Billiken Challenge

Solutions - Summer 2020

Colored cubes

Each corner of a cube has an opposite corner, and each edge of a cube has an opposite edge. Suppose you have a cube with edges colored red and blue, and no pair of opposite edges are the same color. Show that there is a monochromatic path joining some pair of opposite corners.

Solution:

Pick any corner O of the cube. Three edges touch O, so two must be the same color, and suppose that color is blue. Those two blue edges form an L shaped path joining two other corners, call them X and Y. Let Z be the corner opposite X. Y and Z are connected by an edge. If the edge YZ is blue, we are done, since $XOYZ$ is a blue path.

Suppose the edge YZ is red. Then the edge opposite to YZ joins X to a new point W, and S is the corner opposite to Y. Since YZ is red, WX is blue, and therefore $WXOY$ is a blue path joining opposite corners W and Y.

$\square$

This question generalizes to hypercubes. The n-dimensional hypercube has 2^n vertices and $n2^{n-1}$ edges. Again, each vertex has an opposite vertex and each edge has an opposite edge. If you two-color the edges so that each no opposite edges are the same color, is there a monochromatic path joining some two opposite vertices?

The answer is yes for $n = 2, 3, 4, 5$ and unknown for $n \geq 6$.

See `http://garden.irmacs.sfu.ca/op/edge_antipodal_colorings_of_cubes`.

Tower of cubes

Stack infinitely many cubes of side lengths $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ on a table. What is the visible surface area of the stack?

Solution: Each face of the cube of side $\frac{1}{n}$ is visible and has area $\frac{1}{n^2}$. So the total area of visible sides is four times the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Summing this series is the “Basel problem”, solved by Euler in 1734. There are many approaches to compute this sum, which is equal to $\frac{\pi^2}{6}$.

Next, the total top surface of all the visible cubes is 1, because if you view the stack from above you will see a full square.

Therefore, the total surface area of the stack is $1 + 4 \cdot \frac{\pi^2}{6} \approx 7.58$.

$\square$