1. Chapter 2.2 Conditional Statements

- If $p$ and $q$ are statement variables, the **conditional** of $q$ by $p$ is "If $p$ then $q$" or "$p$ implies $q$" and is denoted $p \implies q$. It is false when $p$ is true and $q$ is false; otherwise it is true. We call $p$ the **hypothesis** (or **antecedent**) of the conditional and $q$ the **conclusion** (or **consequent**).

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<th>$p \implies q$</th>
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- A conditional statement that is true by virtue of the fact that its hypothesis is false is called **vacuously true** or **true by default**. In general, when the "if" part of an if-then statement is false, the statement as a whole is said to be true, regardless of whether the conclusion is true or false.
  
  For example: If $0 = 1$, then $1 = 2$.

- **NOTE**: The **order of operations** for evaluating statements is $\sim$ first, then $\lor$ and $\land$, and finally $\implies$.

  For example: Construct the truth table for the statement $p \lor \sim q \implies \sim p$.

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<thead>
<tr>
<th></th>
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<th>$\sim q$</th>
<th>$p \lor \sim q$</th>
<th>$p \lor \sim q \implies \sim p$</th>
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1.1. **In Class Group Work**: Show that $p \lor q \implies r \equiv (p \implies r) \land (q \implies r)$. 


1.2. **Representation of If-Then as Or and The Negation of a Conditional Statement.**

- You can write \( p \rightarrow q \) as \( \sim p \lor q \).
- The negation of "if \( p \) then \( q \)" is logically equivalent to "\( p \) and not \( q \)," that is, 
  \[ \sim (p \rightarrow q) \equiv p \land \sim q. \]

1.3. **In Class Group Work:** First, show that \( p \rightarrow q \equiv \sim p \lor q \). Then, show that \( \sim (p \rightarrow q) \equiv p \land \sim q \). Finally, write down a conditional statement and then negate it.
1.4. **Contrapositive, Converse, Inverse**—Words that made you tremble in high school geometry.

- The **contrapositive** of a conditional statement of the form $p \rightarrow q$ is: If $\sim q \rightarrow \sim p$.
- A conditional statement is logically equivalent to its contrapositive! (This is very useful for proof writing!)
- The **converse** of $p \rightarrow q$ is $q \rightarrow p$.
- The **inverse** of $p \rightarrow q$ is $\sim p \rightarrow \sim q$.
- A conditional statement and its converse are NOT logically equivalent.
- A conditional statement and its inverse are NOT logically equivalent.
- The converse and the inverse of a conditional statement are logically equivalent to each other.

Your example: Write down a conditional statement and its contrapositive, converse, and inverse.

1.5. **Only If** and the Biconditional.

- If $p$ and $q$ are statements, $p$ **only if** $q$ means "if not $q$ then not $p,"$ or equivalently, "if $p$ then $q.""
- Given statement variables, $p$ and $q$, the **biconditional of $p$ and $q$** is "$p$ if, and only if, $q$" and is denoted $p \leftrightarrow q$. It is true if both $p$ and $q$ have teh same truth values and is false if $p$ and $q$ have opposite truth values. The words *if and only if* are sometimes abbreviated **iff**.
- The biconditional has the following truth table:

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<th>$p$</th>
<th>$q$</th>
<th>$p \leftrightarrow q$</th>
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<tbody>
<tr>
<td>T</td>
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</tbody>
</table>

- The order of operations for logical operators is:
  1. Evaluate $\sim$ first.
  2. Evaluate $\lor$ and $\land$ second. When both are present, parenthesis may be needed.
  3. Evaluate $\rightarrow$ and $\leftrightarrow$ third. When both are present, parenthesis may be needed.
- Notice that $p \leftrightarrow q$ is equivalent to $(p \rightarrow q) \land (q \rightarrow p)$.

1.6. **In Class Group Work:** Use a truth table to show $p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$. 
1.7. Necessary and Sufficient Conditions.

- If \( r \) and \( s \) are statements:
  - \( r \) is a **sufficient condition** for \( s \) means "if \( r \) then \( s \)."
  - \( r \) is a **necessary condition** for \( s \) means "if not \( r \) then not \( s \)."
- Because statements and their contrapositives are equivalent, "\( r \) is a necessary condition for \( s \)" also means "if \( s \) then \( r \)."
- Hence, "\( r \) is a necessary and sufficient condition for \( s \)" means "\( r \) if, and only if, \( s \)."

