Billiken Challenge

Solutions - Winter 2020

A difference triangle

There is a unique way to do this, up to symmetry.

6		14		15		3		13
	8		1		12		10	
		7		11		2		
			4		9			
				5				

For smaller cases, up to symmetry, there are two ways to do a triangle of three numbers, two ways to do a triangle of six numbers, and two ways to do a triangle of ten numbers.

It is impossible to do this for larger triangles. A proof is given by G. J. Chang, M. C. Hu, K. W. Lih and T. C. Shieh in "Exact Difference Triangles," Bulletin of the Institute of Mathematics, Academia Sinica, Taipei, Taiwan (vol. 5, June 1977, pages 191- 197).

A differential disaster

Suppose f and g satisfy the naive product rule, so $(fg)' = f'g'$. Then $(f - f')(g - g') = fg - f'g - g'f + f'g' = fg - (fg)' + f'g' = fg$. Divide by fg to get

$$(1 - f'/f)(1 - g'/g) = 1$$

or

$$(1 - (\log(f))')(1 - (\log(g))') = 1$$

Now let $u = \log(f)'$, $v = \log(g)'$. Then $(1 - u)(1 - v) = 1$ so $v = \frac{u}{u-1}$.

Essentially any function u then leads to the solution:

$$f = e^{\int u}; \quad g = e^{\int \frac{u}{u-1}}$$

One simple choice is to let $u = c$, a constant. Then we find the solution $f = e^{cx}$, $g = e^{(\frac{c}{c-1})x}$.

For a concrete example, choose $c = 2$ so that $f = g = e^{2x}$.

Another simple solution occurs when $u = \frac{1}{x}$, leading to $f = x$ and $g = \frac{1}{1-x}$.