

Geometry of curve complex analogues for $Out(F_n)$

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1 May, 2011

Notation

- F_n – free group of rank n
- $Aut(F_n)$ – automorphism group of F_n
- $Inn(F_n)$ – group of inner automorphisms of F_n
- $Out(F_n) = Aut(F_n)/Inn(F_n)$

- $Out(F_n)$ is related to:

$$Out(F_n) \twoheadrightarrow Out(\mathbb{Z}^n) = GL_n(\mathbb{Z})$$

$$MCG(S) \hookrightarrow Out(F_n),$$

where $\pi_1(S) \cong F_n$.

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Vogtmann's Leitmotiv and Bestvina's table

$Out(F_n)$ satisfies a mix of properties, some inherited from the mapping class group, and others from arithmetic groups.

MCGs	$Out(F_n)$	$GL_n(\mathbb{Z})$	Algebraic properties
Teichmüller space	Culler, Vogtmann Outer Space	$GL_n(\mathbb{R})/O_n$ (symmetric spaces)	finiteness properties (co)homology calculations
Thurston normal form	Bestvina, Handel Train Tracks	Jordan normal form	growth rates subgroup fixed points
Harer bordification	Bestvina, Feighn bordification	Borel, Serre bordification	virtual duality group $H^i(G; \mathbb{Z}G)$ vanishes, $i \neq d$
measured laminations	\mathbb{R} -trees	flag manifold	Tits alternative
Harvey curve complex	???	Tits building	rigidity

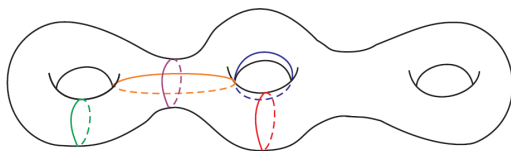
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Curve Complex

surface S :



part of the curve complex $\mathcal{C}(S)$:



Definition

The **curve complex** $\mathcal{C}(S)$:

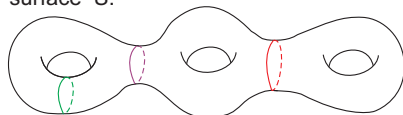
- **vertices:** homotopy classes of essential embedded 1-spheres in S .
- **simplices:** sets of 1-spheres which can be realized disjointly.

Theorem (Masur, Minsky)

$\mathcal{C}(S)$ is Gromov hyperbolic for $g > 1$.

$MCG(S)$ $Out(F_n)$

$$S = \#^n S^1 \times S^1$$

surface S :

simple closed curves

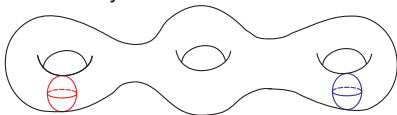
splittings of $\pi_1(S)$ over \mathbb{Z}

separating curves

disjoint curves

curve complex $\mathcal{C}(S)$

$$M = \#^n S^2 \times S^1$$

doubled (solid)
handlebody

spheres

free splittings of $\pi_1(M) = F_n$

separating spheres

disjoint spheres

sphere complex $\mathcal{S}(M)$ free splitting $\mathcal{FS}(M)$

The Splitting Complexes

Definition

The **free splitting complex** \mathcal{FS} , aka the **sphere complex** $\mathcal{S} = \mathcal{S}(M)$:

- **vertices:** free splittings; homotopy classes of essential embedded 2-spheres in M .
- **simplices:** common refinements; sets of 2-spheres which can be realized disjointly.

Definition

The **edge splitting complex** \mathcal{ES} , aka the **separating sphere complex** \mathcal{SS} :
the subcomplex induced by only edge splittings/separating spheres.

$$[(A * B) * C] \xleftrightarrow{[A * B * C]} [A * (B * C)]$$

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The Main Theorem

Theorem (SS)

For $n > 2$, \mathcal{ES} contains a quasi-isometrically embedded copy of \mathbb{Z}^m for every $m \geq 1$.

- 1 \mathcal{ES} is not Gromov hyperbolic. Not 'correct' curve complex analogue
- 2 Quasiflats \mapsto bounded sets in $\mathcal{S}, \mathcal{FF}$
- 3 This supports the analogy between $Out(F_n)$ and MCG
- 4 $\text{asdim } \mathcal{ES} = \infty$, but $Out(F_n) \curvearrowright \mathcal{ES}$ cocompactly (and $\text{asdim } Out(F_n)$ 'should be' finite)
- 5 $\dim(\text{Cone}_\omega(\mathcal{ES})) = \infty$

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The Proof: Motivation

Fix a basepoint $[\langle a \rangle * \langle b, c \rangle]$ in ES_3 .

