^{-1} \| \sum a_i x_i \| \leqslant \| \sum a_i y_i \| \leqslant \sqrt{C} \| \sum a_i x_i \|$ for all scalar sequences $(a_i)$. A basic sequence $(e_i)_{i=1}^\infty$ is *finitely block represented* in a basic sequence $(x_i)_{i=1}^\infty$ if for all $N \in \mathbb{N}$ and $\varepsilon > 0$ there exists a finite block sequence $(y_i)_{i=1}^N$ of $(x_i)_{i=1}^\infty$ which is $(1+\varepsilon)$-equivalent to $(e_i)_{i=1}^N$. For $1 \leqslant p \leqslant \infty$, we say that $\ell_p$ is *finitely block represented* in $(x_i)_{i=1}^\infty$ if the unit vector basis of $\ell_p$ is finitely block represented in $(x_i)_{i=1}^\infty$ (where we use the case $p = \infty$ to mean $c_0$).

We say that a basic sequence $(e_i)_{i=1}^\infty$ is a *spreading model* of a basic sequence $(x_i)_{i=1}^\infty$ if for all finite sequences of scalars $(a_i)_{i=1}^n$ we have that

$$\| \sum_{i=1}^n a_i e_i \| = \lim_{t_1 \to \infty} \cdots \lim_{t_n \to \infty} \| \sum_{i=1}^n a_i x_{t_i} \|.$$

We say that a Banach space $X$ *admits* $(e_i)_{i=1}^\infty$ as a spreading model if $(e_i)_{i=1}^\infty$ is equivalent to a spreading model of some basic sequence in $X$. We say that $X$ admits $\ell_p$ as a spreading model if $X$ admits a spreading model equivalent to the unit vector basis for $\ell_p$. In the literature, the basic sequence $(e_i)_{i=1}^\infty$ as well as the Banach space formed by its closed span are both often referred to as spreading models.

## 3. The definition of the space $X$.

In this section we give the definition of the norming set of the space $X$. We apply a variation of the method of saturation under constraints, introduced by Odell and Schlumprecht in [OS1], [OS2]. The way this method is applied is similar to the one in [ABM] and it allows $\ell_p$ structure to appear hereditarily in the space, for a predetermined set of $p$'s, which is either finite or consists of an increasing sequence and its limit.

**Notation.** Let $G$ be a subset of $c_{00}(\mathbb{N})$.

(i) A finite sequence $(f_q)_{q=1}^d$ of elements of $G$ will be called *admissible* if $f_1 < \cdots < f_d$ and $d \leqslant \min\operatorname{supp} f_1$.

(ii) Assume that $(f_q)_q$ is a sequence of functionals in $G$, and each functional $f_q$ has been assigned a positive integer $s(f_q)$, called the size of $f_q$. Then $(f_q)_q$ will be called *very fast growing* if $(\max\operatorname{supp} f_{q-1})^2 < \min\operatorname{supp} f_q$ and $s(f_q) > \max\operatorname{supp} f_{q-1}$ for all $q > 1$.

Let $\xi_0 \in [2, \omega]$ and let $F = \{p_k : 1 \leqslant k < \xi_0\} \cup \{p_{\xi_0}\} \subset [1, \infty]$ with $p_k \uparrow p_{\xi_0}$ in the case that $\xi_0 = \omega$ and $p_1 < p_2 < \cdots < p_{\xi_0 - 1} < p_{\xi_0}$ otherwise.

We now define the norming set of the space $X$. We do so inductively by defining an increasing sequence of subsets of $c_{00}(\mathbb{N})$. To some of the functionals that we construct we shall assign an order, a size or both. Fix a positive real number $0 < \theta \leqslant 1/4$. Let $W_0 = \{\pm e_j^*\}_{j\in\mathbb{N}}$. To the functionals in $W_0$ we don't assign an order or size. Assume that for some $m \in \mathbb{N} \cup \{0\}$ the set $W_m$ has been defined, below we describe how the set $W_{m+1}$ is defined.

**Functionals of order-0, or of order-$\xi_0$.** Define

$$W_{m+1}^0 = \left\{ \theta \sum_{q=1}^d c_q f_q : \ f_1 < \cdots < f_d \in W_m, \ \sum_{q=1}^d |c_q|^{p'_{\xi_0}} \leqslant 1 \right\}.$$

A functional $f = \theta \sum_{q=1}^d c_q f_q$ as above will be called of order-0. In some cases, for convenience these functionals shall also be referred to as functionals of order-$\xi_0$.

If a functional $f$ of order-0 has the form $f = \theta \sum_{q=1}^d (1/n)^{1/p'_{\xi_0}} f_q$ with $d \leqslant n$, then the size of $f$ is defined to be $s(f) = n$. If a functional $f$ of order-0 is not of this form then we do not assign any size to it.

If $m+1 = 1$ then we define $W_1 = W_0 \cup W_1^0$ , otherwise $m+1 \geqslant 2$ and we shall include more functionals in $W_{m+1}$, as described below.

**Functionals of order-$k$, with $1 \leqslant k < \xi_0$.** Define

$$W_{m+1}^k = \Big\{ \theta \sum_{q=1}^d c_q f_q : \sum_{q=1}^d |c_q|^{p_k'} \leqslant 1,\ (f_q)_{q=1}^d \text{ is an admissible and very fast growing sequence of functionals in } W_m, \text{ each one of which has order strictly smaller than } k \Big\}.$$

A functional $f = \theta \sum_{q=1}^d c_q f_q$ as above will be called of order-$k$ with size $s(f) = \min\{s(f_q) : q = 1, \dots, d\}$.

If $k = 1$ and $p_1 = 1$, we replace the condition $\sum_{q=1}^d |c_q|^{p_1'} \leqslant 1$ with the condition $\max\{|c_q| : \ q = 1, \dots, d\} \leqslant 1$. Note that if $\xi_0$ is finite and $\xi_0 = k_0 + 1$, then very fast growing sequences of functionals of order-$k_0$ are not used. Observe also that some functionals may be of more than one order or have multiple sizes, however, this shall not cause any problems.

If $m+1 \geqslant 2$, let $W_{m+1} = \big(\cup_{0 \leqslant k < \xi_0} W_{m+1}^k\big) \cup W_m$ and $W = \cup_{m=0}^\infty W_m$. The space $X$ is the completion of $c_{00}(\mathbb{N})$ under the norm induced by $W$, i.e. for $x \in c_{00}(\mathbb{N})$ the norm of $x$ is equal to $\sup\{|f(x)| : \ f \in W\}$.

**Remark 3.1.** The following are clear from the definition of the norming set.

(i) For every $f_1 < \cdots < f_d$ in $W$ and real numbers $(c_q)_{q=1}^d$ with $\sum_{q=1}^d |c_{q0}|^{p_{\xi_0}'} \leqslant 1$, the functional $f = \theta \sum_{q=1}^d c_q f_q$ is also in $W$.

(ii) For every $1 \leqslant k < \xi_0$ and every admissible and very fast growing $f_1 < \cdots < f_d$ in $W$, each one of which has order strictly smaller than $k$ and real numbers $(c_q)_{q=1}^d$ with $\sum_{q=1}^d |c_{q0}|^{p_k'} \leqslant 1$, the functional $f = \theta \sum_{q=1}^d c_q f_q$ is also in $W$.

**Remark 3.2.** For every $f \in W$ and subset $E$ of the natural numbers, we have that $f|_E$, the restriction of $f$ onto $E$, is also in $W$. In particular, if $f$ is of order-$k$, then $f|_E$ is also of order-$k$ and $s(f|_E) \geqslant s(f)$. One can also check that the norming set $W$ is closed under changing signs, i.e. if $f \in W$ and $g$ is such that $|f| = |g|$, then $g$ is also in $W$. Therefore, the unit vector basis of $c_{00}(\mathbb{N})$ forms a 1-unconditional basis for $X$.

Recall that functionals of order-0 are also called functionals of order-$\xi_0$.

**Remark 3.3.** For every $m > 0$, $1 \leqslant \zeta \leqslant \xi_0$ and $f \in W_m$, which is of order-$\zeta$, there exist $f_1 < \cdots < f_d$ in $W_{m-1}$ and real numbers $c_1, \dots, c_d$ with $\sum_{q=1}^d |c_q|^{p_\zeta'} \leqslant 1$ such that $f = \theta \sum_{q=1}^d c_q f_q$. If moreover $\zeta = k < \xi_0$, then $(f_q)_{q=1}^d$ is an admissible and very fast growing sequence of functionals, each one of which has order strictly smaller than $k$.

