Chapter 4

Integration

"Everybody knows that mathematics is about miracles, only mathematicians have a name for them: theorems."
Roger Howe

4.1 Definition and Basic Properties

At first sight, complex integration is not really anything different from real integration. For a continuous complex-valued function \( \phi : [a, b] \subset \mathbb{R} \to \mathbb{C} \), we define

\[
\int_a^b \phi(t) \, dt = \int_a^b \text{Re} \phi(t) \, dt + i \int_a^b \text{Im} \phi(t) \, dt.
\] (4.1)

For a function which takes complex numbers as arguments, we integrate over a curve \( \gamma \) (instead of a real interval). Suppose this curve is parametrized by \( \gamma(t), \ a \leq t \leq b \). If one meditates about the substitution rule for real integrals, the following definition, which is based on (4.1) should come as no surprise.

**Definition 4.1.** Suppose \( \gamma \) is a smooth curve parametrized by \( \gamma(t), \ a \leq t \leq b \), and \( f \) is a complex function which is continuous on \( \gamma \). Then we define the integral of \( f \) on \( \gamma \) as

\[
\int_\gamma f = \int_\gamma f(z) \, dz = \int_a^b f(\gamma(t)) \gamma'(t) \, dt.
\]

This definition can be naturally extended to piecewise smooth curves, that is, those curves \( \gamma \) whose parametrization \( \gamma(t), \ a \leq t \leq b \), is only piecewise differentiable, say \( \gamma(t) \) is differentiable on the intervals \([a, c_1], [c_1, c_2], \ldots, [c_{n-1}, c_n], [c_n, b]\). In this case we simply define

\[
\int_\gamma f = \int_a^{c_1} f(\gamma(t)) \gamma'(t) \, dt + \int_{c_1}^{c_2} f(\gamma(t)) \gamma'(t) \, dt + \cdots + \int_{c_{n-1}}^{c_n} f(\gamma(t)) \gamma'(t) \, dt.
\]

In what follows, we’ll usually state our results for smooth curves, bearing in mind that practically all can be extended to piecewise smooth curves.

**Example 4.2.** As our first example of the application of this definition we will compute the integral of the function \( f(z) = \overline{z}^2 = (x^2 - y^2) - i(2xy) \) over several curves from the point \( z = 0 \) to the point \( z = 1 + i \).
(a) Let \( \gamma \) be the line segment from \( z = 0 \) to \( z = 1 + i \). A parametrization of this curve is 
\[ \gamma(t) = t + it, \quad 0 \leq t \leq 1. \]
We have \( \gamma'(t) = 1 + i \) and \( f(\gamma(t)) = (t - it)^2 \), and hence
\[
\int_{\gamma} f = \int_{0}^{1} (t - it)^2 \left(1 + i \right) dt = (1 + i) \int_{0}^{1} t^2 - 2it^2 - t^2 dt = -2i(1 + i)/3 = \frac{2}{3}(1 - i).
\]

(b) Let \( \gamma \) be the arc of the parabola \( y = x^2 \) from \( z = 0 \) to \( z = 1 + i \). A parametrization of this curve is \( \gamma(t) = t + it^2 \), \( 0 \leq t \leq 1 \). Now we have \( \gamma'(t) = 1 + 2it \) and
\[
f(\gamma(t)) = \left(t^2 - (t^2)^2\right) - i 2t \cdot t^2 = t^2 - t^4 - 2it^3,
\]
whence
\[
\int_{\gamma} f = \int_{0}^{1} \left(t^2 - t^4 - 2it^3\right) \left(1 + 2it\right) dt = \int_{0}^{1} t^2 + 3t^4 - 2it^5 dt = \frac{1}{3} + \frac{1}{5} - \frac{2i}{6} = \frac{14}{15} - \frac{i}{3}.
\]