If $\langle w, w' \rangle = \langle b, c \rangle$:

$$\begin{array}{ccc}
 \langle a \rangle * \langle w, w' \rangle & \cdots \cdots \cdots \rightarrow & \langle a, w \rangle * \langle w' \rangle & & \langle aw^k \rangle * \langle b, c \rangle \\
 \parallel & & \parallel & & \parallel \\
 \langle a \rangle * \langle b, c \rangle & & \langle aw^k, w \rangle * \langle w' \rangle & \cdots \cdots \cdots \rightarrow & \langle aw^k \rangle * \langle w, w' \rangle
 \end{array}$$

Whitehead

- Hope: if $w = u_1^{k_1} \dots u_l^{k_l}$, then

$$d([\langle a \rangle * \langle b, c \rangle], [\langle aw \rangle * \langle b, c \rangle]) \sim 2l.$$

- How to detect the u_i ?

Theorem (Whitehead)

If x is a (cyclically reduced) power of a primitive element in F_n , then there is a cut vertex in its Whitehead graph.

Definition (cut vertex)

A **cut vertex** v of Γ :

$$\Gamma = \Gamma_1 \cup \Gamma_2 \text{ and } \Gamma_1 \cap \Gamma_2 = \{v\}.$$

Definition (Whitehead graph $\Gamma_A(x)$)

- Vertices are $A \sqcup A^{-1}$
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Whitehead Graph Example

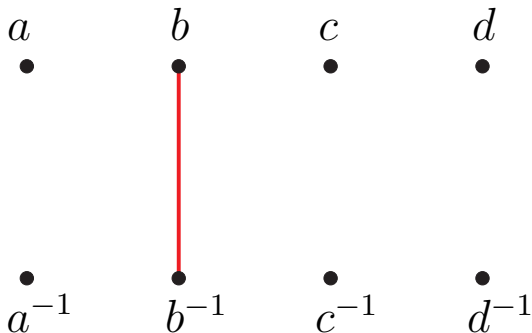
$bbccddb$

a b c d
• • • •

a^{-1} b^{-1} c^{-1} d^{-1}
• • • •

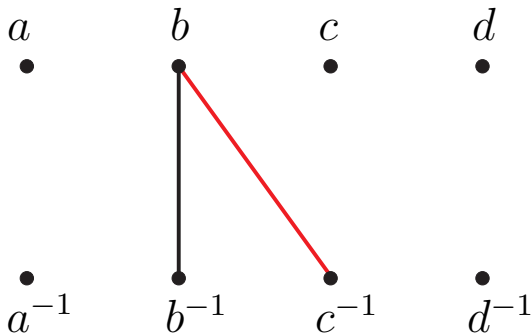
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bb $ccddb$



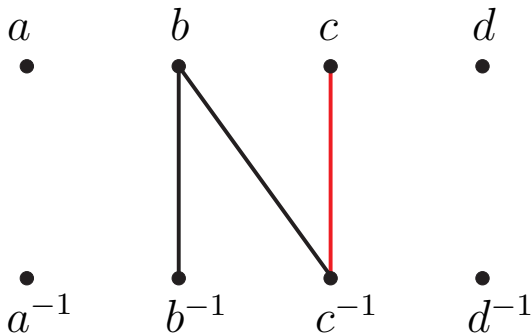
Whitehead Graph Example

b **bc** $cddb$



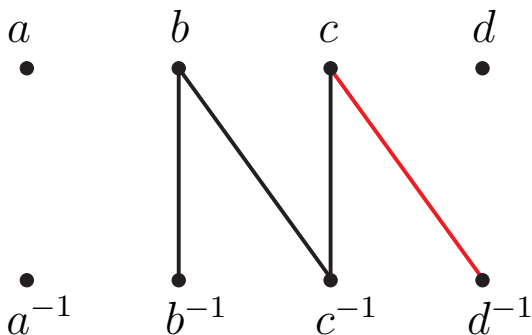
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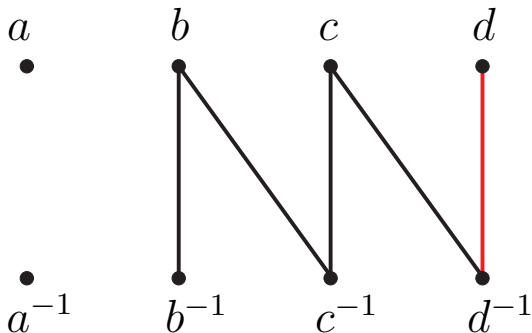
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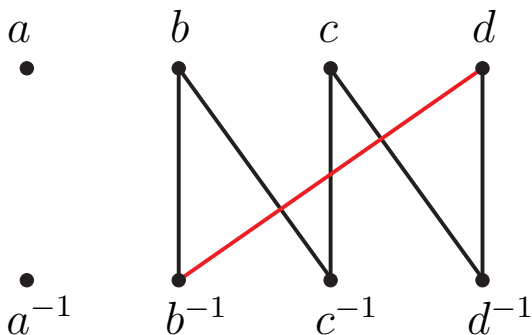
Whitehead Graph Example

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Whitehead Graph Example

$bbccd$ **db**



Simple i -Length

Look at Whitehead graphs of subwords:

Definition (simple i -length of w)

For w not involving a_i , $|w|_i^{\text{simple}} := \max t$ such that $w = w_1 \dots w_t$ where the Whitehead graph $\Gamma_{A-\{a_i\}}(w_j)$ has no cut vertex.