Before proceeding to the study of the properties of the space $X$, let us briefly explain the ingredients of the norming set $W$, without getting into too many details. If $1 < \xi_0 \leqslant \omega$ and we have determined a set $F = \{p_1 < \cdots < p_{\xi_0}\} \subset [1, \infty]$, then every element $f$ of the norming set falls into one of the following three categories:

(i) The functional $f$ is an element of the basis, i.e. $f \in \{\pm e_i\}_i$.
(ii) The functional $f$ is of order-0, i.e. $f = \theta \sum_{q=1}^d c_q f_q$ where $f_1 < \cdots < f_d$ can be any successive elements of the norming set, combined with coefficients $(c_q)_q$ in the unit ball of $\ell_{p'_{\xi_0}}$.
(iii) The functional $f$ is of order-$k$ with $1 \leqslant k < \xi_0$, i.e. $f = \theta \sum_{q=1}^d c_q f_q$ where the sequence $f_1 < \cdots < f_d$ are successive elements of the norming set satisfying certain constraints, while the coefficients $(c_q)_q$ are in the unit ball of $\ell_{p'_k}$.

The functionals of order-0 provide $\ell_{p_{\xi_0}}$ structure to the space and, since the $\ell_{p_{\xi_0}}$ is the smallest of the $\ell_p$ norms for $p \in F$, their construction is not subject to any constraints. On the other hand, for $1 \leqslant k < \xi_0$, the functionals of order $k$ provide $\ell_{p_k}$ structure. One has to define these functionals carefully, in order not to demolish the desired $\ell_{p_\zeta}$ structure, for $k < \zeta \leqslant \xi_0$. This is the role of the constraints, which become more restrictive as $k$ becomes smaller.

One can verify that the norm induced by the norming set $W$ is alternatively described by the implicit formula given in the introduction.

## 4. Basic norm evaluations on block sequences of $X$.

In this section we prove a simple, but useful, lemma and we also prove that block sequences in $X$ have an upper $\ell_{p_1}$ estimate and a lower $\ell_{p_{\xi_0}}$ estimate. We start with some notation, which in conjunction with the next lemma, will be used frequently throughout the paper. Here, the range of a vector is the smallest closed interval containing the support.

**Notation.** Let $x_1 < \cdots < x_m$ be a finite block sequence in $X$ and $f = \theta \sum_{q=1}^d c_q f_q$ be a functional of order-$\zeta$, $1 \leqslant \zeta \leqslant \xi_0$. Define the following :

$$\begin{aligned}
A_1 &= \{q \in \{1, \ldots, d\} : \operatorname{ran} f_q \cap \operatorname{ran} x_j \neq \varnothing \text{ for at most one } 1 \leqslant j \leqslant m\}, \\
A_2 &= \{1, \ldots, d\} \setminus A_1, \\
B &= \{j \in \{1, \ldots, m\} : \text{ there exists } q \in A_1 \text{ with } \operatorname{ran} f_q \cap \operatorname{ran} x_j \neq \varnothing\} \\
A_1^j &= \{q \in A_1 : \operatorname{ran} f_q \cap \operatorname{ran} x_j \neq \varnothing\} \text{ for } j \in B, \\
C_j &= \left\| (c_q)_{q \in A_1^j} \right\|_{\ell_{p'_\zeta}} \text{ for } j \in B, \\
g_j &= \theta \sum_{q \in A_1^j} (c_q / C_j) f_q \text{ for } j \in B \text{ and} \\
E_q &= \{j \in \{1, \ldots, m\} : \operatorname{ran} f_q \cap \operatorname{ran} x_j \neq \varnothing\}, \text{ for } q \in A_2.
\end{aligned}$$

The following lemma follows immediately from our choice of notation. As in Remark 3.3, here we also use the fact that order-0 functionals can be referred to as order-$\xi_0$ functionals.

**Lemma 4.1.** Let $x_1 < \ldots < x_m$ be a finite block sequence in $X$ and $f = \theta\sum_{q=1}^{d} c_q f_q$ be a functional of order-$\zeta$ for some $1 \leqslant \zeta \leqslant \xi_0$. The functionals $(g_j)_{j\in B}$ are order-$\zeta$ functionals in $W$, we have that $(\sum_{j\in B} C_j^{p'_\zeta})^{1/p'_\zeta} \leqslant 1$, and the following holds:

$$\left|f\left(\sum_{j=1}^{m} x_j\right)\right| \leqslant \left(\sum_{j\in B} C_j|g_j(x_j)|\right) + \theta\left(\sum_{q\in A_2}|c_q|\cdot\left|f_q\left(\sum_{j\in E_q} x_j\right)\right|\right). \tag{1}$$

Moreover, if $A_2 = \{q_1 < \cdots < q_r\}$ then $\max E_{q_i} \leqslant \min E_{q_{i+1}}$ for $i = 1,\ldots,r-1$ and $\max E_{q_i} < \min E_{q_{i+2}}$ for $i = 1,\ldots,r-2$. Thus, for each $1 \leqslant j \leqslant m$ there exists at most two sets $E_q$ such that $x_j \in E_q$.

Note that applying Hölder's inequality to (1), gives the following inequality, which in most cases will be more convenient for us than (1).

$$\left|f\left(\sum_{j=1}^{m} x_j\right)\right| \leqslant \left\|(g_j(x_j))_{j\in B}\right\|_{\ell_{p_\zeta}} + \theta\left\|\Big(f_q(\sum_{j\in E_q} x_j)\Big)_{q\in A_2}\right\|_{\ell_{p_\zeta}}. \tag{2}$$

**Proposition 4.2.** Let $x_1 < \cdots < x_m$ be a normalized finite block sequence in $X$ and $(\lambda_j)_{j=1}^m$ be scalars. The following holds:

$$\theta\|(\lambda_j)_j\|_{\ell_{p_{\xi_0}}} \leqslant \left\|\sum_{j=1}^{m}\lambda_j x_j\right\| \leqslant 2\|(\lambda_j)_j\|_{\ell_{p_1}}.$$

*Proof.* We first prove the lower inequality. Note that this is trivial in the case that $p_{\xi_0} = \infty$ , thus we assume that $p_{\xi_0} < \infty$. For each $j \in \{1,\ldots,m\}$ find $f_j$ so that $f_j(x_j) = 1$ and $\operatorname{supp} f_j = \operatorname{supp} x_j$. Without loss of generality, we may assume that $(\sum_{i=1}^m |\lambda_i|^{p_{\xi_0}})^{1/p_{\xi_0}} = 1$ and $\lambda_i \geq 0$ for all $1 \leqslant i \leqslant m$. Thus, $\theta\sum_{j=1}^m |\lambda_j|^{p_{\xi_0}/p'_{\xi_0}} f_j \in W$. Therefore

$$\left\|\sum_{j=1}^{m}\lambda_j x_j\right\| \geqslant \theta\sum_{j=1}^{m}|\lambda_j|^{p_{\xi_0}/p'_{\xi_0}} f_j\Big(\sum_{i=1}^{m}\lambda_i x_i\Big) = \theta\Big(\sum_{i=1}^{m}|\lambda_i|^{p'_{\xi_0}}\Big) = \theta.$$

The upper inequality clearly follows from the following claim that we will prove by induction on $n \in \mathbb{N}$. For all $n \in \mathbb{N}\cup\{0\}$ and $f \in W_n$ (see Remark 3.3) we have

$$f\Big(\sum_{j=1}^{m}\lambda_j x_j\Big) \leqslant 2\left(\sum_{j=1}^{m}|\lambda_j|^{p_1}\right)^{1/p_1}. \tag{3}$$

The case of $f \in W_0 = \{\pm e_j^*\}$ is trivial. Assume that the above holds for some $n \geqslant 0$. Let $f \in W_{n+1}$. Then $f = \theta \sum_{q=1}^d c_q f_q$ is of order-$\zeta$ and $f_1 < \cdots < f_d$ are in in $W_n$ and $\sum_{q=1}^d |c_q|^{p'_\zeta} \leqslant 1$.

