RINGS CHARACTERIZED BY THEIR CYCLIC MODULES AND RIGHT IDEALS: A SURVEY-I

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Dedicated to the memory of John Dauns

Abstract. This survey article addresses the classical questions on determining rings whose cyclic modules or proper cyclic modules have a certain homological property, chain condition, or a combination of such properties. This survey contains proofs of many of the important results and it strives to give complete references for all others.

1. Introduction, Definitions and Notations

It is known that if $R$ is a PID or, more generally, a Dedekind domain then for each nonzero ideal $A$, $R/A$ is self-injective, that is, $R/A$ is injective as $R/A$-module, equivalently, $R/A$ is quasi-injective as $R$-module. The question of classifying commutative noetherian rings $R$ such that each proper homomorphic image is self-injective was initiated by Levy [86], and later continued by Klatt-Levy [79] without assuming the noetherian condition. Later on several authors including Ahsan, Beidar, Boyle, Byrd, Courter, Cozzens, Damiano, Faith, Goel, Goodearl, Gómez Pardo, Guil Asensio, Hajarnavis, Hill, Huynh, Ivanov, Jain, Koecher, López-Permouth, Mohamed, Osofsky, Singh, Skornyakov, Smith, Srivastava, Symonds and Wisbauer described classes of noncommutative rings whose all cyclic modules, a proper subclass of cyclic modules, injective hulls of cyclic modules, right ideals, or a proper subclass of right ideals have properties, such as, injectivity, quasi-injectivity, continuity, quasi-continuity ($= \pi$-injectivity), $CS$ (= complements are summands), weak-injectivity, projectivity, quasi-projectivity, noetherian, or artinian.

In this paper we shall provide a survey of results on rings $R$ over which a family of cyclic right $R$-modules or injective hulls of a family of cyclic modules have a certain property, or each cyclic module has a decomposition into modules with some such properties. To reiterate, these properties include injectivity, quasi-injectivity, continuity, quasi-continuity (or $\pi$-injectivity), $CS$ (= complements are summands), weak-injectivity, projectivity, and quasi-projectivity. We will also consider rings whose proper homomorphic images are artinian or von neumann regular. The rings determined by the properties of their right ideals will be discussed in the forthcoming sequel to this survey article.

All our rings are associative rings with identity and modules are right unital unless stated otherwise. A right $R$-module $E \supseteq M_R$ is called an essential extension of $M$ if every nonzero submodule of $E$ intersects $M$ nontrivially. $E$ is said to be

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a maximal essential extension of $M$ if no module properly containing $E$ can be an essential extension of $M$. If $E \supseteq M$ is an essential extension, we say that $M$ is an essential submodule of $E$, and write $M \subseteq_{e} E$. A submodule $L$ of $M$ is called an essential closure of a submodule $N$ of $M$ if it is a maximal essential extension of $N$ in $M$. A submodule $K$ of $M$ is called a complement if there exists a submodule $U$ of $M$ such that $K$ is maximal with respect to the property that $K \cap U = 0$. A right $R$-module $M$ is called $N$-injective, if every $R$-homomorphism from a submodule $L$ of $N$ to $M$ can be lifted to an $R$-homomorphism from $N$ to $M$. A right $R$-module $M$ is called an injective module if $M$ is $N$-injective for every right $R$-module $N$. By Baer’s criterion, a right $R$-module $M$ is injective if and only if $M$ is $R_R$-injective.

For every right $R$-module $M$, there exists a minimal injective module containing $M$, which is unique up to isomorphism, called the injective hull (or injective envelope) of $M$. The injective hull of $M$ is denoted by $E(M)$. Equivalently, $E(M)$ is a maximal essential extension of $M$. A ring $R$ is called right self-injective if $R$ is injective as a right $R$-module. A right $R$-module $M$ is called quasi-injective if $\text{Hom}_R(-, M)$ is right exact on all short exact sequences of the form $0 \rightarrow K \rightarrow M \rightarrow L \rightarrow 0$.

Johnson and Wong [77] characterized quasi-injective modules as those that are fully invariant under endomorphism of their injective hulls. In other words, a module $M$ is quasi-injective if it is $M$-injective. More generally, as proved by Azumaya, $M$ is $N$-injective if $\text{Hom}_R(E(N), E(M))N \subseteq M$. Consider the following properties;

$(\pi)$: For each pair of submodules $M_1$ and $M_2$ of $N$ with $M_1 \cap M_2 = 0$, canonical projection $\pi_i : M_i \rightarrow M_1 \oplus M_2$, $i = 1, 2$ can be lifted to an endomorphism of $M$.

$(C1)$: Every complement submodule of $M$ is a direct summand of $M$.

$(C2)$: If $N_1$ and $N_2$ are direct summands of $M$ with $N_1 \cap N_2 = 0$ then $N_1 \oplus N_2$ is also a direct summand of $M$.

$(C3)$: Every submodule of $M$ isomorphic to a direct summand of $M$ is itself a direct summand of $M$.

Modules satisfying the property $(\pi)$ are called $\pi$-injective modules and those satisfying $(C1)$ and $(C3)$ are called quasi-continuous modules. It is known that a module is $\pi$-injective if and only if it is quasi-continuous (see [38] and [115]).

Following the definition of continuous rings due to von Neumann, modules satisfying $(C1)$ and $(C2)$ are called continuous modules. Modules satisfying $(C1)$ are called $CS$-modules [19]. $CS$ modules are also known as extending modules (see [32]). In general, we have the following implications:

Injective $\implies$ Quasi-injective $\implies$ Continuous $\implies$ Quasi-continuous $\implies$ CS

A right $R$-module $M$ is called weakly $N$-injective if for each right $R$-homomorphism $\phi : N \rightarrow E(M)$, $\phi(N) \subseteq X \cong M$ for some submodule $X$ of $E(M)$. A right $R$-module $M$ is called weakly injective if $M$ is weakly $N$-injective for each finitely generated module $N$ (see [64]). Dual to injective modules, a right $R$-module $P$ is called a projective module if for any short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, the induced sequence of abelian groups $0 \rightarrow \text{Hom}(P, A) \rightarrow \text{Hom}(P, B) \rightarrow \text{Hom}(P, C) \rightarrow 0$ is also short exact. Equivalently, $P$ is a direct summand of a free module. A right $R$-module $M$ is called quasi-projective if for every submodule $K$ of $M$ the induced sequence $\text{Hom}(M, M) \rightarrow \text{Hom}(M, M/K) \rightarrow 0$ is exact. For any term not defined here, the reader may refer to [83], [84], [115], [32] and [20].
2. Rings Whose Cyclics or Proper Cyclics Are Injective

The study of noncommutative rings characterized by the properties of its cyclic modules has a long history. The first important contribution in this direction is due to Osofsky [97] who considered rings over which all cyclic modules are injective. It is clear that if each \( R \)-module is injective, then \( R \) is semisimple artinian. Osofsky showed that \( R \) is semisimple artinian by simply assuming that cyclic \( R \)-modules are injective.

We begin with the theorem of Osofsky.

**Theorem 1.** (Osofsky, [97]). If each cyclic \( R \)-module is injective then \( R \) must be semisimple artinian.

**Proof.** By the hypothesis each principal right ideal of \( R \) is injective and hence a direct summand of \( R \). Therefore \( R \) is a von Neumann regular ring. Using set-theoretic arguments, Osofsky proved that \( R \) has finite uniform dimension, and hence \( R \) is semisimple artinian. Instead of giving the details of this part here, we will later give proof of a more general result due to Dung-Huynh-Wisbauer (Theorem 9) which would imply that under the given hypothesis, each homomorphic image of \( R \) has finite uniform dimension. \( \square \)

The initial proof of Osofsky in her dissertation was quite elaborate. Later Osofsky gave a shorter proof of this theorem [98]. Indeed, Skornyakov [110] had also attempted to give a proof of this theorem but unfortunately his proof had an error (see [99]).

For commutative rings the classification of commutative noetherian rings whose proper homomorphic images are self-injective was obtained by Levy.

**Theorem 2.** (Levy, [86]). Let \( R \) be a commutative noetherian ring. Then every proper homomorphic image of \( R \) is a self-injective ring if and only if

1. \( R \) is a Dedekind domain, or
2. \( R \) is an artinian principal ideal ring, or
3. \( R \) is a local ring whose maximal ideal \( M \) has composition length 2 and satisfies \( M^2 = 0 \).

**Proof.** Suppose that every proper homomorphic image of \( R \) is self-injective. Assume first that \( R \) is a domain. If \( M \) is a maximal ideal of \( R \), the ring \( R/M^2 \) must be self-injective and the image \( M \) of \( M \) in \( R/M^2 \) satisfies \( M^2 = 0 \). It may be deduced that \( R \) cannot have any ideals between \( M \) and \( M^2 \) (see Lemma, [86]). Now, we invoke a result of Cohen [21] which states that if a noetherian domain \( R \) has the property that for every maximal ideal \( M \), there are no ideals between \( M \) and \( M^2 \) then \( R \) must be a Dedekind domain.

Next, we consider the case when \( R \) is not a domain. Then 0 is not a prime ideal. For each prime ideal \( P \), by hypothesis, \( R/P \) is an injective and hence a divisible \( R/P \)-module. Thus, \( R/P \) is a field and so \( P \) is a maximal ideal. But a commutative noetherian ring in which every prime ideal is maximal must be an artinian ring (see Theorem 1, [21]). Therefore, \( R = R_1 \oplus \ldots \oplus R_n \) where each \( R_i \) is a local artinian ring. Let \( M_i \) be the maximal ideal of \( R \). Again we consider two cases. Suppose first that either \( n > 1 \), or \( n = 1 \) but \( M_1^2 \neq 0 \). Then for each \( i \), \( R/(M_1^2 + \sum_{j \neq i} R_j) \cong R_i/M_i^2 \) is a self-injective ring. Now \( R/M_i^2 \) is a commutative self-injective ring with a maximal ideal \( N = M_i/M_i^2 \) such that \( N^2 = 0 \). Thus, by (Lemma, [86]), \( 0 \subseteq N \subseteq R/M_i^2 \) are all of the ideals of \( R/M_i^2 \). Therefore, for any \( m_i \in M_i \) with \( m_i \notin M_i^2 \), we have...
$M_i = R_i m_i + M_i^2 = R_i m_i + (J(R_i)) M_i$; and Nakayama’s lemma then shows that $M_i = R_i m_i$. Thus $R_i$ is a commutative noetherian ring in which every maximal ideal is principal, therefore by Kaplansky [78], $R_i$ must be a principal ideal ring. Hence $R$ is an artinian principal ideal ring. Finally, suppose that $R$ is a local artinian ring whose maximal ideal $M$ satisfies $M^2 = 0$. We may assume that $M$ has composition length at least 2, since otherwise $R$ would be a principal ideal ring.

Since $R$ is artinian, $M$ contains a minimal ideal $N$ of $R$. Now, $R/N$ is a commutative self-injective ring with a maximal ideal $M/N$ such that $(M/N)^2 = 0$, therefore by (Lemma, [86]), $0 \subseteq M/N \subseteq R/N$ are all the ideals of $R/N$. Hence $M$ has composition length 2.

Conversely, suppose that $R$ is of type (1), (2), or (3). Observe that the proper homomorphic images of all three types of rings are all artinian principal ideal rings and an artinian principal ideal ring is a direct sum of rings $S$ which have exactly one composition series $S \supset Sm \supset Sm^2 \supset \ldots \supset Sm^t = 0$ (see [78]). Therefore, it suffices to show that this ring $S$ is self-injective. Let $f$ be a homomorphism of $Sm^i$ into $S$. Then the composition length of $f(\text{Sm}^i)$ cannot exceed that of $\text{Sm}^t$. Since the above composition series contains all the ideals of $S$, we have $f(\text{Sm}^i) \subseteq \text{Sm}^t$. Consequently, $f(m^i) = m^i x$ for some $x \in S$. The map $r \mapsto rx \,(r \in S)$ is then the extension of $f$ to an endomorphism of $S$, showing that $S$ is self-injective. □

Levy gave example of a non-noetherian domain whose proper homomorphic images are self-injective.

**Example 3.** (Levy, [86]). Let $F$ be a field and $x$ an indeterminate; and let $W$ be the family of all well-ordered sets $\{i\}$ of nonnegative real numbers, the order relation being the natural order of the real numbers. Let $R = \{\Sigma_{j \in \{i\}} a_j x^j : a_j \in F, \{i\} \in W\}$. Note that every element of $R$ whose constant term is nonzero is invertible in $R$. It follows that every nonzero element of $R$ has the form $x^b u$ where $u$ is invertible in $R$. This implies that $R$ has only two types of nonzero ideals: The principal ideals $(x^b)$, and those of the form $(x^b - u) = \{x^c u : c > b \text{ and } u \text{ is invertible or zero}\}$. Let $S = R/I$ where $I \neq 0$. Then $S$ can be considered as the collection of formal power series $\Sigma_{j \in \{i\}} a_j x^j$ with $a_j \in F, \{i\} \in W$ and

(1) $y^I = 0$ if $I = (x^b)$, or

(II) $y^I = 0$ for $c > b$ if $I = (x^b)$. Observe that for $c \leq b$, we have

(A1): If $I = (x^b)$, then $\text{ann}(y^c) = (y^b - c)$ and $\text{ann}(y^{>c}) = (y^b - c)$.

(A2): If $I = (x^b)$, then $\text{ann}(y^c) = (y^{b-c})$ and $\text{ann}(y^{>c}) = (y^b - c)$.

From (A1) and (A2), it follows that whether $S$ is of type (I) or (II), the principal ideals of $S$ satisfy $\text{ann}(\text{ann}(y^c)) = (y^c)$.

To see that $S$ is self-injective, let $f$ be an $S$-homomorphism from an ideal $K$ of $S$ to $S$. If $K = (y^c)$ then $\text{ann}(K) = f(0) = 0$ so that $f(K) \subseteq \text{ann}(\text{ann}(K)) = K$. Hence $f(y^c) = y^c p$ for some $p \in S$. Thus $f$ can be extended to the endomorphism $s \mapsto sp$ of $S$.

Next, let $K = (y^{c_1})$, and choose an infinite sequence $c(1) > c(2) > \ldots$ such that $\lim_{i \to \infty} c(i) = c$. Then $K = \cup_{i=1}^\infty (y^{c(i)})$. For each $i$ the previous paragraph shows that we can choose a “power series” $p_i$ such that $f(y^{c(i)}) = y^{c(i)} p_i$. If $j > i$ so that $y^{c(i)} = y^{c(i)-c(j)} y^{c(j)}$ the fact that $f$ is an $S$-homomorphism shows that $f(y^{c(i)}) = y^{c(i)} p_i$ so that we have $p_j - p_i \in \text{ann}(y^{c(i)})$. If (A1) holds, this means that all the terms of $p_i$ of degree $< bNc(i)$ are equal to the terms of the same degree of $p_j$ while the terms of higher degree do not affect the products $y^{c(i)} p_i$ and $y^{c(i)} p_j$. 
A similar statement is true if \((A2)\) holds. Thus we can assemble a single “power series” \(p\) such that 
\[(p - p_i)\gamma^{(i)} = 0 \text{ for all } i\] 
(It may be verified that the collection of exponents appearing in \(p\) is well-ordered so that \(p\) is in \(S\)). Then the map \(s \rightarrow sp\) extends \(f\) to an endomorphism of \(S\). This shows that \(S\) is self-injective. Thus \(R\) is a non-noetherian domain whose every proper homomorphic image is self-injective.

The result of Levy was extended to noncommutative rings by Hajarnavis. Recall that a ring is called right bounded if every essential right ideal of \(R\) contains an ideal which is essential as a right ideal. A prime Goldie ring \(R\) is called an Asano order if each nonzero ideal of \(R\) is invertible. A ring \(R\) is called a Dedekind prime ring if it is hereditary noetherian Asano order. Hajarnavis considered noetherian bounded prime ring such that every proper homomorphic image is a self-injective ring and proved the following.

**Theorem 4.** (Hajarnavis, [44]). Let \(R\) be a noetherian, bounded, prime ring such that every proper homomorphic image of \(R\) is a self-injective ring. Then \(R\) is a Dedekind prime ring.

Klatt and Levy later described all commutative rings, not necessarily noetherian, all of whose proper homomorphic images are self-injective. Such rings include Prufer domains.

**Theorem 5.** (Klatt and Levy, [79]). A commutative ring \(R\) is pre-self-injective, that is for each non-zero ideal \(A\), \(R/A\) is self-injective if and only if \(R\) is one of the following:

1. A Prufer domain such that the localization \(R_M\) for each maximal ideal \(M\) is an almost maximal rank 1 valuation domain; and every proper ideal is contained in only finitely many maximal ideals, or
2. The finite direct sum of self-injective maximal valuation rings of rank 0, or
3. An almost maximal rank 0 valuation ring, or
4. A local ring whose maximal ideal \(M\) has composition length 2 and satisfies \(M^2 = 0\).

Furthermore, finitely generated modules over pre-self-injective domains are direct sums of cyclic modules and ideals. In this connection we state results of Koethe, Cohen-Kaplansky, and Nakayama.

**Theorem 6.** (Koethe, [82]). Over an artinian principal ideal ring, each module is a direct sum of cyclic modules. Furthermore, if a commutative artinian ring has the property that all its modules are direct sums of cyclic modules, then it is necessarily a principal ideal ring.

**Theorem 7.** (Cohen and Kaplansky, [24]). If \(R\) is a commutative ring such that each \(R\)-module is a direct sum of cyclic modules then \(R\) must be an artinian principal ideal ring.

