On restricting subsets of bases in relatively free groups

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Notation

$F = F(A)$ – free group of rank $n \geq 2$ with basis $A = \{a_1, \ldots, a_n\}$

V – a *fully invariant* subgroup of F
$(\forall f : F \to F, f(V) \subseteq V)$

Cases of interest:

- $V = 1$ is the trivial subgroup, and $F/V = F$ is a free group
- $V = [F, F]$ is the commutator subgroup, and $F/V = F$ is a free abelian group
- $V = F^{(k)}$ is a term of the derived series of F, and F/V is a free solvable group of derived length k
- $V = \gamma_c(F)$ is a term of the lower central series of F, and F/V is a free nilpotent group of class c

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Primitivity

Definition

A subset $S \subset F$ is *primitive in F mod V* if the set of cosets SV of V can be extended to a basis of F/V.

Example

Let $V = [F, F]$. Then $F/K = \mathbb{Z}^n$.
Consider $n = 3$, and let $F = \langle a, b, c \rangle$.
$S = \{bacaab, abcabca\}$ is primitive mod V:

$$SV = \{bacaabV, abcabcaV\} = \{(a^3b^2c)V, (a^3b^2c^2)V\}$$

a^3b^2c, $a^3b^2c^2$, and ab generate $\mathbb{Z}^3$.

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The Main Problem

Let $S \subset F$ be primitive mod V.
Assume $a_l, \ldots, a_n$ is not used to express any $s \in S$:

$$S \subset \hat{F} := F(\{a_1, \ldots, a_{l-1}\})$$

Consider $\hat{V} = V \cap \hat{F}$, a fully invariant subgroup of $\hat{F}$.

Question

When is S also primitive in $\hat{F}$ mod $\hat{V}$?

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Motivation

Arose from our work on $Out(F_n)$:

Question

Is every primitive subset of F_n that does not involve a_n primitive in F_{n-1}?

- What about relatively free groups?
- What about more missing generators?

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Main Theorem

Theorem (SS)

Let $S \subset \hat{F}$ be primitive in F mod V. Then S is also primitive in $\hat{F}$ mod $\hat{V}$ if:

1. $F/V = F$ *is free*
2. F/V *is free abelian*
3. F/V *is free nilpotent*
4. F/V *is free solvable*

Free Case

For each $j = 1, 2, \ldots, n$ define the *free Fox derivative* $D_j \colon \mathbb{Z}F_n \to \mathbb{Z}F_n$ recursively by

$$D_j(a_j) = 1, \qquad D_j(a_i) = 0, i \neq j$$

and

$$D_j(uv) = D_j(u) + uD_j(v) \text{ for all } u, v \in F_n.$$

Example

For $a^2ba \in F(a, b)$ we have

$$\begin{aligned} D_a(a^2b \cdot a) &= D_a(a^2 \cdot b) + a^2b \cdot D_a(a) \\ &= D_a(a \cdot a) + a^2 \cdot D_a(b) + a^2b \\ &= D_a(a) + a \cdot D_a(a) + a^2b \\ &= 1 + a + a^2b \end{aligned}$$

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Primitivity

Theorem (Birman (73) $\Rightarrow$, Umirbaev (94) $\Leftarrow$)

A subset $\{x_1, \ldots, x_k\}$ is primitive in F_n if and only if the $k \times n$ Jacobian matrix

$$J = \begin{pmatrix} D_1(x_1) & \cdots & D_n(x_1) \\ \vdots & \ddots & \vdots \\ D_1(x_k) & \cdots & D_n(x_k) \end{pmatrix}$$

is right invertible in $\mathbb{Z}F_n$.

Restricting Subsets of Bases

Theorem

Let $S = \{x_1, x_2, \ldots, x_k\} \subset \hat{F}$ be primitive in F. Then S is primitive in $\hat{F}$.

Proof:

S is primitive in $F \Rightarrow$ Jacobian J of S is right invertible.

So $\exists\ P = (p_{jl})$ with $p_{jl} \in \mathbb{Z}F_n$ satisfying

$$JP = I_k$$

No $a_l, \ldots, a_n$ in $S \Rightarrow$

- no $a_l, \ldots, a_n$ in J
- columns l to n of J consist of zeros.

Write each entry of P as

$$p_{ij} = q_{ij} + r_{ij}$$

where

- each term in q_{ij} involves at least one of $a_l, \ldots, a_n$
- each term in r_{ij} does not involves $a_l, \ldots, a_n$

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Then for matrices $Q = (q_{jl})$ and $R = (r_{jl})$

$$P = Q + R$$
$$JQ + JR = JP = I_k$$

But each entry of JQ is either 0 or involves $a_l, \dots, a_n$, so $JQ = 0$, and

$$JR = I_k$$

Let $\hat{J}$ and $\hat{R}$ be the matrices obtained from J and R by deleting the last $n - l + 1$ columns and $n - l + 1$ rows, respectively. Then

- $\hat{J}$ is the Jacobian matrix of the set S seen as a subset of $\hat{F}$
- $\hat{R}$ is a matrix over $\mathbb{Z}\hat{F}$.

Also

$$\hat{J}\hat{R} = I_k.$$

By Umirbaev's criterion the set S is a subset of the basis of $\hat{F}$. □

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The Metabelian Case

The Same

$M_n = F/F''$ – free metabelian group
$A_n = F/F'$ – free abelian group
Free Fox derivative D_j induces

$$d_j \colon \mathbb{Z}M_n \to \mathbb{Z}A_n$$

We have

$$d_j(a_j) = 1, \qquad d_j(a_i) = 0, i \neq j$$

and

$$d_j(uv) = d_j(u) + \pi(u)d_j(v) \text{ for all } u, v \in M_n,$$

where $\pi \colon M_n \to A_n$ is the canonical abelianization epimorphism.

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Nilpotent and Solvable Cases

Proposition

For F/V either free nilpotent or free solvable, a subset $S \subset \hat{F}$ is primitive in F mod V if and only if S is primitive in F mod F'.

(Solvable case due to Baumslag)

Reduction of Main Theorem to Free Abelian Case:
Assume S is primitive in F mod V and S avoids $a_l, \ldots, a_n$.
By Proposition, S is primitive in F mod F'.
By abelian case, S is primitive in $\hat{F}$ mod $\hat{F}'$.
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Abelian Case

Extend S to a basis, express as a matrix M.

$$M = \begin{bmatrix} \hat{M} & P \\ 0 & Q \end{bmatrix},$$

Then

$$\det \hat{M} \cdot \det Q = \det M = \pm 1$$

so $\hat{M}$ is invertible.

Question

Does there exist a variety of groups for which the Main Question has a negative answer?

Claim

For any relatively free group of rank n, if S is a subset of a basis which can be extended to a basis $\tilde{S}$ in which m of the generators in $\tilde{S} - S$ are conjugates of the standard basis elements, then S is primitive in rank $n - m$.

Thank You

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