By (2) after Lemma 4.1 then applying the inductive hypothesis, we obtain the following:

$$
\begin{aligned}
\left|f(\sum_{j=1}^m \lambda_j x_j)\right| &\leqslant \left(\sum_{j\in B} |\lambda_j|^{p_\zeta}\right)^{1/p_\zeta} + \theta\left(\sum_{q\in A_2}\left|f_q\left(\sum_{j\in E_q}\lambda_j x_j\right)\right|^{p_\zeta}\right)^{1/p_\zeta}\\
&\leqslant \left(\sum_{j\in B} |\lambda_j|^{p_\zeta}\right)^{1/p_\zeta} + 2\theta\left(\sum_{q\in A_2}\left(\sum_{j\in E_q}|\lambda_j|^{p_1}\right)^{p_\zeta/p_1}\right)^{1/p_\zeta}. \qquad (4)
\end{aligned}
$$

By the last part of Lemma 4.1, for each $j$ there exists at most two distinct $q \in A_2$ such that $j \in E_q$. This fact together with $p_1 \leqslant p_\zeta$ imply that

$$
\left(\sum_{q\in A_2}\left(\sum_{j\in E_q}|\lambda_j|^{p_1}\right)^{p_\zeta/p_1}\right)^{1/p_\zeta} < 2^{1/p_1}\left(\sum_{j=1}^m |\lambda_j|^{p_1}\right)^{1/p_1}. \qquad (5)
$$

Combining relations (4) and (5) together with $0 < \theta \leqslant 1/4$, we obtain the desired bound in (3).

□

## 5. Spreading models of $X$.

In this section we define the $\alpha$-indices in a very similar manner as they have been defined in [ABM], [AM1] and [AM2]. Although previously the $\alpha$-indices were used to describe the action of certain averages of functionals on a block sequence, in our case this is not exactly the same. Here, the indices are used to study the action of functionals of a certain order on a block sequence. However, the principle is the same and we retain this notation. As is the case in these papers, the indices determine the spreading models admitted by a block sequence in the space $X$. As a consequence we prove that every spreading model admitted by a weakly null sequence in $X$ must equivalent to the unit vector basis of $\ell_{p_\zeta}$ for some $\zeta \in [1, \xi_0]$.

**Definition 5.1.** Let $(x_j)_j$ be a block sequence in $X$ and let $1 \leqslant k < \xi_0$. Assume that for every very fast growing sequence $(f_q)_q$ of functionals in $W$, each one of which has order strictly smaller than $k$, and every subsequence $(x_{j_i})_i$ of $(x_j)_j$ we have that $\lim_i |f_i(x_{j_i})| = 0$. Then we say that the $\alpha_{<k}$-index of $(x_j)_j$ is zero and write $\alpha_{<k}\{(x_j)_j\} = 0$. Otherwise we write $\alpha_{<k}\{(x_j)_j\} > 0$.

**Remark 5.2.** Let $(x_j)_j$ be a block sequence in $X$ and $1 \leqslant m < k < \xi_0$. If $\alpha_{<k}\{(x_j)_j\} = 0$ then also $\alpha_{<m}\{(x_j)_j\} = 0$.

The following characterization has appeared in similar forms in [ABM], [AM1] and [AM2]. We omit the proof as it is simple and straightforward.

**Proposition 5.3.** Let $1 \leqslant k < \xi_0$ and $(x_j)_j$ be a block sequence in $X$. The following assertions are equivalent:

(i) $\alpha_{<k}\{(x_j)_j\} = 0$.

(ii) For every $\varepsilon > 0$ there exist $j_0, i_0 \in \mathbb{N}$ such that for every $f \in W$ of order strictly smaller than $k$ with $s(f) \geqslant i_0$ and every $j \geqslant j_0$ we have that $|f(x_j)| < \varepsilon$.

**Lemma 5.4.** Let $1 \leqslant k < \xi_0$ and $(x_j)_j$ be a bounded block sequence in $X$ such that $\alpha_{<k}\{(x_j)_j\} > 0$. Then $(x_j)_j$ has a subsequence with a spreading model that dominates the unit vector basis for $\ell_{p_k}$. That is, there exists $\varepsilon > 0$ and a subsequence $(x_{j_i})_i$ of $(x_j)_j$ such that for every natural numbers $m \leqslant i_1 < \cdots < i_m$ and every real numbers $\lambda_1, \ldots, \lambda_m$ the following holds:

$$\left\| \sum_{t=1}^{m} \lambda_t x_{j_{i_t}} \right\| \geqslant \varepsilon \left\| (\lambda_t)_t \right\|_{\ell_{p_k}}.$$

*Proof.* By the definition of the $\alpha_{<k}$ index, there exists $\varepsilon' > 0$, a subsequence of $(x_j)_j$, again denoted by $(x_j)_j$ and a very fast growing sequence $(f_j)_j$ of functionals of order strictly smaller than $k$, such that $|f_j(x_j)| > \varepsilon'$ for all $j \in \mathbb{N}$. We may also assume that $\operatorname{ran} f_j \subset \operatorname{ran} x_j$ for all $j \in \mathbb{N}$. Set $\varepsilon = \theta\varepsilon'$ and note that for every $m \leqslant j_1 < \cdots < j_m$ and every real numbers $(c_t)_{t=1}^m$ with $\sum_{t=1}^m |c_t|^{p_k'} \leqslant 1$ the functional $f = \theta \sum_{t=1}^m c_t f_{j_t}$ is of order-$k$. Let $m \leqslant j_1 < \cdots < j_m$ be natural numbers and $\lambda_1, \ldots, \lambda_m$ be real numbers. We have the following estimate:

$$\begin{aligned}
\left\| \sum_{t=1}^{m} \lambda_t x_{j_t} \right\| &\geqslant \sup\left\{ \theta \sum_{t=1}^{m} c_t f_{j_t}\left( \sum_{t=1}^{m} \lambda_t x_{j_t} \right) : \sum_{t=1}^{m} |c_t|^{p_k'} \leqslant 1 \right\} \\
&= \sup\left\{ \theta \sum_{t=1}^{m} |c_t \lambda_t| \cdot |f_{j_t}(x_{j_t})| : \sum_{t=1}^{m} |c_t|^{p_k'} \leqslant 1 \right\} \\
&\geqslant \theta\varepsilon' \sup\left\{ \sum_{t=1}^{m} |c_t \lambda_t| : \sum_{t=1}^{m} |c_t|^{p_k'} \leqslant 1 \right\} \\
&= \varepsilon \left( \sum_{t=1}^{m} |\lambda_t|^{p_k} \right)^{1/p_k}.
\end{aligned}$$

□

**Lemma 5.5.** Let $(x_j)_j$ be a normalized block sequence and $2 \leqslant \zeta \leqslant \xi_0$ such that $\alpha_{<k}\{(x_j)_j\} = 0$ for every $1 \leqslant k < \zeta$. Then $(x_j)_j$ has a subsequence with a spreading model that is 2-dominated by the unit vector basis for $\ell_{p_\zeta}$. In particular, there exists a subsequence $(x_{j_i})_i$ of $(x_j)_j$ such that for every natural numbers $m \leqslant i_1 < \cdots < i_m$ and every real numbers $\lambda_1, \ldots, \lambda_m$ the following holds:

$$\left\| \sum_{t=1}^{m} \lambda_t x_{j_{i_t}} \right\| \leqslant 3 \left\| (\lambda_t)_t \right\|_{\ell_{p_\zeta}} .$$

*Proof.* We first consider the case in which $\zeta$ is finite, i.e. $\zeta = k' + 1$ with $1 \leqslant k' < \xi_0$.

Using Proposition 5.3 we pass to a subsequence, again denoted by $(x_j)_j$ such that for any $j \geqslant j_0 \geqslant 2$ and any $f \in W$ of order strictly smaller than $k'$ with $s(f) \geqslant \min \operatorname{supp} x_{j_0}$ we have that

$$|f(x_j)| < (j_0 \max \operatorname{supp} x_{j_0 - 1})^{-1} . \tag{6}$$

We will show by induction on $n$, where $W = \cup_n W_n$ (see Remark 3.3) that for every $m \leqslant j_1 < \cdots < j_m$, every real numbers $\lambda_1, \ldots, \lambda_m$ and every $f \in W_n$ the following holds:

$$\left| f\left( \sum_{t=1}^{m} \lambda_t x_{j_t} \right) \right| \leqslant 2 \left\| (\lambda_t)_{t=1}^m \right\|_{\ell_{p_\zeta}} . \tag{7}$$

For $f \in W_0$ the result holds. Let $f = \theta \sum_{q=1}^d c_q f_q$ be a functional in $W_n$. We distinguish two cases, concerning the order of $f$.