Nakayama [94] gave example of a noncommutative right artinian ring \(R\) whose each right module is a direct sum of cyclic modules but \(R\) is not a principal right ideal ring.

By Ososky’s theorem, if each cyclic \(R\)-module is injective, then each \(R\)-module has finite uniform dimension. This was extended by Ososky-Smith [105] in the following theorem.
Theorem 8. (Osofsky and Smith, [105]). Let $R$ be a ring such that every cyclic right $R$-module is CS. Then every cyclic right $R$-module is a finite direct sum of uniform modules.

We give below a proof of a more general result due to Huynh-Dung-Wisbauer. Its proof follows the same techniques as that of Osofsky-Smith [105]. A module $N$ is called a subfactor of $M$ if $N$ is a submodule of a factor module of $M$. We shall prove Huynh-Dung-Wisbauer’s theorem in the case when $M$ is a cyclic module with the property that each cyclic subfactor of $M$ is a direct sum of a CS module and a module with finite uniform dimension.

Theorem 9. (Huynh, Dung and Wisbauer, [53]). Let $M$ be a cyclic module such that each cyclic subfactor module of $M$ is a direct sum of a CS module and a module with finite uniform dimension. Then $M$ must have finite uniform dimension.

Before we prove the above theorem we will prove a basic lemma which plays a key role not only in the proof of the above theorem but at several places in such problems.

Lemma 10. (Huynh, Dung and Wisbauer, [53]). Let $M$ be a finitely generated CS module. If $M$ contains an infinite direct sum of nonzero submodules $N = \bigoplus_{i=1}^{\infty} N_i$, then the factor module $M/N$ does not have finite uniform dimension.

Proof. Assume to the contrary that $M/N$ has finite uniform dimension, say $n$. Partition $N$ as a disjoint union of infinite sets $P_1, P_2, ..., P_{n+1}$ and set $U_j = \bigoplus_{i \in P_j} N_i$. Then $N = \bigoplus_{j=1}^{\infty} U_j$. Let $E_j$ be a maximal essential extension of $U_j$ for each $j \leq n+1$. Since $M$ is a CS module, each $E_j$ is a direct summand of $M$ and hence finitely generated. This implies $E_j/U_j \neq 0$. Now, $M/N = M/(\bigoplus_{j=1}^{\infty} U_j)$ contains a submodule isomorphic to $E_1/U_1 \oplus E_2/U_2 \oplus ... \oplus E_{n+1}/U_{n+1}$. This yields a contradiction to our assumption that $u \cdot \dim(M/N) = n$. Hence, the factor module $M/N$ must have infinite uniform dimension. \qed

Now we are ready to give the proof of the Theorem 9. Throughout this proof we will denote by $A^e$ a maximal essential extension in a module $B$ (say) of a submodule $A$.

Proof. Assume to the contrary that $M$ does not have finite uniform dimension. By hypothesis $M = M_1 \oplus M_2$ where $M_1$ is CS and $M_2$ is of finite uniform dimension. This implies $u \cdot \dim M_1 = \infty$. Thus without loss of generality we can assume $M$ is a CS module of infinite uniform dimension such that each factor module is a direct sum of a CS module and a module of finite uniform dimension. Let $\bigoplus_{i \in I} M_i$ be an infinite direct sum of submodules in $M$. Then it follows that for every positive integer $k$, $M = M_1^k \oplus ... \oplus M_k^\infty \oplus N_k$ for some submodule $N_k$ of $M$. Note that since $M$ is cyclic, each summand in the decomposition of $M$ is cyclic, and hence there exists maximal submodules $U_i \subset M_i^\infty$. Denote the simple module $M_i^\infty/U_i$ by $S_i$. Then we have

\[(*) : \quad M/(\bigoplus_{i \in I} U_i) = (M_1^\infty \oplus ... \oplus M_k^\infty \oplus N_k)/(U_1 \oplus ... \oplus U_k \oplus S)/\simeq S_1 \oplus ... S_k \oplus X,\]

yielding a direct sum $S = \bigoplus_{i \in I} S_i$ of simple modules in the factor module $M/(\bigoplus_{i \in I} U_i)$. Set $M = M/(\bigoplus_{i \in I} U_i)$. By hypothesis, $M = M_1 \oplus M_2$, where $M_1$ is CS and $M_2$ has finite uniform dimension. We can write $S = (S \cap M_1) \oplus K$. Since $K \cap M_1 = 0$, $K$ is embeddable in $M_2$. Note that $S \cap M_1$ must be infinitely generated because $S$
is infinitely generated. From (1) it follows that every finitely generated submodule of $S$, (and hence of $S \cap M_1$) is a direct summand of $M$ and hence of $M_1$. Let $(S \cap M_1)^e$ denote a maximal essential extension of $S \cap M_1$ in $M_1$ and note that $M_1$ is CS and cyclic because it is a direct summand of a cyclic module. This implies $(S \cap M_1)^e$ is a direct summand of $M_1$ and hence cyclic. Claim: $(S \cap M_1)^e/(S \cap M_1)$ is CS. We write $(S \cap M_1)^e/(S \cap M_1) = C \oplus \bar{D}$, where $C = C/(S \cap M_1)$ is CS and $\bar{D} = D/(S \cap M_1)$ has finite uniform dimension. We proceed to prove that $\bar{D} = 0$. Now $\bar{D} = dR + S \cap M_1$ where $L$ is CS and $\dim L$ is finite. As $S \cap M_1 \subset (S \cap M_1)^e$, $\frac{dR + S \cap M_1}{S \cap M_1}$ is singular and so $dR \cap (S \cap M_1)$ is essential in $dR$. This gives, $\text{Socle}(dR) \subset dR$. We have $\text{Socle}(dR) = \text{Socle}(T) \oplus \text{Socle}(L)$ and $\text{Socle}(dR) \subset dR$, so $\text{Socle}(T) \subset dR$. Thus, $\text{Socle}(T)$ is a finite direct sum of simple modules and hence by (1), $\text{Socle}(T)$ is a direct summand of $T$. Hence, $\text{Socle}(T) = T \subset S \cap M_1$. Moreover, $L/(L \cap (S \cap M_1)) \cong (L + S \cap M_1)/(S \cap M_1) \cong (dR + S \cap M_1)/(S \cap M_1) = \bar{D}$ has finite uniform dimension. Therefore, by Lemma 10, $S \cap M_1$ is finitely generated and hence is a direct summand of $(S \cap M_1)^e$. Thus, $L = L \cap (S \cap M_1) \subset S \cap M_1$ and hence $\bar{D} = 0$. Therefore, $(S \cap M_1)^e/(S \cap M_1)$ is CS. Now, we partition $\mathbb{N}$ as a disjoint union of infinite sets $(P_j)_{j \in \mathbb{N}}$. Then $S \cap M_1 = \bigoplus_{j \in \mathbb{N}} (\bigoplus_{e \in P_j} (S \cap M_1)^e)$. Let $(\bigoplus_{e \in P_j} (S \cap M_1)^e)$ be a maximal essential extension of $(\bigoplus_{e \in P_j} (S \cap M_1)^e)$ in $(S \cap M_1)^e$. For simplicity, we denote $(\bigoplus_{e \in P_j} (S \cap M_1)^e)$ by $Y_j$. Let $Y_j$ be the image of $Y_j$ under the canonical morphism $(S \cap M_1)^e \to (S \cap M_1)^e/(S \cap M_1)$. Since $(S \cap M_1)^e/(S \cap M_1)$ is CS, we may find a maximal essential extension $Y$ of $\bigoplus_{j \in \mathbb{N}} Y_j$ in $(S \cap M_1)^e/(S \cap M_1)$ which is a direct summand of $(S \cap M_1)^e/(S \cap M_1)$. As $Y$ is cyclic, there exists a cyclic submodule $Y \subset (S \cap M_1)^e$ with $Y = (Y + S \cap M_1)/(S \cap M_1)$. For each $j \in \mathbb{N}$, we have $Y_j \subset Y + S \cap M_1 = Y + (Y \cap (S \cap M_1)) + S \cap M_1 = Y + (Y \cap (S \cap M_1)) \oplus V = Y \oplus V$ for a suitable submodule $V$ of $S \cap M_1$. On the other hand, by (1), $Y = Y' \oplus Y''$ where $Y'$ is CS and $Y''$ has finite uniform dimension. We have $\text{Socle}(Y) = \text{Socle}(Y') \oplus \text{Socle}(Y'')$ and $\text{Socle}(Y) \subset Y$, so $\text{Socle}(Y'') \subset Y''$. But, $\text{Socle}(Y'')$ is a finite direct sum of simple modules and hence by (1), $\text{Socle}(Y'')$ is a direct summand of $Y''$. Hence, $\text{Socle}(Y'') = Y'' \subset S \cap M_1$, and for each $j \in \mathbb{N}$, we have $Y_j \subset Y + S \cap M_1 = Y' \oplus V$ with $V = V \oplus Y'' \subset S \cap M_1$. Assume $Y' \cap Y_j = 0$ for some $j \in \mathbb{N}$. Then $Y_j$ can be embedded in $V'$. This implies $Y_j$ is semisimple of finite length, a contradiction. Hence $Y' \cap Y_j \neq 0$ for each $j \in \mathbb{N}$. Now take a minimal submodule $W_j$ in every $Y' \cap Y_j$ and denote by $(\bigoplus_{\mathbb{N}} W_j)^e$ a maximal essential extension of $\bigoplus_{\mathbb{N}} W_j$ in $Y'$. Since $Y'$ is CS, $(\bigoplus_{\mathbb{N}} W_j)^e$ is a direct summand of $Y'$. Then $(\bigoplus_{\mathbb{N}} W_j)^e$ is a cyclic module with an essential socle of infinite length and hence $(\bigoplus_{\mathbb{N}} W_j)^e \subset S \cap M_1$. Thus, $((\bigoplus_{\mathbb{N}} W_j)^e + S \cap M_1)/(S \cap M_1) \neq 0$. We have $\bigoplus_{m=1}^{m} W_j \subset (\bigoplus_{\mathbb{N}} W_j)^e$. This yields $(\bigoplus_{m=1}^{m} W_j) \subset (\bigoplus_{m=1}^{m} Y_j)$ which yields $(\bigoplus_{m=1}^{m} W_j) \subset (\bigoplus_{\mathbb{N}} W_j)^e \cap (\bigoplus_{m=1}^{m} Y_j)$. But as $(\bigoplus_{m=1}^{m} Y_j)$ is also a direct summand of $(\bigoplus_{\mathbb{N}} W_j)^e \cap (\bigoplus_{m=1}^{m} Y_j)$, we conclude that $(\bigoplus_{m=1}^{m} W_j) = (\bigoplus_{\mathbb{N}} W_j)^e \cap (\bigoplus_{m=1}^{m} Y_j)$ for each $m \in \mathbb{N}$. This implies $(\bigoplus_{\mathbb{N}} W_j)^e \cap (\bigoplus_{j \in \mathbb{N}} Y_j) \subset S \cap M_1$ which yields a contradiction to the fact that $(\bigoplus_{j \in \mathbb{N}} Y_j) + S \cap M_1)/(S \cap M_1)$ is essential in $Y$. Therefore, $M$ must have finite uniform dimension. □

Boyle ([11], [12]) initiated the study of right noetherian rings over which each proper cyclic module is injective. Note that this property does not hold for all Dedekind domains. Boyle called these rings right PCI rings and studied right
noetherian right PCI-rings. Right PCI-rings without chain condition were studied by Faith. Later Damiano proved that they are indeed right noetherian.

**Theorem 11.** (Boyle, [11]) A right noetherian right PCI-ring is either a semisimple artinian ring or a simple right hereditary domain.

We first state a lemma for rings $R$ over which proper cyclic modules are quasi-injective. Thus this lemma holds for right PCI-rings also.

**Lemma 12.** (Huynh, Jain, López-Permouth, [54]) Let $R$ be a prime ring such that each proper cyclic right module is quasi-injective. Then $R$ is either artinian or a right Ore domain.

**Proof.** We first show that $R$ is right nonsingular. Assume to the contrary that $Z(R_R) \neq 0$. By Theorem 9, $R$ has finite right uniform dimension and therefore, there is a uniform submodule $U$ of $R_R$. Consider $S = U \cap Z(R_R)$. Since $R$ is prime and $S \neq 0$, we have $S^2 \neq 0$. Let $a \in S$ be such that $aS \neq 0$ and $0 \neq x \in S \cap \ann_R(a)$. Since $x \in Z(R_R)$, $xR \neq R$. Therefore, by hypothesis, $xR$ is quasi-injective. Let $E(S)$ be the injective hull of $S$. As $U$ is uniform, $xR \subset U$ and hence $xR \subset S$. Therefore, $xR$ is a fully invariant submodule of $E(S)$. In particular, $S(xR) \subseteq xR$. Thus $(aS)(xR) = a(SxR) \subseteq a(xR) = 0$, while $aS \neq 0$ and $xR \neq 0$, a contradiction to the primeness of $R$. Therefore, $R$ is right nonsingular. Hence, $R$ is a right Goldie ring. If $R_R$ is uniform the either $R$ is a division ring or a right Ore domain. Assume that $R_R$ is not uniform. Let $U_1 \oplus U_2 \oplus \ldots \oplus U_m \subset R_R$, where $m > 1$ and each $U_i$ is a uniform right ideal of $R$. Let $0 \neq a_1 \in U_1$. Then $a_1R \neq R$. Therefore, $a_1R$ is quasi-injective. Since $R$ is a prime right Goldie ring, all uniform right ideals of $R$ are subisomorphic to each other. Hence, each $U_i$, $i \geq 2$ contains an isomorphic copy $a_iR$ of $a_1R$. It follows that $A = a_1R \oplus a_2R \oplus \ldots \oplus a_mR$ is quasi-injective. Since $A$ is essential in $R_R$, $A$ contains a regular element $b$. Thus, as $A$ is b-R-injective and $bR \cong R$, $A$ is injective. Therefore, $A = R$ and $R$ is right self-injective. Hence $R$ is simple artinian.

Faith proved the following for right PCI-rings without assuming any chain condition.

**Theorem 13.** (Faith, [33]) A right PCI ring $R$ is either semisimple artinian or a simple right semi-hereditary right Ore $V$-domain.

**Proof.** If $A$ is any nonzero ideal, then by Osofsky theorem $R/A$ is semisimple artinian. In particular, every nonzero prime ideal is maximal.

Suppose $R$ is not prime. Then there exists nonzero ideals $A$ and $B$ such that $AB = 0 \subset P$, for any prime ideal $P$. This implies either $A$ or $B$ is contained in $P$. Furthermore, $R/A$ and $R/B$ are semisimple artinian. This means there are only finitely many prime ideals above $A$ and $B$. Since every prime ideal contains either $A$ or $B$, it follows that there are only finitely many prime ideals in $R$. This gives that the prime radical $N = P_1 \cap \ldots \cap P_k$ and $R/N \cong R/P_1 \times \ldots \times R/P_k$. By Osofsky, $R/P_i$ is simple artinian for each $i$. Thus $R/N$ is semisimple artinian. Since $N$ is nil, it implies $N = J(R)$. Hence $R$ is semiperfect. Suppose $R$ is not local. Then $R = e_1R + \ldots + e_nR$, where $e_iR$ are indecomposable right ideals and by hypothesis these are injective. Then $R_R$ is injective. Then by Osofsky theorem $R$ is semisimple artinian. Assume now $R$ is local. We claim $N = 0$. Else, let $0 \neq a \in N$. If $aR \neq R$, then $aR$ is injective and hence a summand of $R$. Since $R$ is local, this gives $aR = R$.
a contradiction because $a \in N$. Thus $aR \cong R$. This too is not possible because $a$ is a nil element. Thus $N = 0$ and we conclude that $R$ is semisimple artinian if it is not prime.

If $R$ is prime then by the above lemma, $R$ is either artinian or a right Ore domain.

Now we proceed to show that $R$ is right semi-hereditary, that is, each finitely generated ideal of $R$ is projective. The proof is by induction. Let $A = aR + bR$. Then $aR + bR/bR$ is cyclic. In case $aR + bR/bR$ is isomorphic to $R$, then $aR + bR = bR \oplus K$, where $K \cong R$. So $aR + bR$ is projective. In the other case, $aR + bR/bR$ is injective and so a direct summand of $R/bR$. This gives $aR + bR/bR \oplus X/bR = R/bR$, that is, $(aR + bR) + X = R$, where $(aR + bR) \cap X = bR$. Thus, $(aR + bR) \times X \cong R \times bR$, proving that $aR + bR$ is projective. By induction, we may deduce that each finitely generated right ideal of $R$ is projective. 

The following is an example of a right and left noetherian domain which is both right and left PCI-ring. We do not know any example of a right PCI-domain which is not a left PCI-domain.

