1.2. Let \( S = \{ -2, -1, 0, 1, 2, 3 \} \). Describe each of the following sets as \( \{ x \in S : \ p(x) \} \), where \( p(x) \) is some condition on \( x \).

(a) \( A = \{ 1, 2, 3 \} \)
(b) \( B = \{ 0, 1, 2, 3 \} \)
(c) \( C = \{ -2, -1 \} \)
(d) \( D = \{ -2, 2, 3 \} \)

1.3. Determine the cardinality of each of the following sets:

(a) \( A = \{ 1, 2, 3, 4, 5 \} \)
(b) \( B = \{ 0, 2, 4, \ldots , 20 \} \)
(c) \( C = \{ 25, 26, 27, \ldots , 75 \} \)
(d) \( D = \{ 1, 2 \}, \{ 1, 2, 3, 4 \} \)
(e) \( E = \{ \emptyset \} \)
(f) \( F = \{ 2, \{ 2, 3, 4 \} \} \)

1.4. Write each of the following sets by listing its elements within braces.

(a) \( A = \{ n \in \mathbb{Z} : -4 < n \leq 4 \} \)
(b) \( B = \{ n \in \mathbb{Z} : n^2 < 5 \} \)
(c) \( C = \{ n \in \mathbb{N} : n^3 < 100 \} \)
(d) \( D = \{ x \in \mathbb{R} : x^2 - x = 0 \} \)
(e) \( E = \{ x \in \mathbb{R} : x^2 + 1 = 0 \} \)

1.5. Write each of the following sets in the form \( \{ x \in \mathbb{Z} : \ p(x) \} \), where \( p(x) \) is a property concerning \( x \).

(a) \( A = \{ -1, -2, -3, \ldots \} \)
(b) \( B = \{ -3, -2, \ldots , 3 \} \)
(c) \( C = \{ -2, -1, 1, 2 \} \)

1.6. The set \( E = \{ 2x : x \in \mathbb{Z} \} \) can be described by listing its elements, namely \( E = \{ \ldots , -4, -2, 0, 2, 4, \ldots \} \). List the elements of the following sets in a similar manner.

(a) \( A = \{ 2x + 1 : x \in \mathbb{Z} \} \)
(b) \( B = \{ 4n : n \in \mathbb{Z} \} \)
(c) \( C = \{ 3q + 1 : q \in \mathbb{Z} \} \)

1.7. The set \( E = \{ \ldots , -4, -2, 0, 2, 4, \ldots \} \) of even integers can be described by means of a defining condition by \( E = \{ y = 2x : x \in \mathbb{Z} \} = \{ 2x : x \in \mathbb{Z} \} \). Describe the following sets in a similar manner.

(a) \( A = \{ \ldots , -4, -1, 2, 5, 8, \ldots \} \)
(b) \( B = \{ \ldots , -10, -5, 0, 5, 10, \ldots \} \)
(c) \( C = \{ 1, 8, 27, 64, 125, \ldots \} \)

1.8. Let \( A = \{ n \in \mathbb{Z} : 2 \leq |n| < 4 \} \), \( B = \{ x \in \mathbb{Q} : 2 < x \leq 4 \} \), \( C = \{ x \in \mathbb{R} : x^2 - (2 + \sqrt{2})x + 2\sqrt{2} = 0 \} \) and \( D = \{ x \in \mathbb{Q} : x^2 - (2 + \sqrt{2})x + 2\sqrt{2} = 0 \} \).

(a) Describe the set \( A \) by listing its elements.
(b) Give an example of three elements that belong to \( B \) but do not belong to \( A \).
(c) Describe the set \( C \) by listing its elements.
(d) Describe the set \( D \) in another manner.
(e) Determine the cardinality of each of the sets \( A, C \) and \( D \).
1.9. For \( A = \{2, 3, 5, 7, 8, 10, 13\} \), let 

\[ B = \{x \in A : x = y + z, \text{ where } y, z \in A\} \text{ and } C = \{r \in B : r + s \in B \text{ for some } s \in B\}. \]

Determine \( C \).