*Case 1:* The functional $f$ is of order-$\eta$ with $\zeta \leqslant \eta \leqslant \xi_0$. By Inequality (2) after Lemma 4.1, we have that

$$\begin{aligned} \left| f\left( \sum_{t=1}^{m} \lambda_t x_{j_t} \right) \right| &\leqslant \left( \sum_{t \in B} |\lambda_t|^{p_\eta} \right)^{1/p_\eta} + \theta \left( \sum_{q \in A_2} \left| f_q \left( \sum_{t \in E_q} \lambda_t x_{j_t} \right) \right|^{p_\eta} \right)^{1/p_\eta} \\ &\leqslant \left( \sum_{t \in B} |\lambda_t|^{p_\eta} \right)^{1/p_\eta} + \theta 2 \left( \sum_{q \in A_2} \left( \sum_{t \in E_q} |\lambda_t|^{p_k} \right)^{p_\eta / p_k} \right)^{1/p_\eta} \quad \text{by (7).} \end{aligned} \tag{8}$$

The fact that $p_\zeta \leqslant p_\eta$ and for each $1 \leqslant t \leqslant m$ there exists at most two distinct $q \in A_2$ such that $t \in E_q$ implies that

$$\left( \sum_{q \in A_2} \left( \sum_{t \in E_q} |\lambda_t|^{p_\zeta} \right)^{p_\eta / p_\zeta} \right)^{1/p_\eta} \leqslant 2^{1/p_\zeta} \left( \sum_{t=1}^{m} |\lambda_t|^{p_\zeta} \right)^{1/p_\zeta} . \tag{9}$$

Combining relations (8) and (9) with $0 < \theta \leqslant 1/4$, we get that $|f(\sum_{t=1}^m \lambda_t x_{j_t})|$ is bounded by the desired value. Note that for convenience we have implicitly assumed that $p_\eta < \infty$, but the case that $p_\eta = \infty$ would only require trivial modification.

*Case 2:* The functional $f$ is of order-$k''$ with $1 \leqslant k'' \leqslant k'$.

Set

$$\begin{aligned} t_0 &= \min\{t : \operatorname{ran} f \cap \operatorname{ran} x_{j_t} \neq \varnothing\} \text{ and} \\ q_0 &= \min\{q : \max\operatorname{supp} f_q \geqslant \min\operatorname{supp} x_{j_{t_0+1}}\}. \end{aligned}$$

We shall prove the following:

$$\theta \left| \sum_{q>q_0} c_q f_q \left( \sum_{t=1}^{m} \lambda_t x_{j_t} \right) \right| < \theta \max\{|\lambda_t| :\ t > t_0\}. \tag{10}$$

Since $(f_q)_{q=1}^d$ is admissible, we have that $d \leqslant \max\operatorname{supp} x_{j_{t_0}}$. Also, $(f_q)_{q=1}^d$ is very fast growing and hence for $q > q_0$ we have that

$$s(f_q) \geqslant \max\operatorname{supp} f_{q_0} \geqslant \min\operatorname{supp} x_{j_{t_0+1}}.$$

Moreover the functionals $f_q$ are of order strictly smaller than $k'$, therefore for $q > q_0$ and $t > t_0$, (6) yields that $|f_q(x_{j_t})| < 1/\left(j_{t_0+1} \max\operatorname{supp} x_{j_{t_0}}\right)$ and since $d \leqslant \max\operatorname{supp} x_{j_{t_0}}$, by keeping $t$ fixed, we obtain $\sum_{q>q_0} |c_q f_q(x_{j_t})| < 1/j_{t_0+1}$. Similarly, summing over the $t$ which are strictly greater than $t_0$, since $m \leqslant j_{t_0+1}$ we obtain:

$$\begin{aligned} \left| \sum_{q>q_0} c_q f_q \left( \sum_{t=1}^{m} \lambda_t x_{j_t} \right) \right| &= \left| \sum_{q>q_0} c_q f_q \left( \sum_{t>t_0}^{m} \lambda_t x_{j_t} \right) \right| \\ &\leqslant \sum_{t>t_0} |\lambda_t| \sum_{q>q_0} |c_q f_q(x_{j_t})| \\ &< \max_{t>t_0} |\lambda_t| (m/j_{t_0+1}) \leqslant \max_{t>t_0} |\lambda_t|. \end{aligned}$$

Thus, (10) holds. We now observe the following:

$$\theta \left| \sum_{q<q_0} c_q f_q \left( \sum_{t=1}^{m} \lambda_t x_{j_t} \right) \right| = \theta \left| \sum_{q<q_0} c_q f_q \left( \lambda_{t_0} x_{j_{t_0}} \right) \right| \leqslant |\lambda_{t_0}|. \tag{11}$$

Moreover, the inductive assumption yields that

$$\left| f_{q_0} \left( \sum_{t=1}^{m} \lambda_t x_{j_t} \right) \right| \leqslant 2 \left\| (\lambda_t)_{t=1}^m \right\|_{\ell_{p_\zeta}}. \tag{12}$$

Combining (10), (11) and (12) with $0 < \theta \leqslant 1/4$ we get that $|f(\sum_{t=1}^m \lambda_t x_{j_t})|$ is bounded by the desired value.

The proof for the case in which $\zeta$ is finite is complete. Assume now that $\zeta = \xi_0 = \omega$ and pass to a subsequence of $(x_j)_j$ generating as a spreading model some sequence $(z_j)_j$. The previous case implies that $(z_j)_j$ is 2-dominated by the unit vector basis of $\ell_{p_k}$ for all $k < \xi_0$ and hence, by taking a limit, it is also 2-dominated by the unit vector basis of $\ell_{p_{\xi_0}}$ which yields the desired result. $\square$

The next result explains that the $\alpha_{<k}$-indices of a given block sequence determine the spreading models admitted by it.

**Proposition 5.6.** Let $(x_j)_j$ be a normalized block sequence in $X$. Then $(x_j)_j$ admits an $\ell_{p_\zeta}$ spreading model, for some $\zeta \in [1, \xi_0]$. The following describes more precisely the spreading models admitted by $(x_j)_j$.

a. Let $2 \leqslant k < \xi_0$, then the following assertions are equivalent:
   (i) $\alpha_{<k}\{(x_j)_j\} > 0$ and $\alpha_{<k'}\{(x_j)_j\} = 0$ for $1 \leqslant k' < k$.
   (ii) There exists a subsequence of $(x_j)_j$ that generates an $\ell_{p_k}$ spreading model, while no subsequence of $(x_j)_j$ generates an $\ell_{p_{k'}}$ spreading model for $1 \leqslant k' < k$.

b. The following are equivalent
   (i) $\alpha_{<1}\{(x_j)_j\} > 0$.
   (ii) There exists a subsequence of $(x_j)_j$ that generates an $\ell_{p_1}$ spreading model.

c. The following are also equivalent:
   (i) For every $1 \leqslant k < \xi_0$ we have that $\alpha_{<k}\{(x_j)_j\} = 0$.
   (ii) Every subsequence of $(x_j)_j$ has a further subsequence generating an $\ell_{p_{\xi_0}}$ spreading model.

Note that in the case $\xi_0$ is finite and $\xi_0 = k_0 + 1$, then c.(i) is equivalent to $\alpha_{<k_0}\{(x_j)_j\} = 0$.

*Proof.* We shall only prove a. as the others are proved similarly, using Proposition 4.2 and Lemmas 5.4, 5.5. Assume that the first assertion of a. holds. Note that on every subsequence of $(x_j)_j$ the $\alpha_{<k-1}$-index is zero, and hence, applying Lemma 5.5, it has a further subsequence which admits a spreading model dominated by the unit vector basis of $\ell_{p_k}$. This in particular implies that no subsequence of $(x_i)_i$ generates an $\ell_{p_{k'}}$ spreading model for $1 \leqslant k' < k$. Moreover, applying Lemma 5.4 we pass to a subsequence $(x_{j_i})_i$, of $(x_j)_j$, generating some spreading model dominating the usual vector basis of $\ell_{p_k}$. Since $\alpha_{<k-1}\{(x_j)_j\} = 0$, Lemma 5.5 implies that this spreading model has to be $\ell_{p_k}$.

We assume now that the second assertion of a. holds. We first note that $\alpha_{<k}\{(x_j)_j\} > 0$. If this were not the case, then on every subsequence of $(x_j)_j$ the $\alpha_{<k}$-index would be zero and hence, by Lemma 5.5, every spreading model admitted by it is dominated by the unit vector basis of $\ell_{p_{k+1}}$. This means that no subsequence of $(x_j)_j$ can generate an $\ell_{p_k}$ spreading model, which is absurd. Therefore the natural number $k_0 = \min\{k \in [1, \xi_0) : \alpha_{<k}\{(x_j)_j\} > 0\}$ is well defined and $k_0 \leqslant k$. We shall prove that $k_0 = k$ and this will complete the proof.