(c) Let \( \gamma \) be the union of the two line segments \( \gamma_1 \) from \( z = 0 \) to \( z = 1 \) and \( \gamma_2 \) from \( z = 1 \) to \( z = 1 + i \). Parameterizations are \( \gamma_1(t) = t, \quad 0 \leq t \leq 1 \) and \( \gamma_2(t) = 1 + it, \quad 0 \leq t \leq 1 \). Hence
\[
\int_{\gamma} f = \int_{\gamma_1} f + \int_{\gamma_2} f = \int_{0}^{1} t^2 \cdot 1 dt + \int_{0}^{1} (1 - it)^2 i dt = \frac{1}{3} + i \int_{0}^{1} 1 - 2it - t^2 dt = \frac{1}{3} + i \left(1 - 2i \frac{1}{2} - \frac{1}{3}\right) = \frac{4}{3} + \frac{2}{3}i.
\]

The complex integral has some standard properties, most of which follow from their real siblings in a straightforward way. To state some of its properties, we first define the useful concept of the length of a curve.

**Definition 4.3.** The length of a smooth curve \( \gamma \) is
\[
\text{length}(\gamma) := \int_{a}^{b} |\gamma'(t)| \; dt
\]
for any parametrization \( \gamma(t), \; a \leq t \leq b \), of \( \gamma \).

The definition of length is with respect to any parametrization of \( \gamma \) because, as we will see, the length of a curve is independent of the parametrization. We invite the reader to use some familiar curves to see that this definition gives what one would expect to be the length of a curve.

**Proposition 4.4.** Suppose \( \gamma \) is a smooth curve, \( f \) and \( g \) are complex functions which are continuous on \( \gamma \), and \( c \in \mathbb{C} \).

(a) \( \int_{\gamma} (f + cg) = \int_{\gamma} f + c \int_{\gamma} g \).

(b) If \( \gamma \) is parametrized by \( \gamma(t), \; a \leq t \leq b \), define the curve \(-\gamma\) through \(-\gamma(t) = \gamma(a + b - t), \; a \leq t \leq b\). Then \( \int_{-\gamma} f = -\int_{\gamma} f \).

(c) If \( \gamma_1 \) and \( \gamma_2 \) are curves so that \( \gamma_2 \) starts where \( \gamma_1 \) ends then define the curve \( \gamma_1 \gamma_2 \) by following \( \gamma_1 \) to its end, and then continuing on \( \gamma_2 \) to its end. Then \( \int_{\gamma_1 \gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f \).


Proof.

(a) This follows directly from the definition of the integral and the properties of real integrals.

(b) This follows with an easy real change of variables \( s = a + b - t \):

\[
\int_{-\gamma} f = \int_a^b f(\gamma(a + b - t)) (\gamma(a + b - t))' \, dt = -\int_a^b f(\gamma(a + b - t)) \gamma'(a + b - t) \, dt
\]

\[
= \int_a^b f(\gamma(s)) \gamma'(s) \, ds = -\int_a^b f(\gamma(s)) \gamma'(s) \, ds = -\int_{\gamma} f .
\]

(c) We need a suitable parameterization \( \gamma(t) \) for \( \gamma_1 \gamma_2 \). If \( \gamma_1 \) has domain \([a_1, b_1]\) and \( \gamma_2 \) has domain \([a_2, b_2]\) then we can use

\[
\gamma(t) = \begin{cases} 
\gamma_1(t) & \text{for } a_1 \leq t \leq b_1, \\
\gamma_2(t - b_1 + a_2) & \text{for } b_1 \leq t \leq b_1 + b_2 - a_2.
\end{cases}
\]