**Example 14.** (Cozzens, [25]). Let $k$ be a universal differential field with derivation $D$ and let $R = k[y,D]$ denote the ring of differential polynomials in the indeterminate $y$ with coefficients in $k$, i.e., the additive group of $k[y,D]$ is the additive group of the ring of polynomials in the indeterminate $y$ with coefficients in field $k$, and multiplication in $k[y,D]$ is defined by: $ya = ay + D(a)$ for all $a \in k$. Let $f = \sum_{i=1}^{n} a_i y^i \in k[y,D]$, $a_n \neq 0$. We define the degree of $f$, $\delta(f) = n$. Clearly we have the following: (i) $\delta(fg) = \delta(f) + \delta(g)$, (ii) for $f, g \in k[y,D]$, there exist $h, r \in k[y,D]$ such that $f = gh + r$ with $r = 0$ or $\delta(r) < \delta(g)$ (a similar algorithm holds on the left). From (ii) it follows that this ring $R = k[y,D]$ is both left and right principal ideal domain. The simple right $R$-modules are precisely of the form $V_\alpha = R/(y - \alpha)R$ where $\alpha \in k$. We claim that $V_\alpha = R/(y - \alpha)R$ is a divisible right $R$-module for all $\alpha \in k$. For this, it suffices to show that $V_\alpha(y + \beta) = V_\alpha$ for all $\alpha, \beta \in k$. Equivalently, given $h \in R$, $\delta(h) = 0$, there exist $f, g \in R$ such that $f(y + \beta) + (y + \alpha)g = h$. We shall determine $a, b \in k$ such that $a((y + \beta) + (y + \alpha)b = h$. This is equivalent to an equation of the form $D(b) + (\alpha - \beta)b = h$. Since $k$ is a universal differential field, there exists $a \in k$ satisfying this equation. Hence, each simple right $R$-module is divisible. Since $R$ is a principal right ideal domain, this implies that each simple right $R$-module is injective. Therefore, $R$ is a right $V$-ring. Similarly, $R$ can be shown to be a left $V$-ring as well. Hence by (Corollary 10, [13]), $R$ is both left as well as a right PCI-domain.

Osofsky provided another example of right and left PCI-ring.

**Example 15.** (Osofsky, [103]). Let $F$ be a field of characteristic $p > 0$, and $\sigma$ an endomorphism of $F$ defined by $\sigma(\alpha) = \alpha^p$ for all $\alpha \in F$. We then form the ring of twisted polynomials with coefficients on the left, $R = F[x, \sigma]$ with $R = \{\sum_{i=0}^{n} \alpha_i x^i : n \in \mathbb{Z}, \alpha_i \in F\}$ under usual polynomial addition and multiplication given by the relation $x^n = \sigma(\alpha)x$ for all $\alpha \in F$. It may be observed that this ring $R$ is a left and right principal ideal domain. If $F$ is separably closed then every simple right (left) $R$-module is divisible and hence injective. Therefore, every proper cyclic right (left) $R$-module is injective, that is, $R$ is a both left and right PCI-domain.

In [26], Cozzens and Faith asked the question if every right PCI-ring is right noetherian. This question was later answered in the affirmative by Damiano [27]. We provide a different proof that uses, in particular Osofsky-Smith theorem.
Theorem 16. (Damiano, [27]). Let $R$ be a right PCI ring. Then $R$ is right noetherian and right hereditary.

Proof. First recall that a right PCI-ring is either semisimple artinian or a simple right Ore domain. Assume $R$ is a right Ore domain which is not a division ring. So each right ideal is essential and socle is zero. Now, for any nonzero right ideal $E$, $R/E$ is a proper cyclic module. Hence by Osofsky-Smith Theorem [105], $R/E$ has finite uniform dimension. Since every cyclic submodule of $R/E$ is injective, $R/E$ is semisimple. This yields $R$ is right noetherian. Now, in view of Theorem 13, $R$ must be right hereditary.

Rings for which every singular right module is injective are called right SI rings. The right PCI and right SI conditions are equivalent for domains. Boyle and Goodearl studied the question of left-right symmetry of PCI-domains. They proved the following.

Theorem 17. (Boyle and Goodearl, [14]). Let $R$ be a right and left noetherian domain. Then $R$ is a right PCI-domain if and only if $R$ is a left PCI-domain.

Proof. We first remark that in order to prove left PCI-domain, it suffices to assume that $R$ is left Ore domain, because a right PCI-domain which is left Ore is left noetherian (See [26], Page 112).

Now we show that a left and right noetherian right PCI-domain is a left PCI-domain. Note each proper cyclic right as well as left module is torsion since $R$ is right and left Ore domain. Furthermore, the functor $\text{Hom}(\cdot, Q(R)/R)$ defines a duality between the category of finitely generated torsion right modules and the category of finitely generated torsion left modules. Because $R$ is a right PCI-domain, each proper cyclic right module is semisimple. This gives each proper cyclic left module is semisimple. Therefore, $R$ is a left PCI domain.

In [27], Damiano had proved that a right PCI ring $R$ is left PCI if and only if $R$ is left coherent. But this is incorrect (see e.g. Remark 6.3, [65]). Jain-Lam-Leroy [65] considered twisted differential polynomial rings over a division ring which are right PCI domains and found equivalent conditions as to when it will be left PCI. However, in general, the question of left-right symmetry of PCI domains is still open.

In [54] Huynh, Jain and Lópe-Permouth had proved that a simple ring $R$ is right PCI if and only if every proper cyclic right $R$-module is quasi-injective. Barthwal-Jhingan-Kanwar showed that $R$ is a simple right PCI domain if and only if each proper cyclic right module is continuous [5]. Huynh, Jain and Lópe-Permouth later strengthened their result by proving the following.

Theorem 18. (Huynh, Jain and Lópe-Permouth, [55]). A simple ring $R$ is Morita equivalent to a right PCI domain if and only if every cyclic singular right $R$-module is quasi-continuous.

As a consequence of this, they also showed the following.

Theorem 19. (Huynh, Jain and Lópe-Permouth, [55]). Let $R$ be a simple ring. If every proper cyclic module is quasi-continuous, then $R$ is Morita equivalent to a right PCI domain and has right uniform dimension at most 2.
Theorem 20. (Huynh, Jain and López-Permouth, [55]). For a right $V$-ring $R$ with $\text{Soc}(R_R) = 0$ the following conditions are equivalent:

(i) Every cyclic singular right $R$-module is quasi-continuous.

(ii) $R$ has a ring-direct decomposition $R = R_1 \oplus R_2 \oplus ... \oplus R_n$, where each $R_i$ is Morita equivalent to a right PCI domain.

Carl Faith called a ring right CSI ring [35] if for each cyclic module $C$, the injective hull $E(C)$ is $\Sigma$-injective. It is still an open question whether every right CSI ring is right noetherian. In the following theorem Faith showed that right CSI rings are right noetherian in some special cases.

Theorem 21. (Faith, [35]). A right CSI ring $R$ is right noetherian under any of the following conditions:

(i) $R$ is commutative.

(ii) $R$ has only finitely many simple modules up to isomorphism, e.g. $R$ is semilocal.

(iii) $R$ satisfies the acc on colocal right ideals.

(iv) $R$ or $R/J(R)$ is right Kasch.

(v) $R/J(R)$ is von Neumann regular, e.g. $R$ is right continuous.

(vi) The injective hull of any countably generated semisimple right $R$-module is $\Sigma$-injective.
3. Rings whose Cyclic or Proper Cyclic Modules are Quasi-injective

In [33] Faith proposed to study the class of rings which satisfy the following condition: (P) Every proper cyclic module is injective modulo its annihilator ideal. Clearly, every right PCI ring satisfies the condition (P). Commutative rings with the condition (P) are precisely the pre-self injective rings studied by Klatt and Levy [79]. It may be observed that in a right self-injective ring with the condition (P), each cyclic module is quasi-injective. These rings were studied by Ahsan [2], and Koehler ([80], [81]) among others and are called \(qc\)-rings. Koehler provided a complete characterization of \(qc\) rings in the following theorem.

**Theorem 22.** (Koehler, [81]). For a ring \(R\) the following are equivalent:

1. Each cyclic right \(R\)-module is quasi-injective.
2. \(R = A \oplus B\) where \(A\) is semisimple artinian and \(B\) is a finite direct sum of self-injective, rank 0, valuation, duo rings with nil radical.
3. Each cyclic left \(R\)-module is quasi-injective.

**Proof.** If \(R\) is prime then by the Lemma 12 above, \(R\) is either simple artinian or Ore domain. But because \(R\) is also right self-injective, \(R\) is simple artinian in both the cases. This implies that each prime ideal is maximal. By the same argument as in Theorem 13, \(R\) is semiperfect. If \(e_iR\) and \(e_jR\) are indecomposable quasi-injective (hence uniform) right ideals such that \(e_iR \times e_jR\) is quasi-injective and \(\text{Hom}_R(e_iR, e_jR) \neq 0\), then \(e_iR \cong e_jR\), and are minimal right right ideals (see Lemma 26 proved later). Write \(R = e_1R \oplus ... \oplus e_nR\) as a direct sum of indecomposable quasi-injective right ideals. We can group all isomorphic indecomposable right ideals as explained above, and write after renumbering, if necessary, \(R = [e_1R] \oplus ... \oplus [e_kR]\) as a direct sum of ideals, where each bracket represents the sum of indecomposable right ideals isomorphic to the term in the bracket. By taking endomorphism rings of \(R\)-modules on both sides, we obtain \(R = M_{n_1}(e_1Re_1) \oplus ... \oplus M_{n_k}(e_kRe_k)\), where each \(M_{n_i}(e_iRe_i)\) represents \(n_i \times n_i\) matrix ring over a division ring \(e_iRe_i\). So for \(n_i \geq 2\), the matrix ring \(M_{n_i}(e_iRe_i)\) is simple artinian. It then follows all matrix rings in the decomposition of \(R\) are simple artinian except those which are local rings. It is trivial to show a right self-injective local ring with nil radical is duo. This proves (1) \(\implies\) (2). That (2) \(\implies\) (1) is straightforward. The right-left symmetry of (2) completes the proof. \(\Box\)

Jain, Singh, Symonds generalized the notion of right \(qc\)-ring and called a ring \(R\) a right \(PCQI\) ring if each proper cyclic right \(R\)-module is quasi-injective. Clearly rings with the property (P) are right \(PCQI\)-rings. Jain et al [76] showed the following.

**Theorem 23.** (Jain, Singh and Symonds, [76]).

(i) A right \(PCQI\) ring \(R\) is either prime or semiperfect.

(ii) If \(R\) is non-prime, non-local then \(R\) is a right \(PCQI\)-ring if and only if either \(R\) is \(qc\)-ring or \(R = \begin{pmatrix} D & D \\ 0 & D \end{pmatrix}\) where \(D\) is a division ring.

(iii) A local right \(PCQI\)-ring with maximal ideal \(M\) is a right valuation ring or \(M^2 = 0\) and \(M_R\) has composition length 2.

(iv) A right \(PCQI\)-domain is a right Ore domain.

(v) A nonlocal semiperfect right \(PCQI\)-ring is also a left \(PCQI\)-ring.
Here also, every nonzero prime ideal is maximal because if $P$ is a prime ideal then $R/P$ is a $qc$-ring and hence simple artinian.

Noting that for every nonzero ideal $A$, $R/A$ is a $qc$-ring and has finitely many prime ideals (see the structure of $qc$-rings). By arguing as before (see Theorem 13) PCQI-ring is either prime or semiperfect. If it is prime, then it is a right Ore domain (see Lemma 12). If $R$ is a nonlocal semiperfect ring, then we proceed to show that it is either semisimple artinian or $2 \times 2$ upper triangular matrix ring over a division ring. The reader is referred to Lemma 5 - Theorem 13 in ([76], pp 462-464) for its proof. For proving (iii), the reader is referred to (Theorem 14, p. 466, [76]) (iv) follows from Lemma 12. (v) is a consequence of the structure of nonlocal semiperfect right PCQI-ring.

The following example of Jain, Singh and Symonds shows that a local right PCQI-ring need not be a left PCQI-ring.

**Example 24.** (Jain, Singh and Symonds, [76]). Let $F$ be a field which has a monomorphism $\sigma : F \to F$ such that $[F : \sigma(F)] > 2$. Take $x$ to be an indeterminate over $F$. Make $V = xF$ into a right vector space over $F$ in a natural way. Let $R = \{(\alpha, x\beta) : \alpha, \beta \in F\}$. Define

$$(\alpha_1, x\beta_1) + (\alpha_2, x\beta_2) = (\alpha_1 + \alpha_2, x\beta_1 + x\beta_2)$$

and

$$(\alpha_1, x\beta_1)(\alpha_2, x\beta_2) = (\alpha_1\alpha_2, x(\sigma(\alpha_1)\beta_2 + \beta_1\alpha_2))$$

Then $R$ is a local ring with maximal ideal $M = \{(0, x\alpha) : \alpha \in F\}$. In fact, $M$ is also a maximal right ideal with $M^2 = 0$ and hence $R$ is a right PCQI-ring. Further, if $\{\alpha\}_{i \in I}$ is a basis of $F$ as a vector space over $\sigma(F)$ then $M = \oplus R(0, x\alpha_i)$ is a direct sum of irreducible left modules $R(0, x\alpha_i)$. Since $|I| > 2$, $R$ is not a left PCQI-ring.

In an attempt to understand rings $R$ satisfying condition (P), Jain and Singh [74] obtained the following.

**Theorem 25.** (Jain and Singh, [74]). Let $R$ be a ring satisfying the condition property (P). Then either $R$ is a right Ore-domain or semiperfect. Further, a semiperfect ring $R$ satisfies the property (P) if and only if $R$ is a right PCQI-ring.

The question of characterizing right Ore-domains with the property (P) is still open.
4. Rings whose Cyclic or Proper Cyclic Modules are Continuous

A ring $R$ is called, respectively, a right cc-ring, a right $\pi$-ring if each cyclic module is continuous, or $\pi$-injective. Note that $\pi$-injective modules are known as quasi-continuous modules. Jain and Mohamed [69] generalized Koehler’s theorem and gave the structure of cc-rings. But first we prove a more general result in the following lemma that holds for rings over which cyclic modules are $\pi$-injective (= quasi-continuous). The reader will come across the application of the next lemma at several places.

**Lemma 26.** Let $R$ be a ring over which each cyclic module is $\pi$-injective (= quasi-continuous). If $e$ and $f$ are indecomposable orthogonal idempotents in $R$ such that $eR \neq \emptyset$, then $eR$ and $fR$ are isomorphic minimal right ideals of $R$.

**Proof.** Let $0 \neq eaf \in eRf$. Now $eafR \times eR \cong (fR/\AnnR(ef)) \cap eR \cong (e+f)R/\AnnR(ef) \cap eR$. This gives by hypothesis, $eafR \times eR$ is $\pi$-injective. Since $eafR \subset eR$, and $eR$ is indecomposable, $eafR = eR$. Then $eR \cong fR/\AnnR(ef) \cap fR$ gives $eR \cong fR$. To prove that $eR$ is minimal, let $0 \neq eb \in eR$. If $eb(1-e) \neq 0$, then as before $eR = eb(1-e)R$. This implies $eR = ebR$. On the other hand, if $eb(1-e) = 0$, then $ebR \neq 0$. Since $ebeR \cap fR = (ebe+f)R$, $ebeR \cap fR$ is $\pi$-injective. Furthermore, $eR \cong fR$ implies $ebeR \cap fR \cong ebeR \times eR$. Therefore, $ebeR \times eR$ is $\pi$-injective, and so as explained earlier $ebeR = eR$. This proves $eR$ is a minimal right ideal.

**Theorem 27.** (Jain and Mohamed, [69]). A ring $R$ is a right cc-ring if and only if $R = A \oplus B$ where $A$ is semisimple artinian and $B$ is a finite direct sum of right valuation right duo rings with nil radical.

**Proof.** Since $R$ a continuous ring $J(R) = Z(R_R)$ and idempotents modulo $J(R)$ can be lifted. Also, $R/J(R)$ is a von-neumann regular ring that has finite uniform dimension (Theorem 9). This gives $R/J(R)$ is semi-simple artinian which yield that $R$ is semi-perfect. In what follows, we shall use that if $A \oplus B$ is direct sum of indecomposable projective right ideals such that for some right ideal $C \subset A$, $A/C$ embeds in $B$ and $A/C \times B$ is continuous, then $C = 0$ and $A \cong B$. Since $R$ is semi-perfect, write $R = e_1R \oplus \ldots \oplus e_nR$ as a direct sum of indecomposable continuous right ideals. We can group all isomorphic indecomposable right ideals as explained above, and write after renumbering if necessary $R = [e_1R] \oplus \ldots \oplus [e_kR]$ as a direct sum of ideals, where each bracket represents the sum of indecomposable right ideals (possibly only one term in the sum) isomorphic to the term in the bracket. By taking endomorphism rings of $R$-modules on both sides, we obtain $R = M_n(e_1Re_1) \oplus \ldots \oplus M_n(e_kRe_k)$, where each $M_n(e_iRe_i)$ represents $n_i \times n_i$ matrix ring over $e_iRe_i$ which is local. It is known that matrix ring of size greater than 1 is continuous if and only if it is self-injective and so $M_n(e_iRe_i)$ is self-injective if $n_i \geq 2$. Furthermore, it can shown that if $eR \times fR$ is continuous with $\HomR(eR, fR) \neq 0$, and every cyclic is continuous then $eR \cong fR$ is minimal (see, for example, Lemma 2.5 and Proposition 2.6 [69]). All this gives $M_n(e_iRe_i)$ is semi-simple artinian for each $n_i \geq 2$. Now consider the ring direct summand $S$ of $R$ which is a local cc-ring. We show $S$ is right duo. Let $aS$ be a right ideal and $a \in S$. If $a \notin J(S)$, then $(1-a)$ is unit and so $(1-a)xS \cong xS$. If $(1-a)xS \subset xS$, we obtain $axS \subset xS$. If $xS \subset (1-a)xS$. Since $(1-a)xS$ is continuous, $xS$ is a direct summand of $(1-a)xS$ which is indecomposable as $S$ is local. Hence
\[xS = (1 - a)xS, \text{ proving that } axS = xS. \text{ If } a \in J(S), \text{ we can similarly show that } axS \subset xS. \] So \(xS\) and thus every right ideal of \(S\) is two-sided. Finally we show \(J(S)\) is nil. This is standard argument of taking a non-nil element \(a \in J(S)\) and finding a maximal element \(P\) in the family of right ideals that does not contain any power of \(a\). This is a prime ideal. Thus \(S/P\) is a prime local \(cc\)-ring and hence right duo. This gives \(S/P\) is a continuous integral domain which is then a division ring. This yields \(P = J(S)\), a contradiction because \(a \in J(S)\). Thus all elements of \(J(S)\) are nil. Hence \(S\) is a local right duo ring with nil radical. The converse is straightforward. \(\square\)

**Remark 28.** It has been mistakenly noted in Osofsky-Smith [105] that Jain-Mohamed theorem was proved for semiperfect rings. The remark toward the end in the paper [69] stated that a cc-ring is always semiperfect.