**Section 1.2: Subsets**

1.10. Give examples of three sets \( A, B \) and \( C \) such that

(a) \( A \subseteq B \subset C \)
(b) \( A \in B, B \in C \) and \( A \notin C \)
(c) \( A \in B \) and \( A \subset C \).

1.11. Let \((a, b)\) be an open interval of real numbers and let \( c \in (a, b) \). Describe an open interval \( I \) centered at \( c \) such that \( I \subseteq (a, b) \).

1.12. Which of the following sets are equal?

\[ A = \{n \in \mathbb{Z} : |n| < 2\} \quad D = \{n \in \mathbb{Z} : n^2 \leq 1\} \]

\[ B = \{n \in \mathbb{Z} : n^3 = n\} \quad E = \{-1, 0, 1\} \]

\[ C = \{n \in \mathbb{Z} : n^2 \leq n\} \]

1.13. For a universal set \( U = \{1, 2, \ldots, 8\} \) and two sets \( A = \{1, 3, 4, 7\} \) and \( B = \{4, 5, 8\} \), draw a Venn diagram that represents these sets.

1.14. Find \( \mathcal{P}(A) \) and \( |\mathcal{P}(A)| \) for

(a) \( A = \{1, 2\} \).
(b) \( A = \{\emptyset, 1, \{a\}\} \).

1.15. Find \( \mathcal{P}(A) \) for \( A = \{0, \{0\}\} \).

1.16. Find \( \mathcal{P}(\mathcal{P}(\{1\})) \) and its cardinality.

1.17. Find \( \mathcal{P}(A) \) and \( |\mathcal{P}(A)| \) for \( A = \{0, \emptyset, \{\emptyset\}\} \).

1.18. For \( A = \{x : x = 0 \text{ or } x \in \mathcal{P}(\{0\})\} \), determine \( \mathcal{P}(A) \).

1.19. Give an example of a set \( S \) such that

(a) \( S \subseteq \mathcal{P}(\mathbb{N}) \)
(b) \( S \in \mathcal{P}(\mathbb{N}) \)
(c) \( S \subseteq \mathcal{P}(\mathbb{N}) \) and \( |S| = 5 \)
(d) \( S \in \mathcal{P}(\mathbb{N}) \) and \( |S| = 5 \)

1.20. Determine whether the following statements are true or false.

(a) If \( \{1\} \in \mathcal{P}(A) \), then \( 1 \in A \) but \( \{1\} \notin A \).
(b) If \( A, B \) and \( C \) are sets such that \( A \subset \mathcal{P}(B) \subset C \) and \( |A| = 2 \), then \( |C| \) can be 5 but \( |C| \) cannot be 4.
(c) If a set \( B \) has one more element than a set \( A \), then \( \mathcal{P}(B) \) has at least two more elements than \( \mathcal{P}(A) \).
(d) If four sets \( A, B, C \) and \( D \) are subsets of \( \{1, 2, 3\} \) such that \( |A| = |B| = |C| = |D| = 2 \), then at least two of these sets are equal.

1.21. Three subsets \( A, B \) and \( C \) of \( \{1, 2, 3, 4, 5\} \) have the same cardinality. Furthermore,

(a) 1 belongs to \( A \) and \( B \) but not to \( C \).
(b) 2 belongs to \( A \) and \( C \) but not to \( B \).
(c) 3 belongs to \( A \) and exactly one of \( B \) and \( C \).
(d) 4 belongs to an even number of \( A, B \) and \( C \).
(e) 5 belongs to an odd number of \( A, B \) and \( C \).
(f) The sums of the elements in two of the sets \( A, B \) and \( C \) differ by 1.

What is \( B \)?

Section 1.3: Set Operations

1.22. Let \( U = \{1, 3, \ldots, 15\} \) be the universal set, \( A = \{1, 5, 9, 13\} \), and \( B = \{3, 9, 15\} \). Determine the following:

(a) \( A \cup B \)  (b) \( A \cap B \)  (c) \( A - B \)  (d) \( B - A \)  (e) \( \overline{A} \)  (f) \( A \cap \overline{B} \).

1.23. Give examples of two sets \( A \) and \( B \) such that \( |A - B| = |A \cap B| = |B - A| = 3 \). Draw the accompanying Venn diagram.