The fact that \( \gamma_1(b_1) = \gamma_2(a_2) \) is necessary to make sure that this parameterization is piecewise smooth. Now we break the integral over \( \gamma_1 \gamma_2 \) into two pieces and apply the simple change of variables \( s = t - b_1 + a_2 \):

\[
\int_{\gamma_1 \gamma_2} f = \int_{a_1}^{b_1 + b_2 - a_2} f(\gamma(t)) \gamma'(t) \, dt
\]

\[
= \int_{a_1}^{b_1} f(\gamma(t)) \gamma'(t) \, dt + \int_{a_1}^{b_1 + b_2 - a_2} f(\gamma(t)) \gamma'(t) \, dt
\]

\[
= \int_{a_1}^{b_1} f(\gamma_1(t)) \gamma_1'(t) \, dt + \int_{b_1}^{b_2 + b_2 - a_2} f(\gamma_2(t - b_1 + a_2)) \gamma_2'(t - b_1 + a_2) \, dt
\]

\[
= \int_{a_1}^{b_1} f(\gamma_1(t)) \gamma_1'(t) \, dt + \int_{a_2}^{b_2} f(\gamma_2(s)) \gamma_2'(s) \, ds
\]

\[
= \int_{\gamma_1} f + \int_{\gamma_2} f .
\]

(d) To prove (d), let \( \phi = \text{Arg} \int_{\gamma} f \). Then

\[
\left| \int_{\gamma} f \right| = e^{-i\phi} \left( \int_{\gamma} f \right) = \text{Re} \left( e^{-i\phi} \left( \int_a^b f(\gamma(t)) \gamma'(t) \, dt \right) \right) = \int_a^b \text{Re} \left( f(\gamma(t)) e^{-i\phi} \gamma'(t) \right) \, dt
\]

\[
\leq \int_a^b \left| f(\gamma(t)) e^{-i\phi} \gamma'(t) \right| \, dt = \int_a^b \left| f(\gamma(t)) \right| \left| \gamma'(t) \right| \, dt
\]

\[
\leq \max_{a \leq t \leq b} \left| f(\gamma(t)) \right| \int_a^b \left| \gamma'(t) \right| \, dt = \max_{z \in \gamma} \left| f(z) \right| \cdot \text{length}(\gamma) .
\]
4.2 Cauchy’s Theorem

We now turn to the central theorem of complex analysis. It is based on the following concept.

**Definition 4.5.** A curve $\gamma \subset \mathbb{C}$ is **closed** if its endpoints coincide, i.e. for any parametrization $\gamma(t)$, $a \leq t \leq b$, we have that $\gamma(a) = \gamma(b)$.

Suppose $\gamma_0$ and $\gamma_1$ are closed curves in the open set $G \subseteq \mathbb{C}$, parametrized by $\gamma_0(t)$, $0 \leq t \leq 1$ and $\gamma_1(t)$, $0 \leq t \leq 1$, respectively. Then $\gamma_0$ is $G$-homotopic to $\gamma_1$, in symbols $\gamma_0 \sim_G \gamma_1$, if there is a continuous function $h : [0, 1]^2 \to G$ such that

$$
\begin{align*}
    h(t, 0) &= \gamma_0(t), \\
    h(t, 1) &= \gamma_1(t), \\
    h(0, s) &= h(1, s).
\end{align*}
$$

The function $h(t, s)$ is called a **homotopy** and represents a curve for each fixed $s$, which is continuously transformed from $\gamma_0$ to $\gamma_1$. The last condition simply says that each of the curves $h(t, s)$, $0 \leq t \leq 1$ is closed. An example is depicted in Figure 4.1.

![Figure 4.1: This square and the circle are (\mathbb{C}\setminus\{0\})-homotopic.](image)

Here is the theorem on which most of what will follow is based.

**Theorem 4.6 (Cauchy’s Theorem).** Suppose $G \subseteq \mathbb{C}$ is open, $f$ is holomorphic in $G$, and $\gamma_0 \sim_G \gamma_1$ via a homotopy with continuous second partials. Then

$$
\int_{\gamma_0} f = \int_{\gamma_1} f.
$$

**Remarks.** 1. The condition on the smoothness of the homotopy can be omitted, however, then the proof becomes too advanced for the scope of these notes. In all the examples and exercises that we’ll have to deal with here, the homotopies will be ‘nice enough’ to satisfy the condition of this theorem.
2. It is assumed that Johann Carl Friedrich Gauß (1777–1855)\(^1\) knew a version of this theorem in 1811 but only published it in 1831. Cauchy published his version in 1825, Weierstraß\(^2\) his in 1842. Cauchy’s theorem is often called the Cauchy–Goursat Theorem, since Cauchy assumed that the derivative of \(f\) was continuous, a condition which was first removed by Goursat\(^3\).