**Lemma 29.** (Jain and Mueller, [71]). Let \(R\) be a ring over which each proper cyclic module is continuous. Then

(a) Each nonzero prime ideal is maximal.

(b) \(R\) is either prime or semiperfect with nil radical.

**Theorem 30.** (Jain and Mueller, [71]). Let \(R\) be a semiperfect ring over which each proper cyclic right module is continuous. Then \(R\) is one of the following types:

(i) \(R = \bigoplus A_i\), where each \(A_i\) is a simple artinian right valuation right duo ring with nil radical, or a local ring whose maximal ideal \(M\) satisfies \(M^2 = 0\) and \(l(M) = 2\).

(ii) \(R = \left( \begin{array}{cc} \Delta & V \\ 0 & D \end{array} \right)\) where \(D\) and \(\Delta\) are division rings and \(V\) is a one-dimensional right vector space over \(D\).

**Proof.** Assume that each proper cyclic \(R\)-module is continuous. If \(R = \bigoplus_{i=1}^n e_iR\) with \(n \geq 3\), then we can show that every cyclic module is continuous and structure of this class of rings, called cc-rings has already been given in an earlier theorem (Theorem 27). The proof is exactly on the same lines as for PCQI-rings. The reader may refer to (Proposition 7, p. 464, [76]).

Case 1. Assume \(R\) is local. Let \(I\) be a non-zero right ideal of \(R\), then \(R/I\) is continuous and hence uniform. We claim that if there exist non-zero right ideals \(A\) and \(B\) of \(R\) such that \(A \cap B = 0\) then \(A, B\) are minimal right ideals and \(S = A \oplus B\) where \(S\) is the right socle of \(R\). If \(X\) is a non-zero right ideal of \(R\) and \(X \subset A\) then \(R/X\) is uniform. But \(A/X \cap B/X = 0\) gives \(A = X\). It is immediate now that \(S = A \oplus B\). Let \(M\) be the unique maximal ideal of \(R\), and let \(x \in M, x \notin S\). Then \(xR\) must be an essential right ideal, for otherwise \(xR\) will be minimal. This implies \(S \subset xR\), and thus \(xR\) cannot be indecomposable. Therefore, \(xR = X_1 \oplus X_2\) for some non-zero right ideals \(X_1, X_2\). But then \(S = X_1 \oplus X_2\), a contradiction since \(x \in S\). Hence \(S = M\), which gives \(M^2 = 0\) and \(l(M) = 2\). Next, if each nonzero right ideal is essential, it follows immediately that \(R\) is a right valuation ring. To prove that \(R\) is right duo, consider right ideal \(aR\) and \(x \in R\). Then either \(xaR \subset aR\) or \(aR \subset xaR\). If \(xaR \subset aR\), for all \(x \in R\), then \(aR\) is two-sided. So assume \(aR \subset xaR\) for some \(x \in R\). In case \(x \notin M\) then \(xaR \cong aR\). Since \(xaR\) is continuous, we get \(xaR = aR\). In case \(x \in M\), consider \(1 - x\) and proceed as before. Thus \(aR\) is a two-sided ideal. Hence \(R\) is a right valuation right duo ring.

Case 2. Consider the case \(R \not\cong e_iR \oplus e_jR\). Assume \(e_iRe_j \neq 0\). If \(e_jRe_i\) is also not zero then \(e_iR, e_jR\) are subisomorphic to each other. It follows then
$e_i R \cong e_j R$. So every nonzero homomorphism $e_j R \rightarrow e_i R$ is a monomorphism. Next, we claim $e_j N e_j = 0$. Else, choose a nonzero element $e_j a e_j \in e_j N e_j$. This induces an $R$-homomorphism $f : e_j R \rightarrow e_j R$ given by $f(e_j x) = e_j a e_j (e_j x)$. $f$ is not monomorphism because $e_j a e_j$ is a nil element. Let $0 \neq g \in \text{Hom}_R(e_j R, e_i R)$. Then $0 \neq gf \in \text{Hom}_R(e_j R, e_i R)$. But $gf$ is not a monomorphism, a contradiction. Hence $e_j N e_j = 0$ and so $e_j R e_j$ is a division ring. This proves $R$ is simple artinian.

Consider now the case $e_j R e_j = 0$. Then $e_j N = e_j N (e_i + e_j) = e_j N e_j = 0$, and so $e_j R$ is a minimal right ideal. This gives $e_j R e_j$ is a division ring. In this case, since $e_i R$ is uniform, for all $0 \neq e_i x e_i \in e_i R e_i$, $e_i x e_i R$ is the unique minimal right ideal in $e_i R$. But then $e_i x e_j R = e_i R e_j R$, for all $0 \neq e_i x e_j \in e_i R e_j$. We proceed to prove that $e_i N e_i = 0$ and thus $e_i R e_i$ is also a division ring. If possible, let $0 \neq e_i x e_i \in e_i N e_i$. Consider the mapping $\sigma : e_i R \rightarrow e_i R$ given by left multiplication with $e_i x e_i$. Since $e_i x e_i$ is nil, $\sigma$ is not one-one. Thus $\ker(\sigma)$ contains the unique minimal right ideal $e_i R e_j R$. Therefore, $e_i x e_i R e_j R = 0$. But then $e_i x e_i R e_j = 0$. Now the unique minimal right ideal $e_i R e_j R$ is contained in every nonzero right ideal. So $e_i R e_j R \subset e_i x e_i R$. Therefore, $e_i R e_j R e_i R e_j = 0$. Since $e_j R e_j$ is a division ring, we obtain $e_i R e_j = 0$, a contradiction. Thus $e_i N e_i = 0$, which gives $e_i R e_i$ is a division ring.

We now prove that $e_i R e_j$ is a one-dimensional right vector space over $e_j R e_j$. Let $N$ denote the radical of $R$. Then $e_i N = e_i N e_i$ because $e_i N e_i = 0$. Since $e_i R e_j$ is a unique minimal right ideal in $e_i R$, $e_i N \supseteq e_i R e_j R$. Because $e_j R e_i = 0$, $e_j R e_j R = e_j R e_j R$. This implies $e_i R e_j R = e_i R e_j$ because $e_j R e_j$ is a division ring. Furthermore, $e_i R e_j$ is a right ideal as shown above and $(e_i R e_j)^2 = 0$. This implies $e_i R e_j \subset e_i N$, and so $e_i R e_j = e_i N = e_i R e_j R = e_i x e_j R e_j$. This proves $e_i R e_j$ is a one-dimensional right vector space over $e_j R e_j$. Hence, $R = \begin{pmatrix} e_i R e_i & e_i R e_j \\ 0 & e_j R e_j \end{pmatrix} \cong \begin{pmatrix} \Delta & V \\ 0 & D \end{pmatrix}$ where $D$ and $\Delta$ are division rings and $V$ is a one-dimensional right vector space over $e_j R e_j$. \qed
5. RINGS WHOSE CYCLIC OR PROPER CYCLIC MODULES ARE QUASI-CONTINUOUS (\(\pi\)-INJECTIVE)

The class of right cc-rings was further generalized by Goel and Jain [37] who studied rings over which each cyclic module is quasi-continuous (in other words, each cyclic module is \(\pi\)-injective). Goel and Jain called such rings \(\pi c\)-rings and obtained the following results.

\textbf{Theorem 31.} (Goel and Jain, [38]).

(i) Let \(R\) be a right self-injective ring. Then \(R\) is a right \(\pi c\)-ring if and only if \(R = A \oplus B\) where \(A\) is semisimple artinian and \(B\) is a finite direct sum of right self-injective right valuation rings.

(ii) Let \(R\) be a semiperfect ring. Then \(R\) is a right \(\pi c\)-ring if and only if \(R = A \oplus B\) where \(A\) is semisimple artinian and \(B\) is a finite direct sum of right valuation rings.

(iii) Let \(R\) be a right \(\pi c\)-ring with zero right singular ideal. Then \(R = A \oplus B\) where \(A\) is semisimple artinian and \(B\) is a finite direct sum of right Ore domains.

By appealing to the Theorem 9, we can prove, in general, the following theorem whence the above theorem comes as special case.

\textbf{Theorem 32.} If \(R\) is a right \(\pi c\)-ring, then \(R = R_1 \oplus R_2 \oplus \ldots \oplus R_k\), where \(R_i\) is either simple artinian or a right uniform ring.

\textbf{Proof.} By Theorem 9, \(R\) is a direct sum of uniform right ideals. Write \(R = e_1 R \oplus \ldots \oplus e_k R\) as a direct sum of uniform \(\pi\)-injective (= quasi-continuous) right ideals. It can be shown that if \(e_i R \times e_j R\) is \(\pi\)-injective such that \(\text{Hom}(e_i R, e_j R) \neq 0\) then \(e_i R \cong e_j R\) minimal right ideals (see [37], Corollary 1.13 and Lemma 2.3). By summing all isomorphic right ideals \(e_i R\) we can show that the sum is indeed two-sided ideal. This proves the desired result. \(\square\)

Huynh and Wisbauer [57] extended above results by studying finitely generated quasi-projective modules whose each factor module is \(\pi\)-injective and gave the complete structure of such modules.

\textbf{Theorem 33.} (Huynh, Wisbauer, [57]) Let \(M\) be a finitely generated quasi-projective module whose each factor module is \(\pi\)-injective. Then there is a decomposition \(M = M_0 \oplus M_1 \oplus M_2\) where \(M_0, M_1, M_2\) are fully invariant submodules such that \(M_0\) is a semisimple module, \(M_1 \cong N_1^{k_1} \oplus \ldots \oplus N_r^{k_r}\) for fully invariant summands \(N_i^{k_i}\) with \(N_i\) non-simple uniserial module, \(\text{End}_R(N_i)\) a division ring, and \(M_2 = U_1 \oplus \ldots \oplus U_k\) for fully invariant uniform modules \(U_i\) with \(\text{End}_R(U_i)\) not a division ring.

In particular, if \(R\) is a ring whose each cyclic module is \(\pi\)-injective then \(R = R_0 \oplus R_1 \oplus \ldots \oplus R_k\) where \(R_0\) is a semisimple ring and \(R_1, \ldots, R_k\) are rings which are uniform as right modules and any \(R_i\) is right uniserial if and only if it is local.

We now consider semiperfect rings over which each proper cyclic right module is \(\pi\)-injective (= quasi-continuous).

We call a ring \(R\) to be a right \(pc\pi i\)-ring if each proper cyclic right \(R\)-module is \(\pi\)-injective. The following lemma follows from the definition of \(\pi\)-injective module.

\textbf{Lemma 34.} Let \(A\) and \(B\) be \(R\)-modules such that \(A\) is embeddable in \(B\). If \(A \times B\) is \(\pi\)-injective then the exact sequence \(0 \to A \to B\) splits.
If \( A, B \) are continuous \( R \)-modules such that \( A \) is embeddable in \( B \) and \( B \) is embeddable in \( A \), then it is well-known that \( A \cong B \). However, this is not true for \( \pi \)-injective modules, in general. But, if both \( A \) and \( B \) are indecomposable projective and \( A \times B \) is \( \pi \)-injective, then \( A \cong B \).

**Lemma 35.** If \( M_1 \) and \( M_2 \) are \( \pi \)-injective such that \( M_1 \times M_2 \) is \( \pi \)-injective and \( E(M_1) \cong E(M_2) \) then \( M_1 \cong M_2 \).

Mueller and Rizvi [93] showed that if \( M_1 \) and \( M_2 \) are direct summands of a quasi-continuous module with \( E(M_1) \cong E(M_2) \) then \( M_1 \cong M_2 \). The above lemma is an easy consequence of this result.

**Theorem 36.** Let \( R = \oplus_{i \in I} e_i R \) be a semiperfect ring with nil radical, where \( e_i R \) are indecomposable right ideals and \( |I| \geq 3 \). Then \( R \) is right a \( pc\pi i \)-ring if and only if \( R = \oplus R_i \) where each \( R_i \) is either simple artinian or a right valuation ring.

**Proof.** Since the number of summands is more than 2, for each pair of indecomposable right ideals \( e_i R, e_j R \), we have \( e_i R \times e_j R \) is \( \pi \)-injective. Suppose \( e_i R e_j \neq 0 \). By above lemmas, \( e_i R \cong e_j R \) and \( e_i R \times e_j R \) is \( \pi \)-injective and the Lemma 26 applies to give \( e_i R \) is minimal. If \( e_i R e_j = 0 \) and \( e_j R e_i = 0 \) for all \( j \neq i \) then \( e_i R \) is a two-sided ideal and is a ring direct summand of \( R \) which is a right valuation ring. This proves the theorem. \( \square \)

**Lemma 37.** Let \( R = e_1 R \oplus e_2 R \) be a semiperfect ring with nil radical \( N \), where \( e_1 R, e_2 R \) are indecomposable right ideals such that \( e_1 R e_2 \neq 0 \) and \( e_2 R e_1 \neq 0 \). Then \( R \) is a right \( \pi \pi i \)-ring if and only if it is simple artinian.

**Proof.** Suppose \( R \) is a right \( \pi \pi i \)-ring. By hypothesis, there exists a nonzero homomorphism \( f : e_1 R \rightarrow e_2 R \). This gives \( R/\ker f \cong e_1 R/\ker f \times e_2 R \). If \( R/\ker f \cong R \) then \( R \cong e_1 R \times \ker f \), and this implies \( \ker f = 0 \), because \( R \) is semiperfect. If \( R/\ker f \neq R \), then \( e_1 R/\ker f \times e_2 R \) is \( \pi \)-injective and \( e_1 R/\ker f \) embeds in \( e_2 R \). This implies \( e_1 R/\ker f \cong e_2 R \) and so \( \ker f = 0 \). In any case, \( e_1 R \) is embeddable in \( e_2 R \) and similarly \( e_2 R \) is embeddable in \( e_1 R \). This yields either \( e_1 R \cong e_2 R \), or \( e_1 R \) embeds in \( e_2 N \) and \( e_2 R \) embeds in \( e_1 R \). Therefore, either \( e_1 R \cong e_2 R \) or \( R \) embeds in \( N \). The latter is impossible. Hence \( e_1 R \cong e_2 R \).

We now show \( e_1 N e_1 = 0 \). Since \( N \) is nil, there exists \( (e_1 x e_1) \) such that \( (e_1 x e_1)^{k-1} \neq 0 \) but \( (e_1 x e_1)^k = 0 \). Consider the mapping \( g : e_1 R \rightarrow e_1 R \) given by left multiplication with \( e_1 x \). Then the image of \( (e_1 x e_1)^{k-1} \) is zero, a contradiction because this mapping naturally induces a mapping from \( e_1 R \) to \( e_2 R \) which as proved above must be one-to-one. Hence \( e_1 N e_1 = 0 \). This yields \( R \) is a \( 2 \times 2 \) matrix ring over a division ring \( e_1 R e_1 \). \( \square \)

**Theorem 38.** Let \( R = e_1 R \oplus e_2 R \) be a semiperfect right \( \pi \pi i \)-ring with nil radical \( N \), where \( e_1 R, e_2 R \) are indecomposable right ideals such that \( e_1 R e_2 \neq 0 \) and \( e_2 R e_1 \neq 0 \). Then \( e_2 R \) is a minimal right ideal and \( \ann_{e_1 R e_1} (e_1 R e_2) = e_1 N e_1 \).

**Proof.** As in the previous theorem any nonzero map from \( e_2 R \) to \( e_1 R \) is a monomorphism. We show \( e_2 N = 0 \). Choose \( e_1 x e_2 \neq 0 \) and if possible let \( e_2 y \neq 0 \). Since \( N \) is nil, we can assume \( (e_2 y)^2 = 0 \). Then \( (e_1 x e_2 y e_2) e_2 y = 0 \). This yields \( e_1 x e_2 y e_2 = 0 \). Again \( e_2 y e_2 = 0 \). This gives \( e_2 y = 0 \) since \( e_2 R e_1 = 0 \), a contradiction. Hence \( e_2 N = 0 \), and so \( e_2 R \) is unique minimal right ideal in \( e_2 R \) because \( e_2 R \) is uniform. The last part is straightforward. For more details the reader may refer to [38]. \( \square \)
Lemma 39. Under the hypothesis of the above lemma \( A = \begin{bmatrix} e_1 Ne_1 & 0 \\ 0 & 0 \end{bmatrix} \) is an ideal in \( S = \begin{bmatrix} e_1 Re_1 & e_1 Re_2 \\ 0 & e_2 Re_2 \end{bmatrix} \) \( \cong R \) and \( S/A \cong \begin{bmatrix} e_1 Re_1/e_1 Ne_1 & e_1 Re_2 \\ 0 & e_2 Re_2 \end{bmatrix} \) is not \( \pi \)-injective as \( S \) or \( S/A \)-module and therefore \( R \) is not a right pc\( \pi \text{-}i \) ring if \( A \neq 0 \).