1.24. Give examples of three sets \( A, B \) and \( C \) such that \( B \neq C \) but \( B - A = C - A \).

1.25. Give examples of three sets \( A, B \) and \( C \) such that

(a) \( A \in B \), \( A \subseteq C \) and \( B \nsubseteq C \)
(b) \( B \in A \), \( B \subset C \) and \( A \cap C \neq \emptyset \)
(c) \( A \in B \), \( B \subseteq C \) and \( A \nsubseteq C \).

1.26. Let \( U \) be a universal set and let \( A \) and \( B \) be two subsets of \( U \). Draw a Venn diagram for each of the following sets:

(a) \( \overline{A} \cup B \)  (b) \( \overline{A} \cap \overline{B} \)  (c) \( \overline{A} \cap B \)  (d) \( A \cup \overline{B} \).

What can you say about parts (a) and (b)? parts (c) and (d)?

1.27. Give an example of a universal set \( U \), two sets \( A \) and \( B \) and accompanying Venn diagram such that \( |A \cap B| = |A - B| = |B - A| = |A \cup \overline{B}| = 2 \).

1.28. Let \( A, B \) and \( C \) be nonempty subsets of a universal set \( U \). Draw a Venn diagram for each of the following set operations.

(a) \( (C - B) \cup A \)
(b) \( C \cap (A - B) \).

1.29. Let \( A = \{\emptyset, \{\emptyset\}, \{\emptyset\}\} \).

(a) Determine which of the following are elements of \( A \): \( \emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\} \).
(b) Determine \( |A| \).
(c) Determine which of the following are subsets of \( A \): \( \emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\} \).

For (d)–(i), determine the indicated sets.

(d) \( \emptyset \cap A \)
(e) \( \{\emptyset\} \cap A \)
(f) \( \{\emptyset, \{\emptyset\}\} \cap A \)
(g) \( \emptyset \cup A \)
(h) \( \{\emptyset\} \cup A \)
(i) \( \{\emptyset, \{\emptyset\}\} \cup A \).

1.30. Let \( A = \{x \in \mathbb{R} : |x - 1| \leq 2\} \), \( B = \{x \in \mathbb{R} : |x| \geq 1\} \) and \( C = \{x \in \mathbb{R} : |x + 2| \leq 3\} \).

(a) Express \( A, B \) and \( C \) using interval notation.
(b) Determine each of the following sets using interval notation:
   \( A \cup B, A \cap B, B \cap C, B - C \).

1.31. Give an example of four different sets \( A, B, C \) and \( D \) such that (1) \( A \cup B = \{1, 2\} \) and \( C \cap D = \{2, 3\} \) and
   (2) if \( B \) and \( C \) are interchanged and \( \cup \) and \( \cap \) are interchanged, then we get the same result.
1.32. Give an example of four different subsets $A, B, C$ and $D$ of $\{1, 2, 3, 4\}$ such that all intersections of two subsets are different.

1.33. Give an example of two nonempty sets $A$ and $B$ such that $\{A \cup B, A \cap B, A - B, B - A\}$ is the power set of some set.

1.34. Give an example of two subsets $A$ and $B$ of $\{1, 2, 3\}$ such that all of the following sets are different: $A \cup B$, $A \cup \overline{B}$, $A \cup B$, $A \cup B$, $A \cap B$, $A \cap \overline{B}$, $\overline{A} \cap B$, $\overline{A} \cap \overline{B}$.

1.35. Give examples of a universal set $U$ and sets $A, B$ and $C$ such that each of the following sets contains exactly one element: $A \cap B \cap C$, $(A \cap B) - C$, $(A \cap C) - B$, $(B \cap C) - A$, $A - (B \cup C)$, $B - (A \cup C)$, $C - (A \cup B)$, $A \cup B \cup C$. Draw the accompanying Venn diagram.

Section 1.4: Indexed Collections of Sets

1.36. For a real number $r$, define $A_r$ to be the interval $[r - 1, r + 2]$. Let $A = \{1, 3, 4\}$. Determine $\bigcup_{a \in A} A_a$ and $\bigcap_{a \in A} A_a$.