An important special case is the one where a curve \(\gamma\) is \(G\)-homotopic to a point, that is, a constant curve (see Figure 4.2 for an example). In this case we simply say \(\gamma\) is \(G\)-contractible, in symbols \(\gamma \sim_G 0\).

![Figure 4.2: This ellipse is \((\mathbb{C} \setminus \mathbb{R})\)-contractible.](image)

The fact that an integral over a point is zero has the following immediate consequence.

**Corollary 4.7.** Suppose \(G \subseteq \mathbb{C}\) is open, \(f\) is holomorphic in \(G\), and \(\gamma \sim_G 0\) via a homotopy with continuous second partials. Then

\[
\int_\gamma f = 0.
\]

The fact that any closed curve is \(\mathbb{C}\)-contractible (Exercise 17a) yields the following special case of the previous special-case corollary.

**Corollary 4.8.** If \(f\) is entire and \(\gamma\) is any smooth closed curve then

\[
\int_\gamma f = 0.
\]

There are many proofs of Cauchy’s Theorem. A particularly nice one follows from the complex Green’s Theorem. We will use the (real) Second Fundamental Theorem of Calculus. We note that with more work, Cauchy’s Theorem can be derived ‘from scratch’, and does not require any other major theorems.

**Proof of Theorem 4.6.** Suppose \(h\) is the given homotopy from \(\gamma_0\) to \(\gamma_1\). For \(0 \leq s \leq 1\), let \(\gamma_s\) be the curve parametrized by \(h(t, s), 0 \leq t \leq 1\). Consider the function

\[
I(s) = \int_{\gamma_s} f
\]

\(^1\)For more information about Gauß, see http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Gauss.html.

\(^2\)For more information about Karl Theodor Wilhelm Weierstraß (1815–1897), see http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Weierstrass.html.

\(^3\)For more information about Edouard Jean-Baptiste Goursat (1858–1936), see http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Goursat.html.
as a function in $s$ (so $I(0) = \int_{\gamma_0} f$ and $I(1) = \int_{\gamma_1} f$). We will show that $I$ is constant with respect to $s$, and hence the statement of the theorem follows with $I(0) = I(1)$. Consider the derivative of $I$. By Leibniz’s Rule,

$$
\frac{d}{ds} I(s) = \frac{d}{ds} \int_0^1 f(h(t, s)) \frac{\partial h}{\partial t} dt = \int_0^1 \frac{\partial}{\partial s} \left( f(h(t, s)) \frac{\partial h}{\partial t} \right) dt.
$$

By the product rule, the chain rule, and equality of mixed partials,

$$
\frac{d}{ds} I(s) = \int_0^1 f'(h(t, s)) \frac{\partial h}{\partial s} \frac{\partial h}{\partial t} + f(h(t, s)) \frac{\partial^2 h}{\partial s \partial t} dt
$$

$$
= \int_0^1 f'(h(t, s)) \frac{\partial h}{\partial t} \frac{\partial h}{\partial s} + f(h(t, s)) \frac{\partial^2 h}{\partial t \partial s} dt
$$

$$
= \int_0^1 \frac{\partial}{\partial t} \left( f(h(t, s)) \frac{\partial h}{\partial s} \right) dt.
$$

Finally, by the Fundamental Theorem of Calculus (applied separately to the real and imaginary parts of the above integral), we have:

$$
\frac{d}{dx} I(s) = f(h(1, s)) \frac{\partial h}{\partial s}(1, s) - f(h(0, s)) \frac{\partial h}{\partial s}(0, s) = 0.
$$

\hfill \Box

### 4.3 Cauchy’s Integral Formula

Cauchy’s Theorem 4.6 yields almost immediately the following helpful result.