Assume that $k_0 < k$ and apply Lemma 5.4 to pass to a subsequence $(x_{j_i})_i$ of $(x_j)_j$ generating some spreading model which dominates the usual basis of $\ell_{p_{k_0}}$. If $k_0 = 1$ then by Proposition 4.2 we conclude that $(x_{j_i})_i$ generates an $\ell_{p_1}$ spreading model, where $1 = k_0 < k$, which is absurd. If $1 < k_0$, then $\alpha_{<k_0-1}\{(x_{j_i})_i\} = 0$ by and Lemma 5.5 we conclude that $(x_{j_i})_i$ generates an $\ell_{p_{k_0}}$ spreading model, which is absurd for the same reasons.

$\square$

**Remark 5.7.** It is not hard to check that for every $1 \leqslant k < \xi_0$, the $\alpha_{<k}$-index of the basis $(e_i)_i$ is zero and hence it only admits $\ell_{p_{\xi_0}}$ as a spreading model.

## 6. Spreading models of infinite dimensional subspaces of $X$.

In the previous section we showed that every spreading model admitted by $X$ must be $\ell_p$ for some $p \in F$. In this section we show that, starting with a block sequence generating some spreading model, one may pass to a block sequence of it generating an other spreading model. We conclude that, in the case in which $F$ is finite, the spreading models admitted by every infinite dimensional subspace of $X$ are exactly the $\ell_p$ for $p \in F$. In the case which $F$ consists of an increasing sequence and its limit $p_{\xi_0}$, the spreading models admitted by every infinite dimensional subspace of $X$ may either be the $\ell_p$ for $p \in F$, or the $\ell_p$ for $p \in F \setminus \{p_{\xi_0}\}$. We start with two lemmas that describe the kind of block vectors one has to consider when switching from one spreading model to an other.

**Lemma 6.1.** Let $k$ be in $[1, \xi_0)$, $x_1 < \cdots < x_K$ be a finite normalized block sequence in $X$ that is 3-dominated by the unit vector basis of $\ell_{p_k}^K$, and set $x = K^{-1/p_k} \sum_{j=1}^K x_j$. If $f$ is a functional of order-0 in $W$ with $s(f) = m$ then the following holds:

$$|f(x)| < \frac{K^{1/p_{\xi_0}}}{K^{1/p_k}} + 2\frac{m^{1/p_{\xi_0}}}{m^{1/p_k}}, \tag{13}$$

where in the case $p_{\xi_0} = \infty$ we set $1/p_{\xi_0} = 0$.

*Proof.* Let $f = \theta \sum_{q=1}^d (1/m)^{1/p'_{\xi_0}} f_q$ be a functional of order-0 with $s(f) = m$ (recall that $d \leqslant m$). For convenience, we assume that $p_{\xi_0} < \infty$, and the proof for the case $\xi_0 = \infty$ requires only trivial modifications. By Lemma 4.1, following the notation used there, applying Hölder's inequality for the pair $(p_{\xi_0}, p_{\xi'_0})$, we obtain the following:

$$\begin{aligned} |f(x)| &\leqslant K^{-1/p_k}\left(\left(\sum_{j\in B}|g_t(x_j)|^{p_{\xi_0}}\right)^{1/p_{\xi_0}}\right. \\ &\qquad \left. +\theta\left(\sum_{q\in A_2}(1/m)^{1/p'_{\xi_0}}\left|f_q\left(\sum_{j\in E_q}x_j\right)\right|\right)\right) \\ &\leqslant K^{-1/p_k}\left(K^{1/p_{\xi_0}} + 3\theta(1/m)^{1/p'_{\xi_0}}\left(\sum_{q\in A_2}(\#E_q)^{1/p_k}\right)\right). \end{aligned} \tag{14}$$

Recall that $\#A_2 \leqslant d \leqslant m$ and the last part of Lemma 4.1 gives that $\sum_{q\in A_2} \#E_q < 2K$. Combing these two facts gives us

$$\begin{aligned}
(1/m)^{1/p'_{\xi_0}} \left(\sum_{q\in A_2} (\#E_q)^{1/p_k}\right) &\leqslant (1/m)^{1/p'_{\xi_0}} m^{1/p'_k} \left(\sum_{q\in A_2} \#E_q\right)^{1/p_k} \\
&< \frac{m^{1/p'_k}}{m^{1/p'_{\xi_0}}} 2^{1/p_k} K^{1/p_k} \\
&= 2^{1/p_k} \frac{m^{1/p_{\xi_0}}}{m^{1/p_k}} K^{1/p_k}. \qquad (15)
\end{aligned}$$

By combining relations (14) and (15) we achieve the desired upper bound. $\square$

**Lemma 6.2.** Let $(x_j)_j$ be a normalized block sequence in $X$ and $2 \leqslant k+1 < \xi_0$ with $\alpha_{<k}\{(x_j)_j\} = 0$. Then there exists a subsequence $(x_{j_i})_i$ of $(x_j)_j$ such that for every $K \leqslant j_{i_1} < \cdots < j_{i_K}$ and every $f \in W$ of order at most $k$ with $s(f) = m$, we have that if $x = K^{-1/p_{k+1}} \sum_{t=1}^K x_{j_{i_t}}$ then

$$|f(x)| < \frac{3 + K^{1/p_{\xi_0}}}{K^{1/p_{k+1}}} + 2\frac{m^{1/p_{\xi_0}}}{m^{1/p_{k+1}}}, \qquad (16)$$

where in the case $p_{\xi_0} = \infty$ we set $1/p_{\xi_0} = 0$.

*Proof.* By Lemma 5.5 we may assume for every $K \leqslant j_1 < \cdots < j_K$ that $(x_{j_i})_{i=1}^K$ is 3-dominated by the unit vector basis of $\ell_{p_{k+1}}^K$. Using Proposition 5.3 we pass to a subsequence, again denoted by $(x_j)_j$ such that for any $j \geqslant j_0 \geqslant 2$, for any $f \in W$ of order strictly smaller than $k$ with $s(f) \geqslant \min\operatorname{supp} x_{j_0}$ we have that

$$|f(x_j)| < (j_0 \max\operatorname{supp} x_{j_0-1})^{-1}. \qquad (17)$$

Let $K \leqslant j_1 < \cdots < j_K$, $x = K^{-1/p_{k+1}} \sum_{i=1}^K x_{j_i}$ and $f = \theta \sum_{q=1}^d c_q f_q$ be a functional of order at most $k$. Using a finite induction on $0 \leqslant k' \leqslant k$, we shall prove that for every $f = \theta \sum_{q=1}^d c_q f_q$ of order at most $k'$ there is $i \in \mathbb{N}$ such that the following holds:

$$|f(x)| < \left(\frac{1-\theta^i}{1-\theta}\right) \frac{2}{K^{1/p_{k+1}}} + \frac{K^{1/p_0}}{K^{1/p_{k+1}}} + 2\frac{m^{1/p_{\xi_0}}}{m^{1/p_{k+1}}}. \qquad (18)$$

The above in conjunction with $0 < \theta \leqslant 1/4$ clearly implies the desired result.

If a functional $f$ is of order-0. Then, by Lemma 6.1 for $i = 1$ we have that (18) holds. Assume that $f = \theta \sum_{q=1}^d c_q f_q$ is of order-$k'$ with $0 < k' \leqslant k$ and that (18) holds for every functional with order strictly smaller than $k'$. Set

$$\begin{aligned}
t_0 &= \min\{t : \operatorname{ran} f \cap \operatorname{ran} x_{j_t} \neq \varnothing\} \text{ and} \\
q_0 &= \min\{q : \max\operatorname{supp} f_q \geqslant \min\operatorname{supp} x_{j_{t_0+1}}\}.
\end{aligned}$$

The same argument used to obtain (10) and (11) in the proof of Lemma 5.5 gives us the following:

$$\left|\theta \sum_{q \neq q_0} c_q f_q(x)\right| < 2/K^{1/p_{k+1}}. \tag{19}$$

By the inductive assumption there exists $i \in \mathbb{N}$ such that:

$$\theta|f_{q_0}(x)| < \theta\left(\left(\frac{1-\theta^i}{1-\theta}\right)\frac{2}{K^{1/p_{k+1}}} + \frac{K^{1/p_0}}{K^{1/p_{k+1}}} + 2\frac{s(f_{q_0})^{1/p_{\xi_0}}}{s(f_{q_0})^{1/p_{k+1}}}\right). \tag{20}$$

By the definition of size for functionals which are not of order-0 we have that $s(f_{q_0}) \geqslant s(f)$ and hence combining (19) and (20) we conclude that:

$$|f(x)| < \left(\frac{1-\theta^{i+1}}{1-\theta}\right)\frac{2}{K^{1/p_{k+1}}} + \theta\left(\frac{K^{1/p_0}}{K^{1/p_{k+1}}} + 2\frac{s(f)^{1/p_{\xi_0}}}{s(f)^{1/p_{k+1}}}\right).$$

□

The next proposition allows us to pass from a block sequence admitting an $\ell_{p_{\xi_0}}$ spreading model to a further block admitting $\ell_{p_1}$ spreading model and from block sequence admitting an $\ell_{p_k}$ spreading model to a further block admitting an $\ell_{p_{k+1}}$ spreading model. In the case that $\xi_0 < \omega$, we use this to show that the spreading models in every subspace are exactly $\ell_p$ for $p \in \{p_1, p_2, \cdots, p_{\xi_0-1}, p_{\xi_0}\}$. In the case that $\xi_0 = \omega$ we require an additional argument to show that we have $\ell_{p_k}$ spreading model for every $k < \omega$ since we are not able to show that every block subspace admits an $\ell_{p_{\xi_0}}$ spreading model.

