

On restricting subsets of bases in relatively free groups

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Notation

$F = F(A)$ – free group of rank $n \geq 2$ with basis $A = \{a_1, \dots, a_n\}$

V – a *fully invariant* subgroup of F
($\forall f : F \rightarrow F, f(V) \subseteq V$)

Cases of interest:

- $V = 1$ is the trivial subgroup,
and $F/V = F$ is a **free group**
- $V = [F, F]$ is the commutator subgroup,
and $F/V = F$ is a **free abelian group**
- $V = F^{(k)}$ is a term of the derived series of F ,
and F/V is a **free solvable group** of derived length k
- $V = \gamma_c(F)$ is a term of the lower central series of F ,
and F/V is a **free nilpotent group** of class c

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Primitivity

Definition

A subset $S \subset F$ is *primitive in $F \bmod V$* if the set of cosets SV of V can be extended to a basis of F/V .

Example

Let $V = [F, F]$. Then $F/V \cong \mathbb{Z}^n$.

Consider $n = 3$, and let $F = \langle a, b, c \rangle$.

$S = \{bacaab, abcabca\}$ is primitive mod V :

$$SV = \{bacaabV, abcabcaV\} = \{(a^3b^2c)V, (a^3b^2c^2)V\}$$

a^3b^2c , $a^3b^2c^2$, and ab generate \mathbb{Z}^3 .

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The Main Problem

Let $S \subset F$ be primitive mod V .

Assume a_l, \dots, a_n is not used to express any $s \in S$:

$$S \subset \hat{F} := F(\{a_1, \dots, a_{l-1}\})$$

Consider $\hat{V} = V \cap \hat{F}$, a fully invariant subgroup of \hat{F} .

Question

When is S also primitive in \hat{F} mod \hat{V} ?

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Motivation

Arose from our work on $Out(F_n)$:

Question

Is every primitive subset of F_n that does not involve a_n primitive in F_{n-1} ?

- What about relatively free groups?
- What about more missing generators?

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Main Theorem

Theorem (SS)

Let $S \subset \hat{F}$ be primitive in $F \bmod V$. Then S is also primitive in $\hat{F} \bmod \hat{V}$ if:

- 1 $F/V = F$ is free
- 2 F/V is free abelian
- 3 F/V is free nilpotent
- 4 F/V is free solvable

Free Case

For each $j = 1, 2, \dots, n$ define the *free Fox derivative* $D_j: \mathbb{Z}F_n \rightarrow \mathbb{Z}F_n$ recursively by

$$D_j(a_j) = 1, \quad D_j(a_i) = 0, i \neq j$$

and

$$D_j(uv) = D_j(u) + uD_j(v) \text{ for all } u, v \in F_n.$$

Example

For $a^2ba \in F(a, b)$ we have

$$\begin{aligned} D_a(a^2b \cdot a) &= D_a(a^2 \cdot b) + a^2b \cdot D_a(a) \\ &= D_a(a \cdot a) + a^2 \cdot D_a(b) + a^2b \\ &= D_a(a) + a \cdot D_a(a) + a^2b \\ &= 1 + a + a^2b \end{aligned}$$

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Primitivity

Theorem (Birman (73) \Rightarrow , Umirbaev (94) \Leftarrow)

A subset $\{x_1, \dots, x_k\}$ is primitive in F_n **if and only if** the $k \times n$ Jacobian matrix

$$J = \begin{pmatrix} D_1(x_1) & \cdots & D_n(x_1) \\ \vdots & \ddots & \vdots \\ D_1(x_k) & \cdots & D_n(x_k) \end{pmatrix}$$

is right invertible in $\mathbb{Z}F_n$.

Restricting Subsets of Bases

Theorem

Let $S = \{x_1, x_2, \dots, x_k\} \subset \hat{F}$ be primitive in F . Then S is primitive in \hat{F} .

Proof:

S is primitive in $F \Rightarrow$ Jacobian J of S is right invertible.

So $\exists P = (p_{ij})$ with $p_{ij} \in \mathbb{Z}F_n$ satisfying

$$JP = I_k$$

No a_1, \dots, a_n in $S \Rightarrow$

- no a_1, \dots, a_n in J
- columns 1 to n of J consist of zeros.

Write each entry of P as

$$p_{ij} = q_{ij} + r_{ij}$$

where

- each term in q_{ij} involves at least one of a_1, \dots, a_n
- each term in r_{ij} does not involve a_1, \dots, a_n

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Then for matrices $Q = (q_{jl})$ and $R = (r_{jl})$

$$P = Q + R$$
$$JQ + JR = JP = I_k$$

But each entry of JQ is either 0 or involves a_1, \dots, a_n , so $JQ = 0$, and

$$JR = I_k$$

Let \hat{J} and \hat{R} be the matrices obtained from J and R by deleting the last $n - l + 1$ columns and $n - l + 1$ rows, respectively. Then

- \hat{J} is the Jacobian matrix of the set S seen as a subset of \hat{F}
- \hat{R} is a matrix over $\mathbb{Z}\hat{F}$.

Also

$$\hat{J}\hat{R} = I_k.$$

By Umirbaev's criterion the set S is a subset of the basis of \hat{F} . □

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The Metabelian Case

The Same

$M_n = F/F''$ – free metabelian group

$A_n = F/F'$ – free abelian group

Free Fox derivative D_j induces

$$d_j: \mathbb{Z}M_n \rightarrow \mathbb{Z}A_n$$

We have

$$d_j(a_j) = 1, \quad d_j(a_i) = 0, i \neq j$$

and

$$d_j(uv) = d_j(u) + \pi(u)d_j(v) \text{ for all } u, v \in M_n,$$

where $\pi: M_n \rightarrow A_n$ is the canonical abelianization epimorphism.

Theorem (Timoshenko (89,92), Roman'kov (91))

A subset $\{x_1, \dots, x_k\}$ is primitive in M_n if and only if

$J := (d_j(x_l)), 1 \leq l \leq k, 1 \leq j \leq n$ is right invertible in the integral ring $\mathbb{Z}A_n$.

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Nilpotent and Solvable Cases

Proposition

For F/V either free nilpotent or free solvable, a subset $S \subset \hat{F}$ is primitive in $F \bmod V$ if and only if S is primitive in $F \bmod F'$.

(Solvable case due to Baumslag)

Reduction of Main Theorem to Free Abelian Case:

Assume S is primitive in $F \bmod V$ and S avoids a_1, \dots, a_n .

By Proposition, S is primitive in $F \bmod F'$.

By abelian case, S is primitive in $\hat{F} \bmod \hat{F}'$.

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Abelian Case

Extend S to a basis, express as a matrix M .

$$M = \begin{bmatrix} \hat{M} & P \\ 0 & Q \end{bmatrix},$$

Then

$$\det \hat{M} \cdot \det Q = \det M = \pm 1$$

so \hat{M} is invertible.

Question

Does there exist a variety of groups for which the Main Question has a negative answer?

(Free Burnside groups?)

Claim

For any relatively free group of rank n , if S is a subset of a basis which can be extended to a basis \tilde{S} in which m of the generators in $\tilde{S} - S$ are conjugates of the standard basis elements, then S is primitive in rank $n - m$.

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Question

Does there exist a variety of groups for which the Main Question has a negative answer?

(Free Burnside groups?)

Claim

For any relatively free group of rank n , if S is a subset of a basis which can be extended to a basis \tilde{S} in which m of the generators in $\tilde{S} - S$ are conjugates of the standard basis elements, then S is primitive in rank $n - m$.

Thank You