1. Chapter 9.4 The Pigeonhole Principle

A flock of $n$ pigeons fly into $m$ pigeon holes. If $m \geq n$ then each pigeon can go to a separate hole. If $m < n$ then there must be at least one hole which contains more than one pigeon.

**Definition.** A function $f : X \rightarrow Y$ is called one-to-one if $f(x) \neq f(y)$ for all $x \neq y$. That is, $f$ maps different points in the domain to different points in the range.

**Theorem 1.1** (Pigeonhole Principle). A function from one finite set to a smaller finite set cannot be one-to-one: There must be at least two elements in the domain that have the same image in the co-domain.

Example: A drawer contains 10 red and 10 white socks. What is the least number of socks that need to be pulled out of the drawer to ensure that at least one matching pair is obtained?
In class work:

1. A drawer contains 10 red, 10 white, 10 black and 10 blue socks. What is the least number of socks that need to be pulled out of the drawer to ensure that at least one matching pair is obtained?

2. Let $A = \{1, 2, 3, 4, 5, 6, 7\}$. $n$ distinct integers will be chosen from $A$. How big does $n$ have to be to ensure that the sum of two of the chosen integers is 9?

3. Let $B = \{1, 2, 3, \ldots, 99, 100\}$. $n$ distinct integers will be chosen from $B$. How big does $n$ have to be to ensure that the sum of two of the chosen integers is 101?
Example: How many times does a 6 sided die need to be rolled to ensure that at least one number is rolled 3 times?

**Theorem 1.2** (Generalized Pigeonhole Principle). For any function $f$ from a finite set $X$ with $n$ elements to a finite set $Y$ with $m$ elements and for any positive integer $k$, if $k < n/m$, then there is some $y \in Y$ such that $y$ is the image of at least $k + 1$ distinct elements of $X$.

**Theorem 1.3** (Generalized Pigeonhole Principle (Contrapositive)). For any function $f$ from a finite set $X$ with $n$ elements to a finite set $Y$ with $m$ elements and for any positive integer $k$, if for each $y \in Y$, $f^{-1}(y)$ has at most $k$ elements, then $X$ has at most $km$ elements; in other words, $n \leq km$. 
(1) A computer programmer wrote 400 lines of code in 12 days. Must there have been a day in which the programmer wrote more than 30 lines of code?

(2) There are 42 students who share 12 computers. Each student uses exactly 1 computer, and no computer is used by more than 6 students. Show that at least 5 computers are used by 3 or more students.
2. **Chapter 9.5: Counting Subsets of a Set: Combinations**

**Definition.** Let \( n \) and \( r \) be nonnegative integers with \( r \leq n \). An \( r \)-combination of a set of \( n \)-elements is a subset of \( r \) of the \( n \) elements. The total number of \( r \)-combinations of a set of \( n \)-elements is denoted:

\[
\binom{n}{r}
\]

This notation is called \( n \) choose \( r \).

Example: What are all the 2 combinations of \( \{a, b, c, d\} \)?

Example: What are all the 3 combinations of \( \{a, b, c, d\} \)?

**Theorem 2.1.** For all \( n, r \in \mathbb{N} \) with \( r \leq n \):

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!}
\]
In class work:

(1) How many ways are there to choose 6 people from a group of 10 to work as a team?

(2) How many ways are there to choose 6 people from a group of 10 to work as a team if there are 2 who insist on being together? That is the team must contain both of them or neither.

(3) How many ways are there to choose 6 people from a group of 10 to work as a team if there are 2 who insist on not working together? That is the team must not contain both of them.