Proof. It follows from above lemma that \( e_1 Re_2 \) is a one-dimensional right vector space over \( e_2 Re_2 \). Thus we may write \( \begin{bmatrix} K & D \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \) as \( S/A \)-modules, where \( K \) and \( D \) are division rings. Since \( K \) and \( D \) are division rings, \( K \) and \( D \) are division rings. Since the second summand can be embedded in the first but the embedding is not onto, \( S/A \) cannot be \( \pi \)-injective. This completes the proof. \( \Box \)

Theorem 40. Let \( R = e_1 R \oplus e_2 R \) be a semiperfect ring with nil radical such that \( e_1, e_2 \) are primitive orthogonal idempotents, \( e_1 Re_2 \neq 0, e_2 Re_1 = 0 \). Then \( R \) is a right pc\( \pi \text{-}i \)-ring if and only if \( R \cong \begin{bmatrix} K & V \\ 0 & D \end{bmatrix} \) where \( K \) and \( D \) are division rings and \( V \) is one-dimensional right space over \( D \).

Proof. By Theorem 38, \( e_2 R \) is a minimal right ideal and hence \( e_2 Re_2 \) is a division ring. Since each proper ring homomorphic image of \( R \) is a \( \pi c \)-ring it follows that \( e_1 Ne_1 = 0 \). Thus \( R \cong \begin{bmatrix} K & V \\ 0 & D \end{bmatrix} \) where \( K \) and \( D \) are division rings and \( V \) is one-dimensional right space over \( D \). Thus the “only if” part of the theorem is completed. The converse is straightforward. \( \Box \)

To complete our discussion we need to give characterization of local pc\( \pi \text{-}i \)-rings, this is contained in the following theorem whose proof is similar to the proof of local rings over which every proper cyclic module is continuous. However, note that the local right pc\( \pi \text{-}i \)-ring need not be right duo.

Theorem 41. Let \( R \) be a local ring with unique maximal ideal \( N \). Then \( R \) is a right pc\( \pi \text{-}i \)-ring if and only if \( R \) is either a right valuation ring or \( N^2 = 0 \), and the composition length of \( N \) is 2.

In summary, we have obtained the following description of non-local right pc\( \pi \text{-}i \)-rings.

Theorem 42. (a) Let \( R \) be a non-local semiperfect right pc\( \pi \text{-}i \)-ring Then either \( R \) is a direct sum of rings of the following types (not necessarily all)

(i) semisimple artinian ring.

(ii) right valuation rings with maximal ideal \( M \) satisfying \( M^2 = 0, \text{ and } l(M) = 2 \).

(iii) \( \begin{bmatrix} K & V \\ 0 & D \end{bmatrix} \) where \( K \) and \( D \) are division rings and \( V \) is one-dimensional right space over \( D \).

Gómez Pardo and Guil Asensio studied rings over which a family of modules that are quasi-continuous with respect to pure-essential sequences [42]. A right module \( M \) is called a pqc-module when (i) every pure submodule of \( M \) is purely essential in a direct summand of \( M \), and (ii) if \( U, V \) are direct summands of \( M \)
such that $U \cap V = 0$ and $U \oplus V$ is pure in $M$, then $U \oplus V$ is a direct summand of $M$. A module $M$ will be called a completely $pqc$-module when every pure quotient of $M$ is a $pqc$-module. The main result in (Theorem 2.9, [42]) shows that if $M$ is a finitely presented completely $pqc$-module, then $M$ has $ACC$ (and $DCC$) on direct summands and so it is a (finite) direct sum of indecomposables.

It will be interesting to study rings over which every proper quotient of $M$ is a finitely presented (i) completely $pqc$-module, or (ii) a completely pure injective module. For definitions of purely essential and other relevant terms, the reader may refer to [42].
6. Rings over which Cyclic Modules are Weakly Injective

In [66] Jain, López-Permouth and Singh studied rings whose each cyclic module is weakly-injective and proved the following.

Theorem 43. (Jain, López-Permouth and Singh, [66]). The following conditions on a ring $R$ are equivalent:

(a) $R$ is right weakly-semisimple, that is, each right $R$-module is weakly-injective.

(b) Each finitely generated right $R$-module is weakly injective and $R$ is right noetherian.

(c) Each cyclic right $R$-module is weakly $R^2$-injective and $R$ is right noetherian.

(d) Each cyclic uniform right $R$-module is weakly $R^2$-injective and $R$ is right noetherian.

Proof. (a) $\implies$ (b). Let $R$ be a right weakly-semisimple ring. We will show that $R$ must be right noetherian. Let $M$ be a quasi-injective right $R$-module. By assumption, $M$ is weakly-injective. Suppose $M$ is not injective. Then by Zorn’s lemma, there exists a submodule $A$ of $E(M)$ and a homomorphism $f : A \to M$ which cannot be extended to any $f' : B \to M$ with $B$ a submodule of $E(M)$ containing $A$ properly. Let $b \in E(M)/A$. As $A \subseteq E(M)$, we have $C = bR \cap A \neq 0$. Let $f_1 : C \to M$ be the restriction of $f$ to $C$. As $M$ is weakly-injective, $bR$ embeds in $M$. Therefore, $M$ is $bR$-injective and $f_1$ extends to $g : bR \to M$. Define $f' : A + bR \to M$ by $f'(a + br) = f(a) + g(br)$ for $a \in A, r \in R$. Since $f'$ extends $f$, this yields a contradiction. Hence, $M$ must be injective. Thus, $R$ is a right $QI$ ring and hence by Boyle [11], $R$ must be right noetherian.

Clearly (b) $\implies$ (c) and (c) $\implies$ (d).

(d) $\implies$ (a). Suppose each cyclic right $R$-module is weakly $R^2$-injective and $R$ is right noetherian. Let $M$ be any right $R$-module and $x_1, \ldots, x_n \in E(M)$. Let $K = \sum_{i=1}^n x_iR$. It follows that $E(M) = E(K) \oplus L$ for some submodule $L$ of $E(M)$ and $E(K)$ has finite uniform dimension. Therefore, $M \cap E(K)$, being essential in $E(K)$, has finite uniform dimension. Let $N$ be a finite direct sum of uniform cyclic modules which is essential in $M \cap E(K)$. Note that if a cyclic right $R$-module is weakly $R^2$-injective then it is weakly-injective. Since over a right q.f.d. ring every direct sum of weakly-injective modules is again weakly-injective, it follows that $N$ is weakly-injective. Thus there exists an automorphism $\sigma : E(K) \to E(K)$ with $K \subseteq \sigma(N) \subseteq \sigma(M \cap E(K))$. Clearly $\sigma$ extends to an automorphism $\phi$ of $E(M)$ which is identity on $L$. This automorphism satisfies $K \subseteq \phi(M)$. Hence $M$ is weakly-injective. Thus, every right $R$-module is weakly-injective. Therefore, $R$ is right weakly-semisimple.

\[\Box\]

7. Rings over which Cyclic Modules are Quasi-Projective

Rings over which each cyclic module is projective are obviously semisimple artinian. The study of rings over which each cyclic module is quasi-projective was initiated by Koehler. She called a ring a right $q^*$-ring if each cyclic module is quasi-projective. Koehler proved the following.

Theorem 44. (Koehler, [80]).

(i) A semiperfect ring $R$ is a left $q^*$-ring if and only if every left ideal in the Jacobson radical of $R$ is an ideal.
(ii) Let $R$ be a left self-injective semiperfect left $q^*$-ring. Then $R = A \oplus B$, where $A$ is semisimple artinian and $B$ is a finite direct sum of pairwise non-isomorphic uniform left ideals.

(iii) If $R$ is a prime semiperfect left $q^*$-ring then $R$ is either semisimple artinian or $R$ is local.

(iv) Let $R$ be a quasi-Frobenius ring. Then $R$ is a left $q^*$-ring if and only if $R$ is a right $q^*$-ring.

(v) A proper matrix ring $M_n(R)$, $(n > 1)$ is a left $q^*$-ring if and only if $R$ is semisimple artinian.

Proof. (i) To prove this we first note the following useful facts proved by Miyashita [90] and Wu-Jans [116]. Let $P$ be a projective module, $\phi : P \to M$ be an endomorphism and $S = \text{End}(P)$. Then

(a) $M$ is quasi-projective if $\ker(\phi)$ is invariant under $S$, and

(b) $\ker(\phi)$ is invariant under $S$ if $\ker(\phi)$ is small in $P$ and $M$ is quasi-projective.

Let $R$ be a semiperfect left $q^*$-ring. Let $I$ be a left ideal of $R$ contained in $J(R)$. Consider $\phi : R \to R/I$. Since $I$ is small in $R$ and $R/I$ is quasi-projective, $I$ is invariant under $\text{End}(R) \cong R$ by (b) above. Hence $I$ is an ideal. Conversely, assume that every left ideal in $J(R)$ is an ideal. Let $K$ be a left ideal in $R$. Then $R/K$ has a projective cover $\phi : P \to R/K$, and $P$ can be considered to be a direct summand of $R$. Since $\ker(\phi)$ is small in $P$, it is small in $R$ and contained in $J(R)$. If $f \in \text{End}(P)$, then there is an $r \in R$ such that $f(x) = xr$ for every $x \in P$. Therefore, $\ker(\phi)$ is invariant under $\text{End}(P)$. Hence by (b) above, $R/I$ is quasi-projective. Therefore, $R$ is a left $q^*$-ring.

(ii) Since $R$ is semiperfect, $R = Re_1 + \ldots + Re_k + Re_{k+1} + \ldots + Re_n$, where $e_1, \ldots, e_n$ are orthogonal indecomposable idempotents. We may assume $Re_1, \ldots, Re_k$ are all the simple components of the decomposition. By (i), $J(R)e_i,e_iRe_j = 0$ if $i \neq j$. Since $\text{Hom}(Re_i,Re_j) = e_iRe_j$, $Re_i \not\subset Re_j$, for $i, j > k$ and $i \neq j$. Let $A = Re_1 + \ldots + Re_k$, and $B = Re_{k+1} + \ldots + Re_n$. To complete the proof we show that $A$ and $B$ are ideals. Let $i \leq k$ and $j > k$. Then $Re_i,e_iRe_j$ is 0 or simple and is contained in $Re_i$. Since $Re_i$ is a simple injective module, and $Re_j$ is not simple and is indecomposable, $Re_i,e_iRe_j = 0$ because $Re_i$ is a simple projective module.

(iii) Since $R$ is semiperfect, $R = Re_1 + \ldots + Re_n$ where $e_1,\ldots,e_n$ are nonzero orthogonal indecomposable idempotents. If $n = 1$, then $R$ is local because $J(R)e_1$ is a unique maximal left ideal in $Re_1$. From (i), $J(R)e_i,J(R)e_j \subset J(R)e_i\cap J(R)e_j = 0$ for $i \neq j$. Hence $J(R)e_i = 0$ for all but at most one $i$, say $i = k$, because $R$ is prime. Since $e_kRe_1 \neq 0$ (because $R$ is prime), there is an epimorphism from $Re_k$ to $Re_i$. So, $Re_k \cong Re_i$ for $i = 1,2,\ldots,n$ because $Re_i$ is simple and projective, and $Re_k$ is indecomposable. Therefore $R$ is semisimple artinian if $n > 1$.

(iv) Let $R$ be a quasi-Frobenius left $q^*$-ring. We first claim that each essential left ideal of $R$ is an ideal. Let $I$ be an essential left ideal of $R$. Then $R/I$ has a projective cover $\phi : P \to R/I$. Let $f : R \to R/I$ be the canonical homomorphism. Then there is an epimorphism $f' : R \to P$ such that $\phi \circ f' = f$ because $P$ is projective and $\ker(\phi)$ is small. Since $P$ is projective and $f'$ is onto, $R = Re_1 \oplus Re_2$ with $Re_1 \cong P$. Also, it can be seen that $I = K \oplus Re_2$ where $K \neq \ker(\phi)$ and $K \subset Re_1$. The left ideal $K$ is small in $R$, and $R$ is a left $q^*$-ring. Thus $K \subset J(R)$, and $K$ must be an ideal by (i). Now as $I$ is an essential left ideal of $R$, the left socle $\text{Soc}(R)$ of $R$ is contained in $I$. The left ideal $J(R)e_1$ is an ideal in $R$. So
Re_2 e, Re_1 \subseteq \text{Socle}_1(R)$ because $J(R)e_2e_2 Re_1 = 0$ and $R$ is semiperfect. Therefore, $I$ is an ideal. Next, we claim that each left ideal of $R$ is quasi-injective. Let $A$ be any left ideal of $R$. If $B$ is a complement of $A$ in $R$ then $A \oplus B$ is an essential left ideal of $R$. We have just now shown that $A \oplus B$ must be an ideal of $R$. Since $R$ is right self-injective, $A \oplus B$ is quasi-injective and hence $A$ is quasi-injective. Thus each left ideal of $R$ is quasi-injective. Now, we proceed to show that $R$ must be right $q^*$. Let $L$ be a right ideal contained in $J(R)$ and $a \in L$. Since $R$ is left self-injective, $J(R) = \{a : \text{ann}_l(a) \text{ is an essential left ideal of } R\}$ and $\text{ann}_r(\text{ann}_l(aR)) = aR$. In addition, $\text{ann}_l(aR) = \text{ann}_l(a)$ which is an essential left ideal of $R$. Therefore, $aR$ is an ideal, that is, $L$ is an ideal. Hence, by (i), $R$ is a right $q^*$-ring. The converse is similar.

(v) For proof of this part, the reader is referred to the Theorem 2.5, [80].

Koehler gave example of a ring which is both left and right artinian, a right $q^*$-ring but not a left $q^*$-ring.

**Example 45.** (Koehler, [80]). Let $R$ be the ring of matrices of the form \[
\begin{bmatrix}
\bar{a} & \bar{b} \\
0 & \bar{c}
\end{bmatrix}
\] such that $\bar{a} \in \mathbb{Z}_4$ and $\bar{b}, \bar{c} \in \mathbb{Z}_2$. Clearly, $J(R)$ consists precisely of the matrices of the form \[
\begin{bmatrix}
\bar{a} & \bar{b} \\
0 & 0
\end{bmatrix}
\] such that $\bar{a} = \bar{0}$ or $2 \in \mathbb{Z}_4$ and $\bar{b} \in \mathbb{Z}_2$. Every right ideal in $J(R)$ is an ideal. However the left ideal $I$ consisting of exactly two elements \[
\begin{bmatrix}
\bar{0} & \bar{0} \\
0 & \bar{0}
\end{bmatrix}
\] and \[
\begin{bmatrix}
\bar{2} & \bar{1} \\
0 & \bar{0}
\end{bmatrix}
\] is not an ideal. Therefore, by (i) of the above theorem, $R$ is a right $q^*$-ring but not a left $q^*$-ring.
8. Hypercyclic, \( q \)-Hypercyclic and \( \pi \)-Hypercyclic Rings

Caldwell [16] initiated study of rings whose each cyclic module has a cyclic injective hull. Such rings are called hypercyclic rings. A ring \( R \) is called restricted hypercyclic if \( R \) is hypercyclic and \( R/J(R) \) is artinian. Caldwell proved the following.

**Theorem 46.** (Caldwell, [16]). A commutative hypercyclic ring must be restricted.

*Proof.* Let \( R \) be a commutative hypercyclic ring. Then \( R \) is self-injective. Therefore, \( R/J(R) \) is a self-injective von Neumann regular ring. Clearly, \( R/J(R) \) is either semisimple artinian or it has an infinite set of orthogonal idempotents. Suppose \( R/J(R) \) has an infinite set of orthogonal idempotents. Since orthogonal idempotents can be lifted orthogonally modulo \( J(R) \), \( R \) has an infinite set of orthogonal idempotents, say \( \{e_i\} \). Let \( I = \Sigma e_i R \). Then \( R/I \) would be non-injective, a contradiction to the fact that if \( \{e_i\} \) is any set of idempotents in a commutative hypercyclic ring \( R \), and if \( I = \Sigma e_i R \) then \( R/I \) is an injective \( R \)-module. \( \square \)

Osofsky continued the study of hypercyclic rings and showed the following.

**Theorem 47.** (Osofsky, [102]). A ring is restricted hypercyclic if and only if it is a ring direct sum of matrix rings over hypercyclic local rings.

Rosenberg and Zelinsky [107] considered rings over which each cyclic module has injective hull of finite length. This led Jain and Saleh [72] to consider rings over which each cyclic module has finitely generated injective hull. A ring \( R \) is called \( q \)-hypercyclic if each cyclic \( R \)-module has a cyclic quasi-injective hull. A ring \( R \) is called \( q \)-hypercyclic if each cyclic \( R \)-module has a cyclic quasi-injective hull.

**Lemma 48.** (Jain and Saleh, [72]).

(a) Let \( R \) be any ring such that \( R/J(R) \) is semisimple artinian and \( J \neq J^2 \). Then any simple \( R \)-module \( A \) that is not injective can be embedded in \( J/J^2 \).

(b) Let \( M \) be an injective \( R \)-module and \( I, K \) be ideals in \( R \), where \( K \subseteq I \). Then \( \text{Hom}(I/K, M) \cong \text{Ann}_M(K)/\text{Ann}_M(I) \) as \( R \)-modules.

(c) Let \( R \) be a ring such that every cyclic \( R \)-module has finitely generated quasi-injective hull. Then every ring homomorphic image of \( R \) has this property.