**Theorem 4.9** (Cauchy’s Integral Formula for a Circle). Let $C_R$ be the counterclockwise circle with radius $R$ centered at $w$ and suppose $f$ is holomorphic at each point of the closed disk $D$ bounded by $C_R$. Then

$$
f(w) = \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{z-w} \, dz.
$$

**Proof.** All circles $C_r$ with center $w$ and radius $r$ are homotopic in $D \setminus \{w\}$, and the function $f(z)/(z-w)$ is holomorphic in an open set containing $D \setminus \{w\}$. So Cauchy’s Theorem 4.6, gives

$$
\int_{C_R} \frac{f(z)}{z-w} \, dz = \int_{C_r} \frac{f(z)}{z-w} \, dz
$$

Now by Exercise 14,

$$
\int_{C_r} \frac{1}{z-w} \, dz = 2\pi i,
$$

and we obtain with Proposition 4.4(d)

$$
\left| \int_{C_R} \frac{f(z)}{z-w} \, dz - 2\pi i f(w) \right| = \left| \int_{C_r} \frac{f(z)}{z-w} \, dz - f(w) \int_{C_r} \frac{1}{z-w} \, dz \right| = \left| \int_{C_r} \frac{f(z)-f(w)}{z-w} \, dz \right|
$$

$$
\leq \max_{z \in C_r} \left| \frac{f(z)-f(w)}{z-w} \right| \cdot \text{length}(C_r) = \max_{z \in C_r} \frac{|f(z)-f(w)|}{r} \cdot 2\pi r
$$

$$
= 2\pi \max_{z \in C_r} |f(z) - f(w)|.
$$
On the right-hand side, we can now take \( r \) as small as we want, and—because \( f \) is continuous at \( w \)—this means we can make \( |f(z) - f(w)| \) as small as we like. Hence the left-hand side has no choice but to be zero, which is what we claimed.

This is a useful theorem by itself, but it can be made more generally useful. For example, it will be important to have Cauchy’s integral formula when \( w \) is anywhere inside \( C_R \), not just at the center of \( C_R \). In fact, in many cases in which a point \( w \) is inside a simple closed curve \( \gamma \) we can see a homotopy from \( \gamma \) to a small circle around \( w \) so that the homotopy misses \( w \) and remains in the region where \( f \) is holomorphic. In that case the theorem remains true, since, by Cauchy’s theorem, the integral of \( f(z)/(z - w) \) around \( \gamma \) is the same as the integral of \( f(z)/(z - w) \) around a small circle centered at \( w \), and Theorem 4.9 then applies to evaluate the integral. In this discussion we need to be sure that the orientation of the curve \( \gamma \) and the circle match. In general, we say a simple closed curve \( \gamma \) is positively oriented if it is parameterized so that the inside is on the left of \( \gamma \). For a circle this corresponds to a counterclockwise orientation.

Here’s the general form:

**Theorem 4.10** (Cauchy’s Integral Formula). Suppose \( f \) is holomorphic on the region \( G \), \( w \in G \), and \( \gamma \) is a positively oriented, simple, closed, smooth, \( G \)-contractible curve such that \( w \) is inside \( \gamma \). Then

\[
f(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - w} \, dz.
\]

We have already indicated how to prove this, by combining Cauchy’s theorem and the special case, Theorem 4.9. All we need is to find a homotopy in \( G \setminus \{w\} \) between \( \gamma \) and a small circle with center at \( w \). In all practical cases we can see immediately how to construct such a homotopy, but it is not at all clear how to do so in complete generality; in fact, it is not even clear how to make sense of the “inside” of \( \gamma \) in general. The justification for this is one of the first substantial theorems ever proved in topology. We can state it as follows:

**Theorem 4.11** (Jordan Curve Theorem). If \( \gamma \) is a positively oriented, simple, closed curve in \( \mathbb{C} \) then \( \mathbb{C} \setminus \gamma \) consists of two connected open sets, the inside and the outside of \( \gamma \). If a closed disk \( D \) centered at \( w \) lies inside \( \gamma \) then there is a homotopy \( \gamma_s \) from \( \gamma \) to the positively oriented boundary of \( D \), and, for \( 0 < s < 1 \), \( \gamma_s \) is inside \( \gamma \) and outside of \( D \).

**Remarks.** 1. The Jordan Curve Theorem is named after French mathematician Camille Jordan (1838-1922)\(^4\) (the Jordan of Jordan normal form and Jordan matrix, but not Gauss-Jordan elimination). It is so named because Jordan claimed a proof in the late 1800s, although his proof was later seen to be incorrect. It was first correctly proved by Oswald Veblen\(^5\).

This theorem, although “intuitively obvious,” is surprisingly difficult to prove. The usual statement of the Jordan curve theorem does not contain the homotopy information; we have borrowed this from a companion theorem to the Jordan curve theorem which is sometimes called the “annulus theorem.” If you want to explore this kind of mathematics you should take a course in topology.

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A nice special case of Cauchy's formula is obtained when \( \gamma \) is a circle centered at \( w \), parametrized by, say, \( z = w + re^{it}, \ 0 \leq t \leq 2\pi \). Theorem 4.10 gives (if the conditions are met)
\[
f(w) = \frac{1}{2\pi i} \int_{0}^{2\pi} \frac{f(w + re^{it}) - re^{it}}{w + re^{it} - w} \, dt = \frac{1}{2\pi} \int_{0}^{2\pi} f(w + re^{it}) \, dt.
\]

Even better, we automatically get similar formulas for the real and imaginary part of \( f \), simply by taking real and imaginary parts on both sides. These identities have the flavor of mean values. Let’s summarize them in the following statement, which is often called a mean-value theorem.

**Corollary 4.12.** Suppose \( f \) is holomorphic on and inside the circle \( z = w + re^{it}, \ 0 \leq t \leq 2\pi \).
Then
\[
f(w) = \frac{1}{2\pi} \int_{0}^{2\pi} f(w + re^{it}) \, dt.
\]
Furthermore, if \( f = u + iv \),
\[
u(w) = \frac{1}{2\pi} \int_{0}^{2\pi} u(w + re^{it}) \, dt \quad \text{and} \quad v(w) = \frac{1}{2\pi} \int_{0}^{2\pi} v(w + re^{it}) \, dt.
\]

**Exercises**

1. Integrate the function \( f(z) = \overline{z} \) over the three curves given in Example 4.2.
2. Evaluate \( \int_{\gamma} \frac{1}{z} \, dz \) where \( \gamma(t) = \sin t + i \cos t, \ 0 \leq t \leq 2\pi \).
3. Integrate the following functions over the circle \( |z| = 2 \), oriented counterclockwise:
   (a) \( z + \overline{z} \).
   (b) \( z^2 - 2z + 3 \).
   (c) \( 1/z^4 \).
   (d) \( xy \).
4. Evaluate the integrals \( \int_{\gamma} x \, dz, \ \int_{\gamma} y \, dz, \ \int_{\gamma} z \, dz \) and \( \int_{\gamma} \overline{z} \, dz \) along each of the following paths. Note that you can get the second two integrals very easily after you calculate the first two, by writing \( z \) and \( \overline{z} \) as \( x \pm iy \).
   (a) \( \gamma \) is the line segment form 0 to 1 - i.
   (b) \( \gamma \) is the counterclockwise circle \( |z| = 1 \).
   (c) \( \gamma \) is the counterclockwise circle \( |z - a| = r \). Use \( \gamma(t) = a + re^{it} \).
5. Evaluate \( \int_{\gamma} e^{3z} \, dz \) for each of the following paths:
   (a) The straight line segment from 1 to i.
   (b) The circle \( |z| = 3 \).
   (c) The parabola \( y = x^2 \) from \( x = 0 \) to \( x = 1 \).
6. Evaluate \( \int_{\gamma} |z^2| \, dz \) where \( \gamma \) is the parabola with parametric equation \( \gamma(t) = t + it^2, \ 0 \leq t \leq 1 \).
7. Compute $\int_\gamma z$ where $\gamma$ is the semicircle from 1 through $i$ to $-1$.