**Proposition 6.3.** Let $(x_j)_j$ be a normalized block sequence in $X$.

(i) If $(x_j)_j$ generates an $\ell_{p_{\xi_0}}$ spreading model, then there exists a further normalized block sequence $(y_j)_j$ of $(x_j)_j$ that generates an $\ell_{p_1}$ spreading model.

(ii) If $1 \leqslant k < \xi_0$ and $(x_j)_j$ generates an $\ell_{p_k}$ spreading model, then there exists a further normalized block sequence $(y_j)_j$ of $(x_j)_j$ that generates an $\ell_{p_{k+1}}$ spreading model.

*Proof.* Let $(x_j)_j$ be a normalized block sequence in $X$, generating an $\ell_{p_\zeta}$ spreading model, for some $1 \leqslant \zeta \leqslant \xi_0$. Note that by Proposition 4.2 and Lemma 5.5 we may assume that for every $K \leqslant j_1 < \cdots < j_K$ we have that $\|x_{j_1} + \cdots + x_{j_K}\| \leqslant 3 \cdot K^{1/p_\zeta}$. We distinguish three cases concerning $\zeta$, namely $\zeta = \xi_0, \zeta = 1$, and $2 \leqslant \zeta < \xi_0$. We shall only consider the first two cases, as the last one is proved in an identical manner as the case $\zeta = 1$ and uses Lemma 6.2 instead of Lemma 6.1.

*Case 1:* $\zeta = \xi_0$. For every $j \in \mathbb{N}$ choose $f_j \in W$ with $f_j(x_j) = 1$ and $\operatorname{ran} f_j \subset \operatorname{ran} x_j$. Choose an increasing sequence of finite subsets of the natural numbers $(E_j)_j$ with $\#E_j \leqslant \min E_j$ and $\lim_j \#E_j = \infty$. For $j \in \mathbb{N}$ define

$y'_j = (\#E_j)^{-1/p_{\xi_0}} \sum_{i \in E_j} x_i$, $y_j = \|y'_j\|^{-1} y'_j$ and $g_j = \theta \sum_{i \in E_j} (\#E_j)^{-1/p'_{\xi_0}} f_i$. Then we have the following:

(a) The sequence $(y_j)_j$ is a normalized block sequence of $(x_j)_j$ and for every $j \in \mathbb{N}$ we have that $g_j(y_j) \geqslant \theta/3$.

(b) The functional $g_j$ is of order-0 with $s(g_j) = \#E_j$ for all $j \in \mathbb{N}$.

Note that $\lim_j s(g_j) = \infty$ and therefore, passing to a subsequence, we may assume that $(g_j)_j$ is a very fast growing sequence of functionals of order-0. We conclude that $\alpha_{<1}\{(x_j)\} > 0$ and by Proposition 5.6 we have that $(y_j)_j$ is the desired sequence.

*Case 2:* $\zeta = 1$. By Proposition 5.6 we have that $\alpha_{<1}\{(x_j)_j\} > 0$ and hence, by passing to a subsequence, there exists $\varepsilon > 0$ and a very fast growing sequence $(f_j)_j$ of order-0 functionals such that $\operatorname{ran} f_j \subset \operatorname{ran} x_j$ and $f_j(x_j) > \varepsilon$ for all $j \in \mathbb{N}$. Choose an increasing sequence of finite subsets of the natural numbers $(E_j)_j$ with $\min E_j \leqslant \#E_j$ and $\lim_j \#E_j = \infty$. For $j \in \mathbb{N}$ define $y'_j = (\#E_j)^{-1/p_1} \sum_{i \in E_j} x_i$, $y_j = \|y'_j\|^{-1} y'_j$ and $g_j = \theta \sum_{i \in E_j} (\#E_j)^{-1/p'_1} f_j$ (if $p_1 = 1$ take $g_j = \theta \sum_{i \in E_j} f_j$ instead). Then we have the following:

(a') The sequence $(y_j)_j$ is a normalized block sequence of $(x_j)_j$ and for every $j \in \mathbb{N}$ we have that $g_j(y_j) \geqslant \varepsilon\theta/3$.

(b') The functional $g_j$ is of order-1 with $s(g_j) \geqslant \max\{s(f_i) : i \in E_j\}$ for all $j \in \mathbb{N}$.

Once more, $\lim_j s(g_j) = \infty$ and as before we conclude that $\alpha_{<2}\{(x_j)\} > 0$. By Proposition 5.6 it remains to observe that $\alpha_{<1}\{(y_j)\}_j = 0$, which is an easy consequence of the definition of the $y_j$'s and Lemma 6.1. □

**Remark 6.4.** The proof of Proposition 6.3 implies that the space $X$ does not admit an $\ell_{p_\zeta}^{\mathcal{S}_2}$ spreading model for any $1 \leqslant \zeta \leqslant \xi_0$. For the definition of an $\ell_p^{\mathcal{S}_k}$ spreading model see [ABM, Definition 1.1].

**Corollary 6.5.** The space $X$ is reflexive.

*Proof.* Proposition 6.3 implies that neither $c_0$ nor $\ell_1$ embed into $X$. By James' well known theorem for spaces with an unconditional basis we conclude that $X$ is reflexive. □

**Remark 6.6.** If $(z_j)_j$ is a spreading model generated by a non-norm convergent (not necesssarily Schauder basic) sequence in $X$, then [AKT, Remark 5, page 581] the reflexivity of the space and Proposition 5.6 imply that, although the sequence $(z_j)_j$ need not be a Schauder basis for $Z = \overline{\langle \{z_j : j \in \mathbb{N}\} \rangle}$, the space $Z$ must be isomorphic to $\ell_p$, for some $p \in F$.

**Lemma 6.7.** Let $1 \leqslant k < \xi_0$, $K \in \mathbb{N}$ and $(x_j)_j$ be a sequence in $X$ generating an $\ell_{p_k}$ spreading model. Then for every $j_0 \in \mathbb{N}$ there exists a normalized vector $x \in \operatorname{span}(x_j)_{j \geqslant j_0}$ and a functional $f$ of order-0 with $s(f) = K$ such that

$$f(x) \geqslant \frac{\theta}{3} K^{1/p_{\xi_0} - 1/p_k}, \tag{21}$$

where in the case $p_{\xi_0} = \infty$ we set $1/p_{\xi_0} = 0$.

*Proof.* We may clearly assume that $(x_j)_j$ is normalized. Proposition 5.6 in conjunction with 5.5 imply that we may choose $j_0 \leqslant j_1 < \cdots < j_K$ such that if $y = K^{-1/p_k} \sum_{i=1}^{K} x_{j_i}$ then $\|y\| \leqslant 3$. Choose $f_1, \ldots, f_K$ with $\operatorname{ran} f_i \subset \operatorname{ran} x_{j_i}$ and $f_i(x_{j_i}) = 1$ for $i = 1, \ldots, K$ and define $f = \theta \sum_{i=1}^{K} (1/K)^{-1/p'_{\xi_0}} f_i$ and $x = y/\|y\|$. □

**Theorem 6.8.** Let $F = \{p_\zeta : 1 \leqslant \zeta \leqslant \xi_0\}$ and let $Y$ be an infinite dimensional subspace of $X$. Then there exists a dense subset $G$ of $F$ such that the spreading models admitted by $Y$ are exactly the $\ell_p$, for $p \in G$. In particular, $\ell_p$ is finitely block represented in every block subspace of $X$ for every $p \in F$.