*Proof.* (a) This is straightforward. (b) The proof follows from the canonical embedding of \( \text{Ann}_M(K)/\text{Ann}_M(I) \) into \( \text{Hom}(I/K, M) \) and the Baer criterion for injective modules. (c) Let \( A \) be a two-sided ideal of \( R \). Let \( \bar{R} = R/A \) and \( \bar{R}/I \) be a cyclic \( \bar{R} \)-module, where \( I = I/A \). We have \( \bar{R}/I \cong R/I \) as \( R \)-modules. Denote by \( P \) the quasi-injective hull of \( R/I \) as an \( R \)-module. \( P \) is a finitely generated \( R \)-module. Since \( P = \text{End}_R(E(R/I))R/I \), \( A \) annihilates \( P \) and hence \( P \) is an \( R/A \)-module and indeed \( P \) is quasi-injective as an \( R/A \)-module. If \( B \subseteq P \), and \( B \) is quasi-injective as an \( R/A \)-module, then \( B \) is also quasi-injective as an \( R \)-module. Hence, \( P \) is the quasi-injective hull of \( R/I \) as an \( R/A \) module. Since \( P \) is a finitely generated \( R \)-module and \( A \) annihilates \( P \), \( P \) is also finitely generated as an \( R/A \)-module. \( \square \)

**Theorem 49.** (Jain and Saleh, [72]). Let \( R \) be a right artinian ring. The following statements are equivalent.

(a) Every cyclic \( R \)-module can be embedded in a finitely generated injective module.

(b) \( \text{Hom}(J/J^2, A) \) is finitely generated for every simple \( R \)-module \( A \).

(c) Every cyclic \( R \)-module has finitely generated quasi-injective hull.

(d) The injective hull of \( R/J^2 \) is a finitely generated \( R \)-module.
Theorem 53. Let $\mathcal{R}$ be a cyclic $\mathcal{R}$-module. Since $q.inj.hull(M) \subseteq E(M)$, and $\mathcal{R}$ is artinian, it follows that $q.inj.hull(M)$ is also finitely generated.

Lemma 51. Only if $R$ is completely prime, which is a contradiction. This proves that $J$ is artinian. Suppose $J$ is not nilpotent, then $J/J^2$ is also finitely generated. Therefore, $Hom_R(J/J^2, I/J^2)$ is a finitely generated $R/J^2$-module for any minimal right ideal $I/J^2$ of $R/J^2$ by Rosenberg-Zelinsky [107]. But then $Hom(J/J^2, I/J^2)$ is a finitely generated $K$-module, where $K = (R/J^2)/(J/J^2) \cong R/J$. Therefore, $Hom_R(J/J^2, I/J^2)$ is a finitely generated $R/J$-module, and so $Hom_R(J/J^2, I/J^2)$ is a finitely generated $R$-module for each simple submodule $I/J^2$ of $R/J^2$. This yields $E_2$, the injective hull of $R/J^2$ as an $R$-module, is finitely generated (see [107], Theorem 1). $(d) \implies (b)$: Let $A$ be a simple $R$-module. If $A$ is injective, then by the above lemma, $Hom(J/J^2, A) = 0$, or $A$ which is finitely generated. If $A$ is not injective, then by above lemma, $A$ can be embedded in $J/J^2$. So there exists a right ideal $I$ of $R$ such that $I \subseteq J$ and $A = I/J^2$. Since $I/J^2$ is a simple submodule of $R/J^2$, and the injective hull of $R/J^2$ as an $R$-module is finitely generated, $Hom_R(J/J^2, I/J^2)$ is finitely generated. Therefore, $Hom_R(J/J^2, A)$ is finitely generated. It remains to see that $(b) \implies (a)$, and this holds by ([107], Theorem 1) and the above lemma. □

Next, we consider rings with Krull dimension over which each cyclic module has cyclic injective hull. Indeed such rings turn out to be artinian rings. We shall need the following.

Lemma 50. (Gordon and Robson, [43]). If $R$ is a ring with Krull dimension $K(R)$ then $K(R) = K(R/P)$, for some prime ideal $P$. In fact $P$ is a minimal prime ideal.

The following is straightforward.

Lemma 51. If $R$ is a valuation ring, and $P = aR$ is a nonzero prime ideal $P \neq J$, then $R/P$ is not a domain.

Proposition 52. (Jain and Saleh, [72]). Let $R$ be a local hypercyclic ring. If $R$ has Krull dimension, then $R$ is right artinian.

Proof. First suppose $J = J(R)$ is nil. Then, since $R$ has Krull dimension, $J$ is nilpotent. This implies $J$ is the only prime ideal of $R$. But then by Lemma 50, $K(R) = K(R/J) = 0$, which proves that $R$ is artinian. Suppose $J$ is not nil. Then by Ossofsky ([102], Theorem 2.12) there exists a nonzero nilpotent ideal $aR \subset J$, $a \in R$ such that $aR$ is the maximal proper two-sided ideal below $J$. Since $R$ has Krull dimension, $R$ satisfies $acc$ on prime ideals. If $J$ is not the only prime ideal, there exists a prime ideal $Q$ such that $Q$ is maximal among all the prime ideals different from $J$. Then $Q \subset aR$. Since $aR$ is nilpotent, $Q = aR$. Thus $Q \neq (0)$, since $aR$ is not zero. So by Lemma 51 and $R$ being valuation, $Q$ is not completely prime. Consider now the prime ring $R/Q$. Since $R/Q$ has Krull dimension, it is a prime Goldie ring. Therefore $Z(R/Q)$, the right singular ideal of $R/Q$, is zero. Thus $ann(x) = 0$, for every $0 \neq x \in R/Q$, since $R$ is a valuation ring. Hence $Q$ is completely prime, which is a contradiction. This proves that $J$ is the only prime ideal. Therefore, $K(R) = K(R/J) = 0$, and hence $R$ is artinian. □

Theorem 53. Let $R$ be a hypercyclic ring. Then $R$ has Krull dimension if and only if $R$ is artinian.
Proof. If $R$ has Krull dimension, then each homomorphic image of $R$ has $acc$ on direct summands. Thus by Osfolsky ([102], Lemma 1.7), $R$ has $acc$ on direct summands. Thus, by Osfolsky, $R$ is a ring direct sum of matrix lings over local hypercyclic rings and hence $R$ is artinian. \hfill \square

Corollary 54. (Jain and Saleh, [72]). A hypercyclic ring with Krull dimension is quasi-Frobenius.

Dinh, Guil Asensio and López-Permouth studied hereditary ring $R$ such that the injective hull $E(R_R)$ is finitely generated (or cyclic) and proved the following which answers in the affirmative a question posed by Dauns in [28] and Gómez-Perdo, Dung and Wisbauer in [40].

Theorem 55. (Dinh, Guil Asensio, López-Permouth, [29]). Let $R$ be a right hereditary ring. If the injective hull $E(R_R)$ is finitely generated (or cyclic), then $R$ is right artinian.

More generally, they obtained the following.

Theorem 56. (Dinh, Guil Asensio, López-Permouth, [29]). Let $R$ be a right hereditary ring. If the injective hull $E(R_R)$ is countably generated, then $R$ is right noetherian.

For a commutative ring $R$, $R$ can be shown to be $q$-hypercyclic (= $qc$-ring) if $R$ is hypercyclic. Whether a hypercyclic ring (not necessarily commutative) is $q$-hypercyclic is considered by showing that a local hypercyclic ring $R$ is $q$-hypercyclic if and only if the Jacobson radical of $R$ is nil. However, It is not known if there exists a local hypercyclic ring with non-nil radical.

Lemma 57. Let $R$ be semiperfect and $q$-hypercyclic. Then $R$ is right self-injective.

Proof. Let $I$ be a right ideal of $R$ such that $R/I$ is the quasi-injective hull of $R$. Let $\phi: R \to R/I$ be the embedding. Since $R/I$ contains a copy of $R$, $R/I$ is injective. Let $\phi(R) = B/I$. Then $B/I \subseteq R/I$. Hence $B \subseteq R$. Since $R \cong B/I$, $B/I$ is projective. Thus $B = I \oplus K$ for some $K_R \subseteq B_R$. Now $R \cong B/I = (I \oplus K)/I \cong K$. Therefore $E(R) \cong E(K)$. But then $I \oplus K \subseteq R$ implies $E(R) = E(I) \oplus E(K) \cong E(I) \oplus E(R)$. Since $E(R) \cong R/I$, $E(R)$ is a finite direct sum of indecomposable modules, by [102]. Thus $E(R)$ has finite Azumaya-Diagram. Therefore, $E(R) \cong E(R) \oplus E(I)$ implies $E(I) = 0$. Hence $I = 0$. Thus $R$ is right self-injective. \hfill \square

Lemma 58. (Jain and Malik, [68]). Let $R$ be $q$-hypercyclic. Then every homomorphic image of $R$ is also $q$-hypercyclic.

Proof. Let $A$ be a two-sided ideal of $R$. Let $\bar{R} = R/A$. Let $\bar{R}/\bar{I}$ be a cyclic $\bar{R}$-module, where $\bar{I} = I/A$. But $\bar{R}/\bar{I} \cong R/I$. Since $A \subseteq I$, annihilates $R/I$. Let $R/K$ be the quasi-injective hull of $R/I$ as an $R$-module. Then $R/K \cong \text{End}_R(E(R/I)/R/I)$. Then it follows that $A$ annihilates $R/K$. Thus $R/K$ may be regarded as an $\bar{R}$-module. Since $R/K$ is quasi-injective as an $R$-module, $R/K$ is quasi-injective as an $R$-module. Since $A$ is a two-sided ideal and annihilates $R/K$, $A \subseteq K$. Hence $R/K \cong (R/A)/(K/A)$. Clearly $\bar{R}/\bar{K}$ is the quasi-injective hull of $\bar{R}/\bar{I}$ as an $\bar{R}$-module. Hence $\bar{R}$ is $q$-hypercyclic. \hfill \square

Jain and Malik [68] studied local $q$-hypercyclic rings and obtained the following results.
Lemma 59. (Jain and Malik, [68]). Let $R$ be a local $q$-hypercyclic ring. Then
(i) Both right and left ideals of $R$ are linearly ordered.
(ii) $R$ is left bounded or right bounded.
(iii) $R$ is a duo ring.

Theorem 60. (Jain and Malik, [68]).
(i) Let $R$ be a local ring. Then $R$ is $q$-hypercyclic if and only if $R$ is a $qc$-ring.
(ii) Let $R$ be a local hypercyclic ring. Then $R$ is $q$-hypercyclic if and only if $J(R)$ is nil.

Proof. (i) Let $R$ be $q$-hypercyclic and let $A$ be a non-zero right ideal of $R$. Then by above lemma, $A$ is a two-sided ideal of $R$. But then by Lemma 58, $R/A$ is a self-injective ring. Thus $R/A$ is a quasi-injective $R$-module, proving that $R$ is a $qc$-ring. The converse is obvious.

(ii) Let $R$ be a local hypercyclic ring with $J(R)$ nil. Then by [102], $R$ is a duo, self-injective, valuation ring. But then $R$ is maximal. Therefore, $R$ is a $qc$-ring and hence a division ring. Therefore $P$ is a maximal ideal of $R$. Thus $P = J(R)$, a contradiction. Hence $J(R)$ is nil.

Theorem 61. (Jain and Malik, [68]).
(i) Let $R$ be a commutative $q$-hypercyclic ring. Then $R$ must be self-injective.
(ii) Let $R$ be a commutative ring. Then $R$ is $q$-hypercyclic if and only if $R$ is a $qc$-ring.
(iii) Let $R$ be a commutative hypercyclic ring. Then $R$ is $q$-hypercyclic

Proof. (i) This is obvious.

(ii) This is similar to the proof of Theorem 60 (i).

(iii) Let $R$ be a commutative hypercyclic ring. Then by ([16], Theorem 2.5), $R$ is a finite direct sum of commutative local hypercyclic rings. So it suffices to show that a commutative local hypercyclic ring is $q$-hypercyclic. Let $R$ be commutative local and hypercyclic. Then by [16], $R$ is valuation and self-injective, and $J(R)$ is nil. Then by ([79], Theorem 2.3), $R$ is maximal. Since $J(R)$ is nil, $R$ has rank 0. Then $R$ is rank 0 maximal valuation ring. Thus $R$ is a $qc$-ring [81], proving the theorem.

The following example of Jain and Malik shows that a $q$-hypercyclic ring need not be hypercyclic.

Example 62. (Jain and Malik, [68]). Let $F$ be a field, and $x$ be an indeterminate over $F$. Let $W = \{ \{ \alpha_i \} : \{ \alpha_i \} \text{ is a well ordered sequence of nonnegative real numbers} \}$. Let $T = \{ \sum_{i=0}^{\infty} a_i x^{\alpha_i} : a_i \in F, \{ \alpha_i \} \in W \}$ and $R = \frac{T}{J(T)}$. Let $S$ be the socle of $R$. Then $R/S$ is a $q$-hypercyclic ring but not a hypercyclic ring.

Jain and Malik also obtained an anologue of Osofsky’s result by considering semiperfect $q$-hypercyclic ring.
Proof. Let $R = e_1R \oplus \ldots \oplus e_nR$, where $e_i$, $1 \leq i \leq n$ are primitive idempotents. We will show that for $i \neq j$, either $e_iR \cong e_jR$, or $\text{Hom}_R(e_iR, e_jR) = 0$. Suppose for some $i \neq j$, $\text{Hom}_R(e_iR, e_jR) \neq 0$. By renumbering, if necessary, we may assume that $i = 1, j = 2$. Let $\alpha : e_1R \to e_2R$ be a non-zero $R$-homomorphism. Then $e_1R/\text{Ker}(\alpha)$ embeds in $e_2R$. Since $e_2R$ is indecomposable, $\text{End}_R(e_1R/\text{Ker}(\alpha)) \cong e_2R$. Hence $B = e_2R \oplus \ldots \oplus e_nR$ contains a copy of $E(e_1R/\text{Ker}(\alpha))$. Now $R/\text{Ker}(\alpha) \cong (e_1R/\text{Ker}(\alpha)) \times e_2R \times \ldots \times e_nR$. Let $A = (e_1R)/\text{Ker}(\alpha)$. Then $B$ is injective and contains a copy of $E(A)$. Hence $\text{Hom}_R(B, E(A))B = E(A)$. Since $R$ is $q$-hypercyclic, for some right ideal $I$, $R/I \cong q.i.h.(R/\text{Ker}(\alpha)) \cong q.i.h.(A \times B) \cong E(A) \times B$. Thus $R/I \cong e_2R \times B$. Then $R/I$ is projective. Hence $R = I \oplus K$ for some right ideal $K$. Then $K \cong R/I \cong e_2R \times e_2R \times \ldots \times e_nR$. Thus $R = I \oplus K \cong I \times e_2R \times e_2R \times \ldots \times e_nR$. Hence by Azumaya Diagram, $e_1R \cong I \times e_2R$. Since $e_1R$ is indecomposable, $I = 0$. Consequently, $R = K$. Then $e_1R \times e_2R \times \ldots \times e_nR \cong e_2R \times e_2R \times \ldots \times e_nR$. Again by Azumaya Diagram, $e_1R \cong e_2R$. Thus for $i \neq j$, either $e_iR \cong e_jR$ or $\text{Hom}_R(e_iR, e_jR) = 0$. Set $[e_kR] = \Sigma e_iR$, $e_kR \cong e_kR$. Renumbering, if necessary, we may write $R = [e_1R] \oplus \ldots \oplus [e_tR]$, $t \leq n$. Then for all $1 \leq k \leq t$, $[e_kR]$ is an ideal. Since for any $k$, $1 \leq k \leq n$, $e_kR$ is indecomposable, $\text{End}_R(e_kR) \cong e_kR\text{End}_k$ is a local ring. Thus $e_kR = \oplus e_kR$ is the $n_k \times n_k$ matrix ring over the local ring $e_kR\text{End}_k$ where $n_k$ is the number of $e_kR$ appearing in $\oplus e_kR$. Since a finite direct sum of $q$-hypercyclic rings is $q$-hypercyclic, the matrix ring is $q$-hypercyclic. \qed

Following the same method as for semiperfect $q$-hypercyclic rings [68], Jain and Saleh obtained the following:

**Lemma 64.** ([Jain and Saleh, [72]]) Let $R$ be a $q$-hypercyclic ring with finite right uniform dimension. Then $R$ is right self-injective.

This lemma is used in the proof of the following theorem.

**Theorem 65.** ([Jain and Saleh, [72]]) Let $R$ be a $q$-hypercyclic ring with Krull dimension. Then $R$ is artinian.

Proof. There exists a prime ideal $P$ of $R$ such that $K(R/P) = K(R)$. Since $R$ is $q$-hypercyclic, $S = R/P$ is a $q$-hypercyclic ring ([68], Lemma 2.6). Therefore, $S$ is right self-injective; that is, $E(S) = S$. Since $S$ is a prime Goldie ring, $Q(S) = E(S) = S$. Thus $S$ is artinian, that is, $K(S) = 0$, which gives $R$ is artinian. \qed

If $R$ is a ring with Krull dimension such that the injective hull of every cyclic $R$-module is finitely generated or the quasi-injective hull of every cyclic $R$-module is finitely generated, it is not known whether $R$ is artinian or not.

**Remark 66.** Let $R$ be a ring with Krull dimension, and let $P$ be a minimal prime ideal of $R$ such that the prime ring $R/P$ has a left classical quotient ring (In particular, if $R$ has also Krull dimension as a left $R$-module). Then, if each cyclic $R$-module has a finitely generated quasi-injective hull, $R$ must be artinian.

We recall that a module $M$ over a ring $R$ is called $\pi$-injective (also called quasi-continuous) if for every pair of $R$-submodules $N_1, N_2$ of $M$ with $N_1 \cap N_2 = 0$ each projection $\pi$: $N_1 \oplus N_2 \to N_i$, $i = 1, 2$, can be lifted to an endomorphism of $M$. 