8. Compute $\int_\gamma e^z$ where $\gamma$ is the line segment from 0 to $z_0$.

9. Find $\int_\gamma |z|^2$ where $\gamma$ is the line segment from 2 to $3+i$.

10. Compute $\int_\gamma z + \frac{1}{z}$ where $\gamma$ is parametrized by $\gamma(t)$, $0 \leq t \leq 1$, and satisfies $\text{Im}\gamma(t) > 0$, $\gamma(0) = -4+i$, and $\gamma(1) = 6+2i$.

11. Find $\int_\gamma |z|$ where $\gamma$ is parametrized by $\gamma(t)$, $0 \leq t \leq 1$, and satisfies $\gamma(0) = i$ and $\gamma(1) = \pi$.

12. Show that $\frac{1}{2\pi} \int_0^{2\pi} e^{ik\theta} d\theta$ is 1 if $k = 0$ and 0 otherwise.

13. Evaluate $\int_\gamma z^{\frac{1}{2}}$ where $\gamma$ is parametrized by $\gamma(t)$, $0 \leq t \leq 1$, and satisfies $\gamma(0) = i$ and $\gamma(1) = \pi$.

14. Let $\gamma$ be the circle with radius $r$ centered at $w$, oriented counterclockwise. You can parameterize this curve as $z(t) = w + re^{it}$ for $0 \leq t \leq 2\pi$. Show that

$\int_{\gamma} \frac{dz}{z - w} = 2\pi i$.

15. Suppose a smooth curve is parametrized by both $\gamma(t)$, $a \leq t \leq b$ and $\sigma(t)$, $c \leq t \leq d$, and let $\tau : [c, d] \to [a, b]$ be the map which “takes $\gamma$ to $\sigma$,” that is, $\sigma = \gamma \circ \tau$. Show that

$\int_c^d f(\sigma(t))\sigma'(t) \, dt = \int_a^b f(\gamma(t))\gamma'(t) \, dt$.

(In other words, our definition of the integral $\int_\gamma f$ is independent of the parametrization of $\gamma$.)

16. Prove that $\sim_G$ is an equivalence relation.

17. (a) Prove that any closed curve is $C$-contractible.

(b) Prove that any two closed curves are $C$-homotopic.

18. Show that $\int_\gamma z^n \, dz = 0$ for any closed smooth $\gamma$ and any integer $n \neq -1$. [If $n$ is negative, assume that $\gamma$ does not pass through the origin, since otherwise the integral is not defined.]

19. Exercise 18 excluded $n = -1$ for a very good reason: Exercises 2 and 14 (with $w = 0$) give counterexamples. Generalizing these, if $m$ is any integer then find a closed curve $\gamma$ so that $\int_\gamma z^{-1} \, dz = 2m\pi i$. (Hint: Follow the counterclockwise unit circle through $m$ complete cycles (for $m > 0$). What should you do if $m < 0$? What if $m = 0$?)

20. Let $\gamma_r$ be the circle centered at $2i$ with radius $r$, oriented counterclockwise. Compute

$\int_{\gamma_r} \frac{dz}{z^2 + 1}$.

(This integral depends on $r$.)
21. Suppose $p$ is a polynomial and $\gamma$ is a closed smooth path in $\mathbb{C}$. Show that
\[ \int_{\gamma} p = 0. \]

22. Compute the real integral
\[ \int_{0}^{2\pi} \frac{d\theta}{2 + \sin \theta} \]
by writing the sine function in terms of the exponential function and making the substitution $z = e^{i\theta}$ to turn the real into a complex integral.

