Billiken Challenge Solutions
July 2019

Palindrome

A palindrome is a number that reads the same backwards and forwards, such as 150323051.

Prove or disprove: $(111\cdots111)^2$ is always a palindrome, no matter how many 1's there are.

Solution: The first nine of these are palindromes:

$$\begin{aligned}
1^2 &= 1\\
11^2 &= 121\\
111^2 &= 12321\\
1111^2 &= 1234321\\
11111^2 &= 123454321\\
111111^2 &= 12345654321\\
1111111^2 &= 1234567654321\\
11111111^2 &= 123456787654321\\
111111111^2 &= 12345678987654321.
\end{aligned}$$

However, the 10th is $1111111111^2 = 1234567900987654321$, which is not a palindrome. These first nine are called the "wonderful Demlo numbers", studied by the Indian mathematician D. R. Kaprekar in the 1930's and named after the Demlo train station, some 30 miles from Bombay.

2019

Show that some multiple of 2019 is a number all of whose digits are ones.

Solution: Let's call a number of the form $1111\cdots1111$ a rep-one. Consider the 2020 rep-ones $1, 11, 111, 1111, \ldots, 1111\cdots1111$, where the last number has 2020 ones. Reduce these rep-ones modulo 2019, that is, compute the remainder after division by 2019 for each rep-one. Since this gives 2020 remainders each less than 2019, by the pigeonhole principle two of these remainders must be the same. The difference of the two rep-ones is then divisible by 2019 and is also a number of the form:

$$1111\cdots11110000\cdots0000$$

which we can write as:

$$1111\cdots1111 \times 10000\cdots0000.$$

Since 2019 divides this product, and is not divisible by 2 or 5, 2019 must divide the left term, $1111\cdots1111$. A computer search reveals that:

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1111111111111111111111111111111111111111111111111111111111111111111111111
1111111111111111111111111111111111111111111 = 2019 * 55032744482967365
5825216003522095646909911397281382422541412140223432942600847 5
0426503769742997083264542402729624126355181332893071377469594 4
0867316053051565681580540421550822739530020362115458697925265 5
3299213031753893566672169941114963403224918826701887623135765 7
8063948049089208078806890099609267514170931704364096637499312 0
9069396290793021847999559738044136261075339827197182323482472 0
7088217489406196687028782125364591932199658796984205602333388 3
6607781630069891585493368554289802432447306147157558747454735 5
6766275934180837598371030763304165978757360629574596885146662 2
6404710802927742006493863848990149138737548841560728633536954 4
879203125859886632546365087226 9
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where the rep-one has 672 ones.

This problem is derived from an exercise in Rosen's *Discrete Mathematics*.