- could be 0.
- No cut vertex \implies not a power of a primitive.

Conjugate reduced i -Length

But for $w = (\text{short word})^{(\text{long word})}$, i -length can be huge:
 $|w|_i^{\text{simple}} \sim 2|(\text{long word})|_i$. True even if w is primitive....

Definition (CR i -length of w)

For w not involving a_i , $|w|_i^{\text{CR}} := \min k$ such that

$$W =_{\text{reduced}} V_1^{u_1} \dots V_l^{u_l}$$

and

$$k = (l - 1) + \sum_{j=1}^l |v_j|_i^{\text{simple}}.$$

- considers *all* possible conjugations
- only need 'obvious' conjugations
- **full i -length** is for sets and bases

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Main Computational Theorems

Theorem (SS)

For any $x_j \in \vec{x}$ containing a_i ,

$$|\vec{x}|_i - 2 \leq |x_j|_i \leq |\vec{x}|_i + 2.$$

Theorem (SS)

For any vertex $[(\vec{x}, S)]$ and any S -auto ϕ with $\phi|_{\langle \vec{x}_S \rangle} = id$,

$$|\vec{x}|_i - 12 \leq |\phi\vec{x}|_i \leq |\vec{x}|_i + 12.$$

Theorem (SS)

$$d([(a, S_a)], [(\vec{x}, S_x)]) \geq \frac{|\vec{x}|_i}{24} - 4.$$

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Constructing a quasiflat

Fix basepoint $\langle \mathbf{a}_1, \dots, \mathbf{a}_{n-1} \rangle * \langle \mathbf{a}_n \rangle$. Let

$$p_t := \mathbf{a}_1^{t+1} \dots \mathbf{a}_{n-1}^{t+1} \mathbf{a}_1^{t+1} \mathbf{a}_2^{t+1} \mathbf{a}_1^{t+1}.$$

For $m \geq 1$ and $\vec{k} \in \mathbb{Z}^m$, let $w_{\vec{k}} := p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$.

Define $\Psi : \mathbb{Z}^m \rightarrow \mathcal{ES}$ by

$$(k_1, \dots, k_m) \mapsto [\langle \mathbf{a}_1, \dots, \mathbf{a}_{n-1} \rangle * \langle \mathbf{a}_n w_{\vec{k}} \rangle]$$

Theorem

$$\frac{1}{11}d(\vec{k}, \vec{l}) - \frac{21}{11} \leq d(\Psi(\vec{k}), \Psi(\vec{l})) \leq 2d(\vec{k}, \vec{l}) + m.$$

Lower Bound

Theorem (SS)

For

$$\omega = w_{\vec{k}}^{-1} w_{\vec{l}} = p_m^{-k_m} \dots p_2^{-k_2} p_1^{-k_1} p_1^{l_1} p_2^{l_2} \dots p_m^{l_m}$$

and $d = d(\vec{k}, \vec{l}) = \sum_{i=1}^m |l_i - k_i|$, we have

$$|p_t|_n \geq 1; \quad |\omega|_n \geq \frac{d}{11} - \frac{21}{11}.$$

Proof: Characterize all 'obvious' families of canceling pairs.

Upper Bound

Theorem

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Proof.

Note if $w \in H := \langle a_1, \dots, a_{n-1} \rangle$ then $H^w = H$.

$$\begin{aligned} \Psi(\vec{k}) &= [H * \langle a_n p_1^{k_1} p_2^{k_2} \dots p_m^{k_m} \rangle] \\ &= [H * \langle p_1^{k_1} p_2^{k_2} \dots p_m^{k_m} a_n \rangle] \\ &\rightsquigarrow [H * \langle p_1^{k_1} p_2^{k_2} \dots p_m^{k_m} a_n p_1^{h_1 - k_1} \rangle] \\ &= [H * \langle p_2^{k_2} \dots p_m^{k_m} a_n p_1^{h_1} \rangle] \\ &\rightsquigarrow \dots \text{ (} m \text{ times)} \\ &\rightsquigarrow [H * \langle a_n p_1^{h_1} \dots p_m^{k_m} \rangle] = \Psi(\vec{l}). \end{aligned}$$

Takes $\sum_{i=1}^m (2|l_i - k_i| + 1) = 2d(\vec{k}, \vec{l}) + m$ steps. □

Upper Bound

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