On restricting subsets of bases in relatively free groups

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Motivation

$F_n = \langle a_1, a_2, \ldots, a_n \rangle$ – free group of rank n

One potential analogue of Tits buildings/curve complex: *splitting complex*

To understand the geometry of the *edge splitting graph*, we introduced i-length, a measure of complexity of words in F_n.

Theorem (Whitehead)

If x is a power of a primitive element in F_n, then there is a cut vertex in $\Gamma(\{x\})$.

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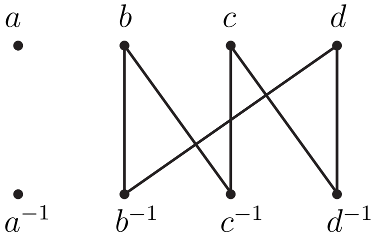

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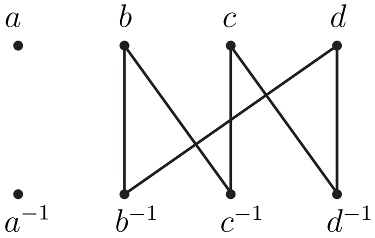

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No cut vertex, not primitive in $F(b, c, d)$

Simple i-Length

Definition (simple i-length of w)

For w not involving a_i, $|w|_i^{simple} :=$ *max* t such that $w = w_1 \dots w_t$ where the Whitehead graph $\Gamma_{A-\{a_i\}}(w_i)$ has no cut vertex.

- *conjugate-reduced* i-length is similar, but takes conjugation into account

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One of our key Lemmas shows:

Lemma

For any basis $\mathbf{x}$ of F_n, any $x \in \mathbf{x}$, and any subword w of $\alpha_{\mathbf{x}} x$ not involving a_i, $|w|_i^{cr} = 0$.

Corollary

If x is primitive in F_n and does not involve a_n, then its Whitehead graph with respect to $\{a_1, \ldots, a_{n-1}\}$ has a cut vertex.

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Is every such x primitive in F_{n-1}?

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What about relatively free groups?

More General Notation

$F = F(A)$ – free group of rank $n \geq 2$ with basis $A = \{a_1, \ldots, a_n\}$
V – a *fully invariant* subgroup of F

Cases of interest:

- $V = 1$ is the trivial subgroup, and $F/V = F$ is a free group
- $V = F^{(k)}$ is a term of the derived series of F, and F/V is a free solvable group of derived length k
- $V = \gamma_c(F)$ is a term of the lower central series of F, and F/V is a free nilpotent group of class c

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Definition

A subset $S \subset F$ is *primitive in F mod V* if the corresponding set of cosets SV of V can be extended to a basis of F/V.

Let $S \subset F$ be primitive mod V.
Assume a_n is not used to express any $s \in S$:

$$S \subset \hat{F} := F(A - \{a_n\})$$

Consider $\hat{V} = V \cap \hat{F}$, a fully invariant subgroup of $\hat{F}$.

Question

When is S also primitive in $\hat{F}$ mod $\hat{V}$?

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Our Results

Theorem (SS)

Let $S \subset \hat{F}$ be primitive in F mod V. Then S is also primitive in $\hat{F}$ mod $\hat{V}$ if:

1. $F/V = F$ *is free*
2. F/V *is free abelian*
3. F/V *is free metabelian*
4. F/V *is free nilpotent of class c and* $|S| \leq n - c$
5. F/V *is free nilpotent of class 2*

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Claim

1. *Result still holds if $\hat{F}$ misses more than 1 element.*
2. *Result holds when F/V is free solvable and $|S| = n - 1$.*

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Proof for free groups: Fox Calculus

For each $j = 1, 2, \dots, n$ define the *free Fox derivative* $D_j : \mathbb{Z}F_n \to \mathbb{Z}F_n$ recursively by

$$D_j(a_j) = 1, \qquad D_j(a_i) = 0, i \neq j$$

and

$$D_j(uv) = D_j(u) + uD_j(v) \text{ for all } u, v \in F_n.$$

Example

For $a^2ba \in F(a,b)$ we have

$$\begin{aligned} D_a(a^2b \cdot a) &= D_a(a^2 \cdot b) + a^2b \cdot D_a(a) \\ &= D_a(a \cdot a) + a^2 \cdot D_a(b) + a^2b \\ &= D_a(a) + a \cdot D_a(a) + a^2b \\ &= 1 + a + a^2b \end{aligned}$$

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Theorem (Birman (73) $\Rightarrow$, Umirbaev (94) $\Leftarrow$)

A subset $\{x_1, \ldots, x_k\}$ is primitive in F_n if and only if the $k \times n$ Jacobian matrix

$$J = \begin{pmatrix} D_1(x_1) & \cdots & D_n(x_1) \\ \vdots & \ddots & \vdots \\ D_1(x_k) & \cdots & D_n(x_k) \end{pmatrix}$$

is right invertible in $\mathbb{Z}F_n$.