23. Show that $F(z) = \frac{i}{2} \log(z + i) - \frac{i}{2} \log(z - i)$ is a primitive of $\frac{1}{1 + z^2}$ for $\text{Re}(z) > 0$. Is $F(z) = \arctan z$?

24. Prove the following integration by parts statement. Let $f$ and $g$ be holomorphic in $G$, and suppose $\gamma \subset G$ is a smooth curve from $a$ to $b$. Then
\[ \int_{\gamma} fg' = f(\gamma(b))g(\gamma(b)) - f(\gamma(a))g(\gamma(a)) - \int_{\gamma} f'g. \]

25. Suppose $f$ and $g$ are holomorphic on the region $G$, $w \in G$, and $f(z) = g(z)$ for all $z \in \gamma$. Prove that $f(z) = g(z)$ for all $z$ inside $\gamma$.

26. This exercise gives an alternative proof of Cauchy’s integral formula (Theorem 4.10), which does not depend on Cauchy’s Theorem 4.6. Suppose $f$ is holomorphic on the region $G$, $w \in G$, and $\gamma$ is a positively oriented, simple, closed, smooth, $G$-contractible curve such that $w$ is inside $\gamma$.

   (a) Consider the function $g : [0, 1] \rightarrow \mathbb{C}$, $g(t) = \int_{\gamma} \frac{f(w + t(z - w))}{z - w} dz$. Show that $g' = 0$. (Hint: Use Theorem 1.20 (Leibniz’s rule) and then find a primitive for $\frac{\partial f}{\partial t}(z + t(w - z))$.)

   (b) Prove Theorem 4.10 by evaluating $g(0)$ and $g(1)$.

27. Prove Corollary 4.7 using Theorem 4.10.

28. Suppose $a$ is a complex number and $\gamma_0$ and $\gamma_1$ are two counterclockwise circles (traversed just once) so that $a$ is inside both of them. Explain geometrically why $\gamma_0$ and $\gamma_1$ are homotopic in $\mathbb{C} \setminus \{a\}$.

29. Let $\gamma_r$ be the counterclockwise circle with center at 0 and radius $r$. Find $\int_{\gamma_r} \frac{dz}{z - a}$. You should get different answers for $r < |a|$ and $r > |a|$. (Hint: In one case $\gamma_r$ is contractible in $\mathbb{C} \setminus \{a\}$.

In the other you can combine Exercises 14 and 28.)

30. Let $\gamma_r$ be the counterclockwise circle with center at 0 and radius $r$. Find $\int_{\gamma_r} \frac{dz}{z^2 - 2z - 8}$ for $r = 1$, $r = 3$ and $r = 5$. (Hint: Since $z^2 - 2z - 8 = (z - 4)(z + 2)$ you can find a partial fraction decomposition of the form $\frac{1}{z^2 - 2z - 8} = \frac{A}{z - 4} + \frac{B}{z + 2}$. Now use Exercise 29.)
31. Use the Cauchy integral formula to evaluate the integral in Exercise 30 when \( r = 3 \). (Hint: The integrand can be written in each of following ways:

\[
\frac{1}{z^2 - 2z - 8} = \frac{1}{(z - 4)(z + 2)} = \frac{1}{z + 2} = \frac{1}{z - 4}.
\]

Which of these forms corresponds to the Cauchy integral formula for the curve \( \gamma_3 \)?

32. Evaluate \( \int_{|z|=2} \frac{e^z}{z(z-3)} \) and \( \int_{|z|=4} \frac{e^z}{z(z-3)} \).

33. Find \( \int_{|z+1|=2} \frac{z^2}{4-z^2} \).

34. What is \( \int_{|z|=1} \frac{\sin z}{z} \)?