For example: If John is eligible to vote, then he is at least 18 years old.

2. 2.3 Valid and Invalid Arguments

- An **argument** is a sequence of statements, and an **argument form** is a sequence of statement forms. All statements in an argument and all statement forms in an argument form, except for the final one, are called **premises** (or **assumptions** or **hypotheses**). The final statement or statement form is called the conclusion. The symbol \( \therefore \), which is read "therefore," is normally placed just before the conclusion.
- To say that an **argument form** is **valid** means that no matter what particular statements are substituted for the statement variables in its premises, if the resulting premises are all true, then the conclusion is also true. To say that an **argument** is valid means that its form is valid.
- A valid argument is is such that the truth of its conclusion follows necessarily from the truth of its premises. It is impossible to have a valid argument with true premises and a false conclusion.
- Testing an argument form for validity
  1. Identify the premises and conclusion of the argument form.
  2. Construct a truth table showing the truth values of all the premises and the conclusion.
  3. A row of the truth table in which all the premises are true is called a **critical row**. If there is a critical row in which the conclusion is false, then it is possible for an argument of the given form to have true premises and a false conclusion, and so the argument form is invalid. if the conclusion in every critical row is true, then the argument form is valid.

For example: Determine the validity of the following argument:

\[
\begin{align*}
  p &\rightarrow q \lor \sim r \\
  q &\rightarrow p \land r \\
  \therefore & p \rightarrow r
\end{align*}
\]
2.1. Modus Ponens and Modus Tollens.

- An argument form consisting of two premises and a conclusion is called a syllogism. The first and second premises are called the major premise and minor premise, respectively.
- The **modus ponens** argument form has the following form:
  
  If \( p \) then \( q \).
  
  \[ p \quad \therefore q. \]

- **Modus tollens** has the following form:
  
  If \( p \) then \( q \).
  
  \[ \sim q \quad \therefore \sim p. \]

Your example: Write down an example of modus ponens and an example of modus tollens.
2.2. Additional Valid Argument Forms: Rules of Inference.

- A rule of inference is a form of argument that is valid. Modus ponens and modus tollens are both rules of inference. Here are some more...

| Generalization      | \( p \rightarrow q \) | \( \therefore p \lor q \)
|---------------------|------------------------|------------------------
| Specialization      | \( p \land q \)       | \( \therefore p \)
| Proof by Division into Cases | \( p \lor q \) | \( p \rightarrow q \) |
|                     | \( p \rightarrow q \) | \( q \rightarrow r \) |
|                     | \( \therefore r \)     | \( \therefore p \rightarrow r \)
| Contradiction Rule  | \( \sim p \rightarrow c \) | \( \therefore p \lor q \)

2.3. In Class Group Work. The famous detective Percule Hoirot was called in to solve a baffling murder mystery. He determined the following facts:

1. Lord Hazelton, the murdered man, was killed by a blow on the head with a brass candlestick.
2. Either Lady Hazelton or a maid, Sara, was in the dining room at the time of the murder.
3. If the cook was in the kitchen at the time of the murder, then the butler killed Lord Hazelton with a fatal dose of strychnine.
4. If Lady Hazelton was in the dining room at the time of the murder, then the chauffeur killed Lord Hazelton.
5. If the cook was not in the kitchen at the time of the murder, then Sara was not in the dining room when the murder was committed.
6. If Sara was in the dining room at the time the murder was committed, then the wine steward killed Lord Hazelton.

Is it possible for the detective to deduce the identity of the murderer from these facts? If so, who did murder Lord Hazelton? (Assume there was only one cause of death).
2.4. Fallacies.

- A fallacy is an error in reasoning that results in an invalid argument. Three common fallacies are using ambiguous premises, and treating them as if they were unambiguous, circular reasoning (assuming what is to be proved without having derived it from the premises), and jumping to a conclusion (without adequate grounds).

- For an argument to be valid, every argument of the same form whose premises are all true must have a true conclusion. It follows that for an argument to be invalid means that there is an argument of that form whose premises are all true and whose conclusion is false.

- Here are two more error types

<table>
<thead>
<tr>
<th>Converse Error</th>
<th>$p \rightarrow q$</th>
<th>$q$</th>
<th>$\therefore p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inverse Error</td>
<td>$p \rightarrow q$</td>
<td>$\sim p$</td>
<td>$\therefore \sim q$</td>
</tr>
</tbody>
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