Theorem

Let $S = \{x_1, x_2, \ldots, x_k\} \subset \hat{F}$ *be primitive in* F*. Then* S *is primitive in* $\hat{F}$*.*

Proof:

S is primitive in $F \Rightarrow$ Jacobian J of S is right invertible.

So $\exists\ P = (p_{jl})$ with $p_{jl} \in \mathbb{Z}F_n$ satisfying

$$JP = I_k$$

No a_n in $S \Rightarrow$

- no a_n in J
- n-th column of J consists of zeros.

Write each entry of P as

$$p_{jl} = q_{jl} + r_{jl}$$

where

- each term in q_{jl} involves a_n
- each term in r_{jl} does not involves a_n

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Then for matrices $Q = (q_{jl})$ and $R = (r_{jl})$

$$P = Q + R$$
$$JQ + JR = JP = I_k$$

But each entry of JQ is either 0 or involves a_n, so $JQ = 0$, and

$$JR = I_k$$

Let $\tilde{J}$ and $\tilde{R}$ be the matrices obtained from J and R by deleting n-th row and n-th column correspondingly. Then

- $\tilde{J}$ is the Jacobian matrix of the set S seen as a subset of F_{n-1}
- $\tilde{R}$ is a matrix over $\mathbb{Z}F_{n-1}$.

Also

$$\tilde{J}\tilde{R} = I_k.$$

By Umirbaev's criterion the set S is a subset of the basis of $\hat{F}$. □

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The Metabelian Case

The Same

$M_n = F/F''$ – free metabelian group
$A_n = F/F'$ – free abelian group
Free Fox derivative D_j induces

$$d_j : \mathbb{Z}M_n \to \mathbb{Z}A_n$$

We have

$$d_j(a_j) = 1, \qquad d_j(a_i) = 0, i \neq j$$

and

$$d_j(uv) = d_j(u) + \pi(u)d_j(v) \text{ for all } u, v \in M_n,$$

where $\pi : M_n \to A_n$ is the canonical abelianization epimorphism.

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Theorem (Timoshenko (89,92), Roman'kov (91))

A subset $\{x_1, \ldots, x_k\}$ is primitive in M_n if and only if $J := \big(d_j(x_l)\big), 1 \leq l \leq k, 1 \leq j \leq n$ is right invertible in the integral ring $\mathbb{Z}A_n$.

Corollary

Let $S \subset \hat{F}$ be primitive in F mod F''. Then S is primitive in $\hat{F}$ mod $\hat{F}''$.

Question

Can we extend this technique to free solvable groups? (must deal with wild automorphisms)

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The Nilpotent Case

$\gamma_c(F)$ – c-th term in the lower central series of F.

- $\gamma_1(F) = F$
- $\gamma_{n+1}(F) = [\gamma_n(F), F]$

$F/\gamma_{c+1}(F)$ – free nilpotent group of class c.

Theorem (Gupta-Gupta, 92)

Let $F/\gamma_{c+1}(F)$ be the free nilpotent group of class $c \geq 2$. If S is primitive in F mod $\gamma_{c+1}(F)$ and $k \leq n - c + 1$, then S lifts to a set $\tilde{S}$ which is primitive in F.

Theorem (SS)

Moreover, if $k \leq n - c$, and $S \subset \hat{F}$, then $\tilde{S}$ can be chosen to be inside $\hat{F}$.

Corollary

Let $S \subset \hat{F}$ be primitive in F mod $\gamma_{c+1}(F)$. If $n > 2$ and $k \leq n - c$ then S is primitive in $\hat{F}$ mod $\gamma_{c+1}(\hat{F})$.

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Theorem (Gupta-Gupta, 92)

Let $F/\gamma_{c+1}(F)$ be the free nilpotent group of class $c \geq 2$. If S is primitive in F mod $\gamma_{c+1}(F)$ and $k \leq n - c + 1$, then S lifts to a set $\tilde{S}$ which is primitive in F.

Theorem (SS)

Moreover, if $k \leq n - c$, and $S \subset \hat{F}$, then $\tilde{S}$ can be chosen to be inside $\hat{F}$.

Corollary

Let $S \subset \hat{F}$ be primitive in F mod $\gamma_{c+1}(F)$. If $n > 2$ and $k \leq n - c$ then S is primitive in $\hat{F}$ mod $\gamma_{c+1}(\hat{F})$.

The Nilpotent Case

$\gamma_c(F)$ – c-th term in the lower central series of F.

- $\gamma_1(F) = F$
- $\gamma_{n+1}(F) = [\gamma_n(F), F]$

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