Goel and Jain [38] considered rings with finite uniform dimension such that any cyclic $R$-module is $\pi$-injective, or more generally has cyclic $\pi$-injective hull. These results generalize the results for semiperfect rings over which each cyclic $R$-module has injective or quasi-injective hull.

Recall that if $K = \text{Hom}_R(E(A), E(A))$, then the $q.inj.\text{hull}(A) = KA$ and the $\pi$-injective hull of $A$, denoted by $\pi(A) = VA$, where $V$ is the subring of $K$ generated by the idempotents of $K$.

**Lemma 67.** (Goel and Jain, [38]). Let $M$ be $\pi$-injective and $E(M) = \oplus A_i$, be a direct sum of submodules. Then $M = \oplus (A_i \cap M)$.

**Lemma 68.** Let $R$ be a ring with finite uniform dimension. If the $\pi$-injective hull of $R$ is cyclic, then $R$ is $\pi$-injective.

*Proof.* See [38]. $\square$

**Remark 69.** A ring homomorphic image of a $\pi$-hypercyclic ring is also a $\pi$-hypercyclic ring.

**Lemma 70.** Let $R$ be a ring with finite uniform dimension. Then $R$ is $\pi$-hypercyclic if and only if $R = e_1 R \oplus \ldots \oplus e_n R$, where $e_i R$ are valuation rings.

*Proof.* See [38]. Only if part: The proof follows from the fact that each $e_i R$ is uniform and for every submodules $A$ and $B$ of $e_i R$, $e_i R/A \cap B$ is also uniform. $\square$

The proof of the following is straightforward.

**Lemma 71.** (a) Let $A$ be essential in an injective module $B$. Then the $\pi$-injective hull of $A \times B$, $\pi(A \times B) = E(A) \times B$.

(b) Let $R$ be an artinian ring with radical $J$ and let $I$ be a right ideal of $R$. If $R/I = \oplus_{i=1}^n N_i$ then the composition length of $R/J \geq n$.

A ring $R$ is called right $\pi$-hypercyclic if each cyclic right $R$-module has cyclic $\pi$-injective hull. Following the techniques given earlier for rings over which each proper cyclic module is continuous it follows that each proper cyclic $R$-module is $\pi$-injective if and only if each proper cyclic $R$-module is continuous where $R$ is a semiperfect nonlocal ring with nil radical.

Jain and Saleh [73] called a ring $\pi$-hypercyclic if each cyclic module has a cyclic $\pi$-injective hull. They studied $\pi$-hypercyclic rings with finite uniform dimension and proved the following.

**Theorem 72.** (Jain and Saleh, [73]). Let $R$ be a $\pi$-hypercyclic indecomposable ring with finite uniform dimension other than 1. Then $R = M_n(A)$, where $A$ is a right valuation ring, $n > 1$ if and only if $R$ is self-injective.

*Proof.* Let $R$ be self-injective. Thus there exist primitive orthogonal idempotents $e_i$, $1 \leq i \leq n$ are primitive idempotents such that $R = e_1 R \oplus \ldots \oplus e_n R$. Let $\alpha : e_1 R \rightarrow e_2 R$ be a non-zero $R$-homomorphism. Then $R/I \cong e_2 R \times e_2 R \times \ldots \times e_n R$ for some right ideal $I$ of $R$. Thus $R/I$ is projective and hence $R = I \oplus K$ where $K \cong e_2 R \times e_2 R \times \ldots \times e_n R$. Since $R$ is self-injective, Azumaya Diagram gives $e_1 R \cong I \times e_2 R$ which forces $I$ to be zero as $e_1 R$ is indecomposable. Thus $e_1 R \cong e_2 R$. Let $[e_1 R] = \Sigma e_i R, e_i R \cong e_k R$. Since $R$ is indecomposable, $R = [e_1 R]$. Therefore, $R$ is a matrix ring over a local ring $A \cong eRe$ where $e = e_1$. It remains to show that $eRe$ is a right valuation ring. We first show that for any right ideal $I \subset eR$,
$eR/I$ is uniform. Then it follows that the submodules of $eR$ are linearly ordered. Let $A$ and $B$ be right ideals of $eRe$. Then $AeR \subset (eR)^2$ and $BeR \subset (eR)^2$. Since the submodules of $eR$ are linearly ordered, $AeR \subset BeR$ or $BeR \subset AeR$ and so $AeRe \subset BeRe$ or $BeRe \subset AeRe$, that is, $A \subset B$ or $B \subset A$. The converse follows from the fact that an $n \times n$ matrix ring $M_n(A)$, $n > 1$ is self $\pi$-injective if and only if it is self-injective. □
9. Cyclic Modules being Direct Sum of Projective, Injective, CS, and Noetherian

The study of rings via decomposition properties of cyclic modules has been considered by many authors. Chatters studied rings whose each cyclic module is a direct sum of a projective module and a noetherian module and proved the following.

**Theorem 73.** (Chatters, [17]). A ring $R$ is right noetherian if and only if every cyclic module is a direct sum of a projective module and a noetherian module.

In [51] Huynh and Dung showed the following.

**Theorem 74.** (Huynh and Dung, [51]). A ring $R$ is right artinian if and only if each cyclic right $R$-module is the direct sum of an injective module and a finitely cogenerated module.

Huynh extended it and proved the following.

**Theorem 75.** (Huynh, [46]). A ring $R$ is hereditarily artinian if and only if each cyclic right $R$-module is the direct sum of an injective module and a finite module.

Huynh, Dung and Smith obtained the following.

**Theorem 76.** (Huynh, Dung and Smith, [52]). The following statements are equivalent for a ring $R$:

(i) Every right ideal is the direct sum of an injective module and a finitely generated semisimple right ideal.

(ii) Every essential right ideal is the direct sum of an injective module and a finitely generated semisimple right ideal.

(iii) $R$ is a direct sum of minimal right ideals and injective right ideals of length 2.

(iv) Every cyclic right $R$-module is the direct sum of an injective module and a semisimple module.

(v) Every right $R$-module is the direct sum of an injective module and a semisimple module.

(vi) $R$ is an artinian serial ring such that $(J(R))^2 = 0$.

(vii) Any of the left-sided analogues of (i)-(v).

**Theorem 77.** (Osofsky and Smith, [105]). A ring $R$ is right noetherian if every cyclic right module is a direct sum of a projective module and an injective module.

As a consequence, it follows that if each cyclic right $R$-module is injective or projective then the ring $R$ is right noetherian.

Goel, Jain and Singh [39] had considered rings whose each cyclic module is either injective or projective. They obtained the following.

**Theorem 78.** (Goel, Jain and Singh, [39]). If each cyclic right $R$-module is injective or projective then $R = A \oplus B$ where $A$ is semisimple artinian and $B$ is a simple right semi-hereditary right Ore-domain whose each proper cyclic module is semisimple.

Smith in his paper [113] independently proved the same result. The structure of rings whose each cyclic module is a direct sum of a projective module and an injective module was completely described by Huynh in [49].
In 1991, Smith asked the question whether a ring is right noetherian if each cyclic module is a direct sum of a projective module and a module that is either injective or noetherian. This question has been recently answered in the affirmative by Huynh and Rizvi [56].

**Theorem 79.** (Huynh and Rizvi, [56]). A ring $R$ is right noetherian if and only if every cyclic module is a direct sum of a projective module and a module $Q$ where $Q$ is either injective or noetherian.

This clearly extends the above two results of Chatters [17] and Osofsky-Smith [105].

The theorem of Huynh and Rizvi also implies that a ring $R$ is right noetherian if and only if every cyclic right $R$-module is either injective or noetherian.
10. Cyclic Modules Embeddable (Essentially) in Free Modules

It is well-known that a ring $R$ is quasi-Frobenius ($QF$ for short) if and only if each right $R$-module embeds in a projective or, equivalently, in a free module. More generally, a ring $R$ is called a right $FGF$ ring if each finitely generated right $R$-module embeds in a free module.

It is known that if $R$ is both left and right $FGF$ then $R$ is, again, a $QF$ ring, but it is an open problem whether a right $FGF$ ring is $QF$. This problem appeared first in Levy’s paper [85] as a question for right Ore rings, and it was later formulated in the present form by Faith. Osofsky proved that a right $PF$ ring is semiperfect and has finite essential right socle [100]. Using this, Bjork [10] proved that a right $FGF$ right self-injective ring must be $QF$. This was also obtained, independently, by Tolskaya (cf. [34]). Menal [88] used a modification of Osofsky’s arguments to prove that if each cyclic right $R$-module embeds in a free module and the injective hull $E(R_R)$ is projective then $R$ is $QF$. Jain and López-Permouth studied rings under a tighter embedding hypothesis, more specifically, rings whose cyclic modules are essentially embeddable in projective modules (direct summands of $R_R$) [63]. Such rings are called right $CEP$ (right $CES$ rings). Examples of right $CEP$ rings include $QF$-rings and right uniserial rings. Indeed a ring $R$ is a $QF$-ring if and only if $R$ is both a right and left $CEP$-ring. Jain and López-Permouth [63] provided the following characterization of $QF$-rings.

**Theorem 80.** (Jain and López-Permouth, [63]). For an arbitrary ring, the following are equivalent:

(a) $R$ is $QF$.
(b) $R$ is $CEP$ and $QF$-3.
(c) Every cyclic $R$-module has a projective injective hull.

For a semiperfect $CEP$ ring, Jain and López-Permouth proved the following.

**Theorem 81.** (Jain and López-Permouth, [63]). A semiperfect ring $R$ is $CEP$ if and only if the following hold:

(a) $R$ is right artinian.
(b) Every indecomposable projective module is uniform, and
(c) Every indecomposable projective module is weakly $R$-injective.

The following example due to Jain and López-Permouth [63] is an example of a local $CEP$ ring which is neither right uniserial nor quasi-Frobenius.

**Example 82.** (Jain and López-Permouth, [63]). Let $S$ be a ring having only three right ideals, namely, $(0)$, $J(S)$ and $S$ and not right self-injective. Let $R = S \times S$ be the trivial extension of $S$ by itself. Then $R$ is a local $CEP$ ring which is neither right uniserial nor quasi-Frobenius.

In [3] Al-Huzali, Jain and López-Permouth asked if each right $CEP$ ring (and hence right $CES$ ring) is semiperfect. This question was answered in the affirmative by Gómez Pardo and Guil Asensio [41]. Gómez Pardo and Guil Asensio first proved the following.

**Theorem 83.** (Gómez Pardo and Guil Asensio, [41]). Let $R$ be a ring and $P_R$ a finitely generated projective module. Suppose that $\Omega(R)$ denotes a set of representatives of the isomorphism classes of simple right $R$-modules and $C(P)$ denotes a set of representatives of the isomorphism classes of simple submodules of $P$. Assume
that $|\Omega(R)| \leq |C(P)|$ and that every cyclic submodule of $E(R_P)$ is essentially embeddable in a projective module. Then $P_R$ cogenerates the simple right $R$-modules and has finite essential socle.

In particular, the above theorem says that if $R$ is a ring such that $E = E(R_R)$ is a cogenerator and every cyclic submodule of $E_R$ is essentially embeddable in a projective module, then $R_R$ has finite essential socle. The proof given by Gómez Pardo and Guil Asensio involves an adaptation of Osofsky’s counting argument.

As a consequence of the above theorem, Gómez Pardo and Guil Asensio obtained the following which answers in the affirmative the question asked by Al-Huzali, Jain and López-Permouth.

**Theorem 84.** (Gómez Pardo and Guil Asensio, [41]). Every right CEP ring is right artinian. In particular, every right CES ring is right artinian.

**Proof.** It follows from the above theorem that $R_R$ has finite essential socle. Since every cyclic right $R$-module embeds in a finitely generated free module, it has also finite essential socle. It is well-known that this implies $R$ is right artinian. □

As a consequence of this result, all results obtained in ([63], [62], [3]) for semiperfect right CEP or CES ring hold for all right CEP and CES ring. In [62] Jain and López-Permouth had shown the structure of semiperfect right CES ring. In view of the result of Gómez Pardo the result of Jain and López-Permouth gives the structure of any right CES ring.

**Theorem 85.** ([62], [41]). Let $R$ be a ring. Then the following conditions are equivalent:

(i) $R$ is right CES.

(ii) $R$ is of one of the following types:

(a) $R$ is (artinian) uniserial as a right $R$-module,

(b) $R$ is an $n \times n$ matrix ring over a right self-injective ring of type (a), or

(c) $R$ is a direct sum of rings of types (a) or (b).
11. Restricted Artinian Rings and Restricted Regular Rings

A ring $R$ is called a right restricted artinian ring if each proper homomorphic image of $R$ is a right artinian ring. Cohen studied commutative restricted artinian rings and proved the following.

**Theorem 86.** (Cohen, [21]). Let $R$ be a commutative ring. Then
(a) $R$ is restricted artinian if and only if $R$ is noetherian and every proper prime ideal of $R$ is maximal.
(b) $R$ is restricted artinian but not artinian if and only if $R$ is a noetherian integral domain not a field, in which every proper prime ideal is maximal.

The following is a simple observation.

**Lemma 87.** Let $S$ be a ring such that $S/M$ is a simple artinian ring for each maximal ideal $M$ of $S$. If the zero ideal of $S$ is a finite product of maximal ideals, then $S$ is right (left) artinian if and only if $S$ is right (left) noetherian.

**Theorem 88.** (Ornstein, [96]). A ring $R$ is a right restricted artinian ring if and only if
(i) $R$ is a right restricted noetherian ring.
(ii) For each prime ideal $P \neq (0)$ in $R$, the ring $R/P$ is a simple artinian ring.

**Proof.** Suppose $R$ is a right restricted artinian ring then clearly $R$ satisfies (i) and (ii). Conversely, let $I$ be a nonzero ideal of $R$ and let $R' = R/I$. If $I$ is not prime then by (i), in $R'$ the zero ideal is a finite product of prime ideals. If $P'$ is any prime ideal in $R'$, its inverse image $P$ is a prime ideal in $R$, and $R'/P' \cong R/P$ is simple artinian by (ii). By above lemma and (i), it follows that $R'$ is right artinian. □

**Theorem 89.** (Ornstein, [96]). Let $R$ be a non-simple prime ring. Then $R$ will be right restricted artinian but not right artinian if and only if $R$ is right restricted noetherian and for each prime ideal $P \neq (0)$, $R/P$ is a simple artinian ring.

**Theorem 90.** (Ornstein, [96]). Let $R$ be a prime ring containing a minimal right ideal. The following are equivalent:
(i) $R$ is right restricted artinian but not right artinian.
(ii) The socle $S$ is an infinite direct sum of minimal right ideals, and $R/S$ is right artinian.

**Proof.** A prime ring $R$ with a minimal right ideal is right primitive. Clearly, its socle $S$ is a simple ring which is contained in every nonzero ideal of $R$, and is therefore a minimal ideal. Assume (i). Then $R/S$ is right artinian. But if $S$ should be a finite direct sum of minimal right ideals, this would force $R$ to be right artinian, and we have (ii). Conversely, the first condition of (ii) implies that $R$ is not artinian, while the second condition implies that each proper homomorphic image is artinian. For let $A$ be an ideal in $R$, $A \neq (0)$. Then $S \subset A$ and $R/A \cong (R/S)/(A/S)$ is a homomorphic image of a right artinian ring. □

Let $R$ be a non-prime right restricted artinian ring. Then there exist a finite number $n > 0$ of prime ideals $P_1, P_2, ..., P_n$ (they are necessarily maximal ideals) of $R$ such that $P_1P_2...P_n = (0)$.

**Theorem 91.** (Ornstein, [96]). A non-prime right restricted artinian ring $R$ has only a finite number of prime ideals, $P_1, P_2, ..., P_n$, $n \geq 1$ which are also the only
maximal ideals of $R$. Their intersection $A = \cap_{i=1}^{n} P_i$ is nilpotent and is precisely the Jacobson radical $J(R)$. It follows that a non-prime right restricted artinian ring has a nilpotent radical.

**Proof.** Let $P \subset R$ be a prime ideal. Since $(0) = P_1 \ldots P_n \subset P$, it follows that $P_i \subset P$ for some $i$, and therefore $P = P_i$ by the maximality of $P_i$. From $A^n = (P_1 \cap \ldots \cap P_n)^n \subset P_1 \ldots P_n = (0)$, we have that $A$ is nilpotent and so $A \subset J(R)$. Since $R/P_i$ is simple artinian, $(R/P_i)J(R) = (0)$, so that $J(R) \subset P_i$ for each $i$ and $J(R) \subset A$, $J(R) = A$. □

As a consequence, we have

**Theorem 92.** (Ornstein, [96]). A non-prime ring with zero Jacobson radical is right restricted artinian if and only if it is right artinian.

A well-known result of Hopkins and Levitzki states that every right artinian ring is right noetherian. This motivated Camillo and Krause to ask the following problem: Is a ring $R$ right noetherian if for any nonzero right ideal $A$ of $R$, $R/A \not\cong R$ is an artinian right $R$-module? We will call a ring $R$ a right Camillo-Krause ring if for each nonzero right ideal $I$ of $R$, $R/I$ is a right artinian module. We start with the following useful observation.

**Theorem 93.** (Ornstein, [96]). If a right Camillo-Krause ring is not right artinian then it must be a right Ore domain.