*Proof.* Let $Y$ be a block subspace of $X$. We observe the following:

(i) Every spreading model admitted by $Y$ is equivalent to the unit vector basis of $\ell_{p_\zeta}$ for some $1 \leqslant \zeta \leqslant \xi_0$. In particular, there exists $1 \leqslant \zeta_0 \leqslant \xi_0$ such that $Y$ admits an $\ell_{p_{\zeta_0}}$ spreading model.

(ii) If $1 \leqslant k < \xi_0$ and $Y$ admits an $\ell_{p_k}$ spreading model and, then $Y$ also admits an $\ell_{p_{k+1}}$ spreading model.

(iii) If $Y$ admits an $\ell_{p_{\xi_0}}$ spreading model then $Y$ also admits an $\ell_{p_1}$ spreading model.

(iv) There exists $1 \leqslant k_0 < \xi_0$ such that $Y$ admits an $\ell_{p_{k_0}}$ spreading model.

The statement (i) follows from Proposition 5.6. Statements (ii) and (iii) follow from Proposition 6.3, while (iv) follows from the first and the third. We now distinguish two cases regarding whether $\xi_0$ is finite or not.

*Case 1:* If $\xi_0$ is finite, statement (i), statement (ii), and a finite inductive argument yield that $Y$ admits an $\ell_{p_{\xi_0}}$ spreading model. By (iii) we have that $Y$ admits an $\ell_{p_1}$ spreading model. Once more, by a finite induction we obtain that $G = F$ is the desired set.

*Case 2:* If $\xi_0 = \omega$ we shall prove that for every $1 \leqslant k < \xi_0$, $Y$ admits an $\ell_{p_k}$ spreading model. This in particular implies that $G = F$ or $G = F \setminus \{p_{\xi_0}\}$ is the desired set. By (ii) it is sufficient to show that $Y$ admits an $\ell_{p_1}$ spreading model. By Proposition 5.6 it is enough to find a normalized block sequence $(x_j)_j$ in $Y$ and a very fast growing sequence of functionals $(f_j)_j$ of order-0 with $f_j(x_j) > \theta/4$, i.e. $\alpha_{<1}\{(x_j)_j\} > 0$.

Choose a normalized vector $x_1$ in $Y$ and a functional $f \in W$ with $f(x_1) = 1$ and set $f_1 = \theta f$. Then $f_1$ is of order-0 with $s(f) = 1$ and $f_1(x_1) > \theta/4$. Assume that we have chosen normalized vectors $x_1 < \cdots < x_j$ and a very fast growing sequence of functionals $f_1, \ldots, f_j$ of order-0 with $f_i(x_i) > \theta/4$ for $i = 1, \ldots, j$. By (iv), there exists $1 \leqslant k_0 < \xi_0$ such that $Y$ admits an $\ell_{p_{k_0}}$ spreading model. Fix $K > \max \operatorname{supp} f_j$ and choose $k_0 \leqslant k < \xi_0$ such that $K^{1/p_{\xi_0} - 1/p_k} > 3/4$ (recall that $\lim_k p_k = p_{\xi_0}$). By (ii) we may choose a sequence $(y_i)_i$ in $Y$ generating an $\ell_{p_k}$ spreading model. Choose $i_0 \in \mathbb{N}$ with

$\min \operatorname{supp} y_{i_0} \geqslant (\max \operatorname{supp} x_j)^2$ and apply Lemma 6.7 to find the desired pair $x_{j+1}, f_{j+1}$. □

**Remark 6.9.** In the case that $F$ is finite, then clearly the spreading models admitted by every block subspace of $X$ are exactly the $\ell_p$, for $p \in F$. In the case that $F$ consists of an increasing sequence and its limit $p_{\xi_0}$, then it is easily checked that exactly one of the following holds:

(i) The spreading models admitted by every block subspace of $X$ are exactly the $\ell_p$, for $p \in F$.
(ii) There exists a block subspace $Y$ of $X$, such that the spreading models admitted by every further block subspace of $Y$ are exactly the $\ell_p$, for $p \in F \setminus \{p_{\xi_0}\}$.

We were unable to determine which one of the above holds, in either case however on some subspace $Y$ of $X$, the set of spreading models admitted by every further subspace of $Y$ is stabilized.

## 7. The set of Krivine $p$'s of the space $X$.

In this section we prove that for any $p \notin F = \{p_\zeta : 1 \leqslant \zeta \leqslant \xi_0\}$, $\ell_p$ is not finitely block represented in the space $X$. We conclude that the set of $p$'s that are finitely block represented in every block subspace of $X$ is exactly the set $F$, which is not connected.

We begin with the following Lemma, whose proof we omit as it follows from the same argument as the proof of Lemma 6.1.

**Lemma 7.1.** Let $p \in [p_1, p_{\xi_0}] \setminus F$. Suppose $\varepsilon > 0$ and $(x_j)_{j=1}^N$ is a finite block sequence in $X$ which is $(1+\varepsilon)$-equivalent to the unit vector basis of $\ell_p^N$. If $1 \leqslant \zeta \leqslant \xi_0$ is such that $p < p_\zeta$ and $f$ is a functional of order-$\zeta$, then we have the following estimate:

$$\left| f\left(\frac{1}{N^{1/p}} \sum_{j=1}^N x_j\right)\right| < (1+\varepsilon)\left(\frac{N^{1/p_\zeta}}{N^{1/p}} + 2\theta\right). \tag{22}$$

The next lemma follows directly from the above lemma.

**Lemma 7.2.** Suppose that $(x_j)_{j=1}^N$ is a finite block sequence in $X$ that is $(1+\varepsilon)$-equivalent to the unit vector basis of $\ell_p^N$, $1 \leqslant k < \xi_0$ satisfies $p < p_{k+1}$ and $N$ satifies

$$N^{1/p_{k+1} - 1/p} + 2\theta < (1+\varepsilon)^{-2}.$$

If $f \in W$ satisfies $f(N^{-1/p} \sum_{j=1}^N x_j) \geqslant 1/(1+\varepsilon)$ then $f$ has non-zero order less than or equal to $k$.

We are now ready to prove the second main theorem.

**Theorem 7.3.** For all $p \in [1, \infty] \setminus F$ there exists $K \in \mathbb{N}$ and $\varepsilon > 0$ such that no block sequence $(x_j)_{j=1}^K$ in $X$ is $(1+\varepsilon)$-equivalent to the unit vector basis of $\ell_p^K$.

*Proof.* Let $p \in [1,\infty] \setminus F$. If $p \notin [p_1, p_{\xi_0}]$, then the result clearly follows from Proposition 4.2. Otherwise, we have that $p \in [p_1, p_{\xi_0}] \setminus F$. Find $k \in \mathbb{N}$ so that $p_k < p < p_{k+1}$. Find $N, M \in \mathbb{N}$ and $\varepsilon > 0$ as follows:

Choose $N \in \mathbb{N}$ such that

$$N^{1/p} > 2 + \theta(N-2)^{1/p} \text{ and} \tag{23}$$

$$\frac{N^{1/p_{k+1}}}{N^{1/p}} < 1 - 2\theta. \tag{24}$$

Now that $N$ is fixed, we choose $\varepsilon > 0$ such that:

$$\frac{1}{1+\varepsilon} N^{1/p} > 2 + (1+\varepsilon)\theta(N-2)^{1/p}, \tag{25}$$

$$N^{1/p} > (1+\varepsilon)^2 (N-1)^{1/p} \text{ and} \tag{26}$$

$$\frac{N^{1/p_{k+1}}}{N^{1/p}} < \frac{1}{(1+\varepsilon)^2} - 2\theta. \tag{27}$$

We set

$$\Theta = \frac{1}{1+\varepsilon} N^{1/p} - (1+\varepsilon)(N-1)^{1/p}. \tag{28}$$

Notice (26) implies that $\Theta > 0$. Finally, let $M \in \mathbb{N}$ so that

$$M^{1/p_k}\Theta > (1+\varepsilon) M^{1/p}. \tag{29}$$

Let $K = (N-1)M + 1$ and consider the following normalized block sequence which, towards a contradiction, we assume is $(1+\varepsilon)$-equivalent to the unit vector basis of $\ell_p^K$ and that $M < \min \operatorname{supp} x_1$.

$$x_1 < x_2^1 < x_3^1 < \cdots < x_N^1 < x_2^2 < x_3^2 < \cdots < x_N^{M-1} < x_2^M < \cdots < x_N^M.$$

(i.e. $x_j^m < x_i^m$ for $i < j$ and $x_N^m < x_2^{m+1}$). Let us mark the following, which is obviously true.