1.37. Let $A = \{1, 2, 5\}$, $B = \{0, 2, 4\}$, $C = \{2, 3, 4\}$ and $S = \{A, B, C\}$. Determine $\bigcup_{X \in S} X$ and $\bigcap_{X \in S} X$.

1.38. For a real number $r$, define $A_r = \{r^2\}$, $B_r$ as the closed interval $[r - 1, r + 1]$ and $C_r$ as the interval $(r, \infty)$. For $S = \{1, 2, 4\}$, determine
   (a) $\bigcup_{a \in S} A_a$ and $\bigcap_{a \in S} A_a$
   (b) $\bigcup_{a \in S} B_a$ and $\bigcap_{a \in S} B_a$
   (c) $\bigcup_{a \in S} C_a$ and $\bigcap_{a \in S} C_a$.

1.39. Let $A = \{a, b, \ldots, z\}$ be the set consisting of the letters of the alphabet. For $\alpha \in A$, let $A_\alpha$ consist of $\alpha$ and the two letters that follow it, where $A_r = \{z, a, \alpha\}$ and $A_\alpha = \{z, a, b\}$. Find a set $S \subseteq A$ of smallest cardinality such that $\bigcup_{\alpha \in S} A_\alpha = A$. Explain why your set $S$ has the required properties.

1.40. For $i \in \mathbb{Z}$, let $A_i = \{i - 1, i + 1\}$. Determine the following:
   (a) $\bigcup_{i=1}^{5} A_{2i}$
   (b) $\bigcup_{i=1}^{5} (A_{i} \cap A_{i+1})$
   (c) $\bigcup_{i=1}^{5} (A_{2i-1} \cap A_{2i+1})$.

1.41. For each of the following, find an indexed collection $(A_n)_{n \in \mathbb{N}}$ of distinct sets (that is, no two sets are equal) satisfying the given conditions.
   (a) $\bigcap_{n=1}^{\infty} A_n = \{0\}$ and $\bigcup_{n=1}^{\infty} A_n = [0, 1]$
   (b) $\bigcap_{n=1}^{\infty} A_n = (-1, 0, 1)$ and $\bigcup_{n=1}^{\infty} A_n = \mathbb{Z}$.

1.42. For each of the following collections of sets, define a set $A_n$ for each $n \in \mathbb{N}$ such that the indexed collection $(A_n)_{n \in \mathbb{N}}$ is precisely the given collection of sets. Then find both the union and intersection of the indexed collection of sets.
   (a) $\{[1, 2 + 1], [1, 2 + 1/2], [1, 2 + 1/3], \ldots\}$
   (b) $\{(-1, 2), (-3/2, 4), (-5/3, 6), (-7/4, 8), \ldots\}$.

1.43. For $r \in \mathbb{R}^+$, let $A_r = \{x \in \mathbb{R} : \{x\} < r\}$. Determine $\bigcup_{r \in \mathbb{R}^+} A_r$ and $\bigcap_{r \in \mathbb{R}^+} A_r$.

1.44. Each of the following sets is a subset of $A = \{1, 2, \ldots, 10\}$:
   $A_1 = \{1, 5, 7, 9, 10\}$, $A_2 = \{1, 2, 3, 8, 9\}$, $A_3 = \{2, 4, 6, 8, 9\}$,
   $A_4 = \{2, 4, 8\}$, $A_5 = \{3, 6, 7\}$, $A_6 = \{3, 8, 10\}$, $A_7 = \{4, 5, 7, 9\}$,
   $A_8 = \{4, 5, 10\}$, $A_9 = \{4, 6, 8\}$, $A_{10} = \{5, 6, 10\}$,
   $A_{11} = \{5, 8, 9\}$, $A_{12} = \{6, 7, 10\}$, $A_{13} = \{6, 8, 9\}$.
   Find a set $I \subseteq \{1, 2, \ldots, 13\}$ such that for every two distinct elements $j, k \in I$, $A_j \cap A_k = \emptyset$ and $|\bigcup_{i \in I} A_i|$ is maximum.

1.45. For $n \in \mathbb{N}$, let $A_n = (-\frac{1}{n}, 2 - \frac{1}{n})$. Determine $\bigcup_{n \in \mathbb{N}} A_n$ and $\bigcap_{n \in \mathbb{N}} A_n$. 