**Proof.** Let $R$ be a right Camillo-Krause ring that is not right artinian. Let $a \neq 0$ be an element of $R$. We have $R/ann_r(a) \cong aR$. Suppose $ann_r(a) \neq (0)$, then $aR$ is right artinian and therefore by assumption $aR \not\cong R$. Since $aR \neq (0)$, $R/aR$ is also right artinian, and so is $R$, a contradiction. Therefore, $ann_r(a) = (0)$, and hence $R$ is an integral domain. Now suppose that there exist two nonzero proper right ideals $A$ and $B$ such that $A \cap B = (0)$. Then $(A + B)/B \cong A$ is right artinian, and hence $R$ is right artinian, which is a contradiction. Therefore, $R$ is a right Ore domain. □

**Theorem 94.** (Shamsuddin, [109]). Let $R$ be an affine $k$-algebra over an uncountable algebraically closed field. Suppose that $R$ is a right Camillo-Krause domain which is not right primitive. Then $R$ is right noetherian.

**Proof.** Let $I$ be a non-zero prime right ideal of $R$. By a result of Michler [89], to prove that $R$ is right noetherian, it suffices to show that $I$ is finitely generated. Since $R/I$ is right artinian, there exists a right ideal $K$ of $R$ such that $K/I$ is a simple module. Since $R$ is not right primitive, the annihilator $P$ of $K/I$ is non-zero. Now, $KP \subseteq I$. Because $I$ is right prime, we have $P \subseteq I$. Since $R/P$ is right primitive, it is right noetherian. Hence $I/P$ is finitely generated. Since $R/P$ is right primitive, $R/P$ is simple artinian and hence $R/P \cong M_n(D)$ for some division ring $D$. Since $R$ is affine and $k$ is uncountable, $D$ is algebraic over $k$. But as $k$ is algebraically closed, we have $D = k$. Thus, $R/P \cong M_n(k)$. Hence $dim_k(R/P)$ is finite, so it follows from a theorem of Cohn [22] that $P$ is finitely generated as a right ideal. Hence it follows that $I$ is finitely generated and therefore, $R$ must be right noetherian. □

However, the question whether a right Camillo-Krause ring is right noetherian, is still open.
A ring $R$ is said to satisfy the right (left) restricted minimum condition (RMC for short) if for each essential right (left) ideal of $E$ of $R$, $R/E$ is an artinian right (left) $R$-module.

We first prove the following lemma which is quite useful for the study of modules and rings with the right restricted minimum condition.

**Lemma 95.** (Huynh et al., [50]). If $M$ is a module with the restricted minimum condition, then $M/Soc(M)$ has finite Goldie dimension.

**Proof.** Let $A \subseteq B$ be submodules of $M$ such that $A$ is essential in $B$. By Zorn’s Lemma, there is a submodule $L$ of $M$ such that $A \cap L = 0$ and $A \oplus L$ is essential in $M$. By assumption, $M/(A \oplus L)$ is artinian. Since $B/A \cong (B \oplus L)/(A \oplus L)$, $B/A$ is artinian. Now, by Zorn’s Lemma there is a submodule $H$ of $M$ such that $H \cap Soc(M) = 0$ and $H \oplus Soc(M)$ is essential in $M$. Then $M/(H \oplus Soc(M))$ is artinian. Now we claim that $H$ has finite Goldie dimension. Assume to the contrary that $H$ contains an infinite direct sum $X = X_1 \oplus X_2 \oplus \ldots$ of non-zero submodules $X_i$. Since $X_i \cap Soc(M) = 0$, each $X_i$ contains an essential submodule $Y_i$ with $Y_i \neq X_i$. Then $Y = Y_1 \oplus Y_2 \oplus \ldots$ is clearly an essential submodule of $X$. By the above $X/Y$ is artinian. But this is impossible, because $X/Y \cong X_1/Y_1 \oplus X_2/Y_2 \oplus \ldots$ with non-zero $X_i/Y_i$. This shows that $H$ has finite Goldie dimension. From this it follows that $M/Soc(M)$ has finite Goldie dimension. \hfill \Box

**Theorem 96.** (Dung, [31]). If $R$ is a right self-injective ring such that $R$ satisfies the right restricted minimum condition then $R$ is quasi-Frobenius.

**Proof.** Let $R$ be a right self-injective ring. Then $\bar{R} = R/J(R)$ is von Neumann regular right self-injective and satisfies the right restricted minimum condition. Let $\bar{S} = Soc_{\bar{R}}(\bar{R})$. By above lemma, $\bar{R}/\bar{S}$ has finite Goldie dimension. As $\bar{S}_R$ is finitely generated, $R/J(R)$ has finite Goldie dimension. Hence, $R/J(R)$ is semisimple artinian. Let $A$ be any nonzero right ideal of $R$. If $A \cap J(R) = 0$, then $A$ is embedded in $R/J(R)$, so $A_R$ is semisimple. If $A \cap J(R) \neq 0$, consider $x \neq 0 \in A \cap J(R)$. We have $R/ann_r(x) \cong xR$. But $ann_r(x)$ is an essential right ideal of $R$, therefore $xR$ must be right artinian. Thus, in any case $A$ has nonzero socle. Hence $Soc(R_R)$ is essential in $R_R$. Therefore, $R/Soc(R_R)$ is right artinian. This implies $R$ is right artinian and hence $R$ is quasi-Frobenius. \hfill \Box

Later, Huynh [47] proved that if $R$ is a left self-injective ring such that $R$ satisfies the right restricted minimum condition and, in addition, $R$ is semiprime or $J(R)$ is nil then $R$ is quasi-Frobenius.

Chatters studied hereditary noetherian ring $R$ which satisfies the restricted minimum condition and proved the following.

**Theorem 97.** (Chatters, [18]).

(i) Let $R$ be a left Noetherian left hereditary ring, and let $I$ be a finitely generated essential right ideal of $R$. Then $R$ satisfies the descending chain condition for finitely generated right ideals which contain $I$.

(ii) Let $R$ be a hereditary Noetherian ring. Then $R$ satisfies both the right and left restricted minimum condition.

The following example, due to Small [111], shows that a left hereditary left noetherian ring does not necessarily satisfy the left restricted minimum condition.
Example 98. (Small, [111]). Let $R$ be the ring of all matrices of the form \[
\begin{pmatrix}
a & 0 \\
b & c
\end{pmatrix}
\]
where $a$ is an integer and $b$ and $c$ are rationals. This ring $R$ is left hereditary and left noetherian. Let $I$ be the ideal of $R$ consisting of all matrices of the form \[
\begin{pmatrix}
0 & 0 \\
b & c
\end{pmatrix}.
\]
$I$ is an essential left ideal of $R$, but $R/I$ is a ring isomorphic to the ring of integers, and it follows that $R/I$ does not satisfy the left restricted minimum condition.

Recently, Somsup et al. [114] have considered serial rings with restricted minimum condition.

Lemma 99. (Somsup et al., [114]). Let $R$ be a semiprime serial ring with the right restricted minimum condition. Then $R$ is noetherian.

Proof. First note that $R$ has right Krull dimension at most 1. Since $R$ is a semiprime ring with Krull dimension (at most 1), $R$ is a right Goldie ring (see Lemma 6.2.8, [93]). Let $R = e_i R \oplus \ldots \oplus e_n R$, where $\{e_i\}$ is a set of mutually orthogonal primitive idempotents of $R$. Suppose that $\text{Soc}(e_i R) \neq 0$, $i = 1, \ldots, k \leq n)$ and $\text{Soc}(e_j R) = 0$, $j = k + 1, \ldots, n$. Since $R$ has the right restricted minimum condition, we have $e_i R$ is artinian for each $i = 1, \ldots, k$. Set $A = e_1 R \oplus \ldots \oplus e_k R$, $B = e_{k+1} R \oplus \ldots \oplus e_n R$. Then $A$ is clearly a two-sided ideal of $R$ with $A_R$ artinian and $R_R = A \oplus B$. We will show that $B$ is also a two-sided ideal of $R$. It is clear that $BA = 0$. Therefore $(AB)^2 = A(BA)B = 0$. Since $R$ is semiprime, it follows that $AB = 0$. This shows that $B$ is also a left ideal of $R$, i.e. $B$ is a two-sided ideal. Thus $R$ has a ring decomposition $R = A \oplus B$ where $A$ is semiprime artinian and $\text{Soc}(B_B) = 0$. Therefore, splitting off the socle we may assume $\text{Soc}(R) = 0$. The restricted minimum condition implies that $Z(R) = 0$. So $R$ is semihereditary. Let $E_1 \subseteq E_2 \subseteq \ldots \subseteq R$ be an ascending chain of finitely generated essential right ideals. Choose $c$ regular in $E_1$. Then we have a chain of left $R$-modules $Rc^{-1} \supseteq (E_1)^* \supseteq (E_2)^* \supseteq \ldots \supseteq R$ so that $R \supseteq (E_1)^* c \supseteq (E_2)^* c \supseteq \ldots \supseteq Rc$. This chain of left ideals terminates by the restricted minimum condition. Therefore, $E_n^* c = E_{n+1}^* c$. Thus $E_n^* = E_{n+1}^*$ and hence $E_n = E_{n+1}$ by taking stars again and using the fact that the $E_i$'s are projective. Since $R$ is a semiprime right Goldie ring with ACC on finitely generated essential right ideals, $R$ must be right noetherian and hence two-sided noetherian [?].

Theorem 100. (Somsup et al., [114]). Let $R$ be a serial ring which satisfies both the left and right restricted minimum condition. Then $R$ must be a noetherian ring.

Proof. Note that $R$ has a right and left Krull dimension at most 1. Let $N$ be a prime radical of $R$. Then $N$ is nilpotent, by Lenagan [?]. Consider the ring $\bar{R} = R/N$. Then $\bar{R}$ is semiprime and satisfies both the left and right restricted minimum condition. Hence by above result, $\bar{R}$ is noetherian. By a result of Warfield [?], $\bar{R}$ has a ring decomposition $\bar{R} = S \oplus \bar{B}$ where $S$ is an artinian ring, $\bar{B}$ is a direct sum of hereditary prime rings. In particular, $\text{Soc}(\bar{B}_B) = 0$. Assume that $\bar{S} = S/N$, where $S$ is a two-sided ideal of $R$. To complete the proof, it suffices to show that $S$ is artinian and there are idempotents $e$ and $f$ of $R$ such that $S = eR = Rf$. By the above arguments, $\text{Soc}(R/S)_R = 0$. Since $R$ has the right restricted minimum condition, it follows that there is an idempotent $e$ of $R$ such that $S = eR$. Using the same arguments, we can find an idempotent $f$ of $R$ such that $S = Rf$. Thus we have $R = S \oplus R'$, with $N \subset S$. Since $R'$ is a semiprime serial ring with the left
and right restricted minimum condition, \( R' \) is noetherian. For the ring, \( S \), since \( S/N \) is semiprime artinian, it implies that \( N = J(S) \). Thus \( J(S) \) is nilpotent and \( S/J(S) \) is artinian. From this, we conclude that \( S \) is a semiprimary ring. It follows that \( S \) is an artinian ring since \( S \) has left and right restricted minimum condition. Therefore, \( R \) is a noetherian ring.

\( \square \)

Jain and Saroj Jain studied rings whose each proper homomorphic image is a von Neumann regular ring and called such rings restricted regular rings. They proved the following.

**Theorem 101.** (Jain and Saroj Jain, [61]). Let \( R \) be a nonprime right noetherian ring. Then \( R \) is a restricted regular ring if and only if

1. \( R \) is semisimple artinian, or
2. \( R \) has exactly one non-trivial ideal, namely, the Jacobson radical \( J(R) \), and is isomorphic to an \( n \times n \) matrix ring over a local ring, or
3. \( R \) has exactly three non-trivial ideals, namely, \( J(R) \), \( \text{ann}_1(J(R)) \) and \( \text{ann}_r(J(R)) \) and is isomorphic to \( \begin{pmatrix} U & N \\ 0 & V \end{pmatrix} \), where \( U, V \) are simple artinian and \( N \) is an irreducible \( U-V \) bimodule.

**Theorem 102.** (Jain and Saroj Jain, [61]). If \( R \) is a prime right noetherian restricted regular ring then \( R \) is semisimple and each non-trivial ideal is a unique product of maximal ideals.

The next result characterizes right duo restricted regular rings without chain conditions.

**Theorem 103.** (Jain and Saroj Jain, [61]). A right duo ring \( R \) is a restricted regular ring if and only if \( R \) is strongly regular or \( R \) has exactly one proper ideal.

Almost perfect domains, that is, integral domains whose proper homomorphic images are perfect were introduced by Bazzoni and Salce [6] in connection with the study of the existence of strongly flat covers over commutative integral domains [9]. Since a one-dimensional Noetherian domain is an almost perfect domain, it is natural to look for conditions ensuring that an almost perfect domain is Noetherian. Salce raised a question as to when any ring, not necessarily commutative domain, is an almost perfect ring?

In a recent work Abuhlail-Jain-Laradji [1] showed, among other results, that such rings are either perfect or prime and radical of prime local ring is nil.
12. Questions, Exercises and Open Problems

(1) (Koethe, [82]) Describe rings over which each right and left module is a direct sum of cyclic modules.

(2) Is every right PCI-domain also a left PCI-domain?

The only result known in this direction is the one due to Boyle and Goodearl which says that a left and right noetherian domain is a left PCI-domain if and only if it is a right PCI-domain.

(3) Let \( R = K[t, \sigma, \delta] \) be a twisted differential polynomial ring over a division ring \( K \). Suppose \( R \) is a left \( V \)-domain. Then \( R \) is a right PCI-domain if and only if \( \sigma \) is onto. Does there exist an example when \( R \) is a left \( V \)-domain with \( \sigma \) not onto (see Theorem 6.2, [65]).

(4) (Camillo and Krause) Is a right Ore domain necessarily right noetherian if every cyclic right \( D \)-module is projective or artinian? This is equivalent to: Is a ring \( R \) right noetherian if for any nonzero right ideal \( A \) of \( R \), \( R/A \) is an artinian right \( R \)-module?

(5) (Faith, [33]) Characterize a right Ore domain whose every proper cyclic module \( C \) is injective modulo its annihilator ideal.

(6) (Faith, [35]) Is every right CSI ring right noetherian?

Faith has shown that a right CSI ring is right noetherian under some additional conditions [35].

(7) Describe a ring over which each cyclic module is a proper homomorphic image of an injective module.

(8) Is a prime ring whose each cyclic module is quasi-continuous a right nonsingular ring?

(9) Is it true that a cyclic module whose quotients are \( CS \), a finite direct sum of uniform modules? (this is true for projective modules)

(10) (Faith, [34]) Is every right \( FGF \) ring also a \( QF \) ring?

A ring \( R \) is called a right \( FGF \) ring if each finitely generated right \( R \)-module embeds in a free module. It is known that if \( R \) is both left and right \( FGF \) then \( R \) is a \( QF \) ring. Bjork [10] proved that a right \( FGF \) right self-injective ring must be \( QF \). This was also obtained, independently, by Tolskaya (cf. [34]).

(11) (Huynh, [47]) If \( R \) is a left self-injective ring such that \( R/E \) is an artinian right \( R \)-module for any essential right ideal \( E \) of \( R \), then must \( R \) be a quasi-Frobenius ring?

(12) Let \( R \) be a serial ring such that \( R/E \) is an artinian right \( R \)-module for any essential right ideal \( E \) of \( R \). Must \( R \) be a noetherian ring?

(13) Describe non-simple prime right noetherian ring \( R \) such that each proper ring homomorphic image is von Neumann regular.

(14) Is it true that a local hypercyclic ring \( R \) has a nil radical?

(15) Give a characterization of right PCQI-domain. They are known to be right noetherian for simple domains or \( V \)-domains.

(16) (Salce, [108]) Study rings such that each \( R \)-homomorphic image of \( R_R \) is a perfect module.

Commutative rings with this hypothesis has been studied by Salce in [108]. It has been shown by Abuhlail-Jain-Laradji [1] that such rings are...
either prime or perfect. In case it is prime local or non-domain then the radical is nil.

(17) Study rings over which every proper quotient of $M$ is a finitely presented (i) completely $pqc$-module, or (ii) completely pure injective module. See page 21 for the definition of completely $pqc$-module. Gómez Pardo and Guil Asensio showed that if $M$ is a finitely presented completely $pqc$-module, then $M$ has ACC (and DCC) on direct summands and so it is a (finite) direct sum of indecomposable modules (see [42],Theorem 2.9).

(18) (Dinh, Guil Asensio, López-Permouth, [29]) Let $E$ be a finitely generated module such that any pure quotient is pure-injective. Is $E$ a direct sum of indecomposable pure-injective modules?

Dinh, Guil Asensio and López-Permouth [29] showed that if $R$ is a ring of cardinality at most $2^{\aleph_0}$ and $E$ is a countably generated injective right $R$-module such that every quotient of $E$ is injective then $E$ is a direct sum of indecomposable modules.

Notes added later

(1) Askar Tuganbaev has pointed out that the answer to the Question 7 is: Any right self-injective ring. Let $R$ be a right self-injective ring, $M$ be a cyclic right $R$-module, $R_1$ and $R_2$ be two copies of the right $R$-module $R$, $h$ be an epimorphism from $R_1$ onto $M$, $F$ be the direct sum of $R_1$ and $R_2$, and let $f$ be the projection from $F$ onto $R_1$ with kernel $R_2$. The module $F$ is injective, and $h \circ f$ is an epimorphism from $F$ onto $M$ with nonzero kernel. Conversely, if $R$ is a homomorphic image of an injective right $R$-module, then $R$ is a right self-injective ring.

(2) Tuganbaev has pointed out that the answer to the Question 8 is “no”, since there exist uniserial prime rings with zero-divisors (see Theorem 3 in [30]). Such rings cannot be right or left nonsingular. Clearly, every cyclic right module over a right uniserial ring is quasi-continuous.

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