(a) For each $m$ with $1 \leqslant m \leqslant M$ the block sequence $x_1 < x_2^m < \cdots < x_N^m$ is $(1+\varepsilon)$-equivalent to the unit vector basis of $\ell_p^N$.

Fix $m$ with $1 \leqslant m \leqslant M$. For notational reasons we set $x_1^m = x_1$. Find $g_m \in W$ with $g_m(\sum_{i=1}^N x_i^m) \geqslant \frac{N^{1/p}}{(1+\varepsilon)}$. By Lemma 7.2 and (27) we conclude that $g_m$ has non-zero order less than or equal to $k$. Let

$$g_m = \theta \sum_{q=1}^{d_m} c_{m,q} f_{m,q}.$$

be the functionals decomposition according to Remark 3.3, i.e.:

(b) the coefficients $(c_{m,q})_{q=1}^{d_m}$ are in the unit ball of $\ell_{p_k'}$ for $m = 1, \ldots, M$ and

(c) the sequence $(f_{m,q})_{q=1}^{d_m}$ is an admissible and very fast growing sequence of functionals, each one of which has order strictly smaller than $k$.

Define

$$q_m = \min\{q : \min\operatorname{supp} x_2^m \leqslant \max\operatorname{supp} f_{m,q}\}$$

We will prove the following three claims:

(i) $g_m(x_1) > \Theta$, $g_m(x_N^m) > \Theta$ and $d_m \leqslant \max\operatorname{supp} x_1$.
(ii) The number $q_m$ exists, $\max\operatorname{supp} f_{m,q_m} < \min\operatorname{supp} x_N^m$ and $q_m < d_m$.
(iii) $(\min\operatorname{supp} x_2^m)^2 < \min\operatorname{supp} f_{m,q_m+1}$ and $\min\operatorname{supp} x_2^m < s(f_{m,q_m+1})$.

Item (i): Using that $g_m(\sum_{i=1}^N x_i^m) \geqslant \frac{N^{1/p}}{(1+\varepsilon)}$, (a) and (28) we have

$$\begin{aligned} g_m(x_1) &= g_m\left(\sum_{j=1}^N x_j^m\right) - g_m\left(\sum_{j=2}^N x_j^m\right) \\ &\geqslant \frac{1}{1+\varepsilon}N^{1/p} - (1+\varepsilon)(N-1)^{1/p} = \Theta > 0. \end{aligned} \tag{30}$$

The same argument works to show that $g_m(x_N^m) > \Theta$. If $d_m > \max\operatorname{supp} x_1$ then $\max\operatorname{supp} x_1 < \min\operatorname{supp} g_m$ which implies that $g_m(x_1) = 0$. This contradiction tells us that $d_m \leqslant \max\operatorname{supp} x_1$.

Item (ii): If $q_m$ did not exist then $\min\operatorname{supp} x_2^m > \max\operatorname{supp} f_{m,q}$ for all $q$ and so $g_m(\sum_{j=1}^N x_j^m) = g_m(x_1) \leqslant 1$. On the other hand we clearly have $g_m(\sum_{j=1}^N x_j^m) > 2$ and so $q_m$ exists.

If $\max\operatorname{supp} f_{m,q_m} \geqslant \min\operatorname{supp} x_N^m$ then we have:

$$\begin{aligned} g_m\left(\sum_{j=1}^N x_j^m\right) &= g_m(x_1) + \theta c_{m,q_m} f_{m,q_m}\left(\sum_{j=2}^{N-1} x_j^m\right) + g_m(x_N^m) \\ &\leqslant 2 + (1+\varepsilon)\theta(N-2)^{1/p} < \frac{N^{1/p}}{(1+\varepsilon)}. \end{aligned} \tag{31}$$

The last inequality uses (25). This contradicts that fact that $g_m(\sum_{j=1}^N x_j^m) \geqslant \frac{N^{1/p}}{(1+\varepsilon)}$.

Using item (i) we have $g_m(x_N^m) > \Theta$. This fact combined with the fact that $\max\operatorname{supp} f_{m,q_m} < \min\operatorname{supp} x_N^m$ gives us $q_m < d_m$.

Item (iii): By definition of $q_m$ and the fact that $(f_{m,q})_{q=1}^{d_m}$ is very fast growing

$$(\min\operatorname{supp} x_2^m)^2 \leqslant (\max\operatorname{supp} f_{m,q_m})^2 < \min\operatorname{supp} f_{m,q_m+1} \text{ and}$$

$$\min\operatorname{supp} x_2^m < s(f_{m,q_m+1}).$$

This proves (iii).

Note that $q_m + 1 \leqslant d_m$ by item (ii). Define

$$f_m := g_m\big|_{\operatorname{ran} x_N^m} = \theta \sum_{q=q_m+1}^{d_m} c_{m,q} f_{m,q}\big|_{\operatorname{ran} x_N^m}.$$

We claim that

$$\frac{1}{M^{1/p'_k}}\sum_{m=1}^{M} f_m \in W. \tag{32}$$

We first assume that (32) holds and finish the proof of our theorem as follows:

$$M^{1/p}(1+\varepsilon) \geqslant \frac{1}{M^{1/p'_k}}\sum_{m=1}^{M} f_m\left(\sum_{m=1}^{M} x_N^m\right) = \frac{1}{M^{1/p'_k}}\sum_{m=1}^{M} f_m(x_N^m) \geqslant M^{1/p_k}\Theta.$$

This contradicts (29).

All that remains to prove is (32). Note that

$$\frac{1}{M^{1/p'_k}}\sum_{m=1}^{M} f_m = \theta\sum_{m=1}^{M}\sum_{q=q_m+1}^{d_m}\left(c_{m,q}/M^{1/p'_k}\right) f_{m,q}\big|_{\operatorname{ran} x_N^m}.$$

Using (b) we obtain that $\sum_{m=1}^{M}\sum_{q=q_m+1}^{d_m}(c_{m,q}/M^{1/p'_k})^{p'_k} \leqslant 1$ and therefore it suffices to show that $((f_{m,q})_{q=q_m+1}^{d_m})_{1\leqslant m\leqslant M}$ is an admissible and very fast growing sequence of functionals, each one of which has order strictly smaller than $k$, which will imply that $f$ is a funtional in $W$ of order-$k$. First we check admissibility:

$$\begin{aligned}\sum_{m=1}^{M} d_m &\leqslant M \max\operatorname{supp} x_1\\ &\leqslant \min\operatorname{supp} x_2^1 \cdot \min\operatorname{supp} x_2^1\\ &\leqslant \min\operatorname{supp} f_{1,q_1+1}.\end{aligned} \tag{33}$$

The first inequality follows from item (i), the second from that fact that $M < \max\operatorname{supp} x_1 < \min\operatorname{supp} x_2^1$ and the third comes from item (iii) (for $m=1$).

Note that by (b) the functionals under consideration have order strictly smaller than $k$ and for each $m$ with $1\leqslant m\leqslant M$ the collection $(f_{m,q})_{q=q_m+1}^{d_m}$ is very fast growing. At last, it suffices to show for each $m\in\mathbb{N}$ with $2\leqslant m\leqslant M$ that

$$(\max\operatorname{supp} f_{m-1,d_{m-1}})^2 < \min\operatorname{supp} f_{m,q_m+1} \text{ and}$$

$$\max\operatorname{supp} f_{m-1,d_{m-1}} < s(f_{m,q_m+1}).$$

This, however, follows from item (iii) since

$$\max\operatorname{supp} f_{m-1,d_{m-1}} < \min\operatorname{supp} x_2^m.$$

This proves the claim and finishes the proof of the theorem. □

We are interested in three problems related to the present work.

**Problem 1.** Let $1<p_1<p_2<\infty$. Is the space $X_{p_1,p_2}$ constructed in this paper super-reflexive?

**Problem 2.** Let $1 \leqslant p_1 < p_2 \leqslant \infty$. Does there exist a space so that in every block subspace the Krivine set is $[p_1, p_2]$? More generally, which types of closed sets can be hereditary Krivine sets?

**Problem 3.** Let $F \subset [2, \infty)$ be finite. Does there exist a Banach space $X$ such that for every infinite dimensional subspace $Y$ of $X$, $\ell_p$ is finitely represented in $Y$ if and only if $p \in \{2\} \cup F$? In particular, does our construction satisfy this?

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