Geometry of curve complex analogues for $Out(F_n)$

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Notation

F_n	–	free group of rank n
$Aut(F_n)$	–	automorphism group of F_n
$Inn(F_n)$	–	group of inner automorphisms of F_n
$Out(F_n)$	=	$Aut(F_n)/Inn(F_n)$

- $Out(F_n)$ is related to:

$$Out(F_n) \twoheadrightarrow Out(\mathbb{Z}^n) = GL_n(\mathbb{Z})$$

$$MCG(S) \hookrightarrow Out(F_n),$$

where $\pi_1(S) \cong F_n$.

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Vogtmann's Leitmotiv and Bestvina's table

$Out(F_n)$ satisfies a mix of properties, some inherited from the mapping class group, and others from arithmetic groups.

MCGs	$Out(F_n)$	$GL_n(\mathbb{Z})$	Algebraic properties
Teichmüller space	Culler, Vogtmann Outer Space	$GL_n(\mathbb{R})/O_n$ (symmetric spaces)	finiteness properties (co)homology calculations
Thurson normal form	Bestvina, Handel Train Tracks	Jordan normal form	growth rates subgroup fixed points
Harer bordification	Bestvina, Feighn bordification	Borel, Serre bordification	virtual duality group $H^i(G;\mathbb{Z}G)$ vanishes, $i \neq d$
measured laminations	$\mathbb{R}$-trees	flag manifold	Tits alternative
Harvey curve complex	**???**	Tits building	rigidity

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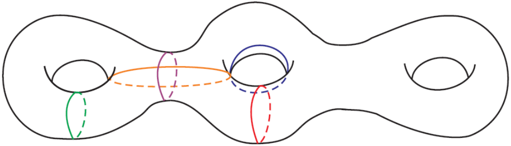
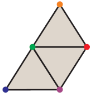
Curve Complex

surface S:



part of the curve complex $\mathcal{C}(S)$:



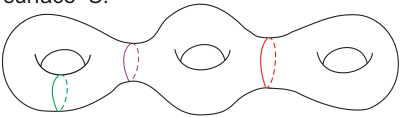
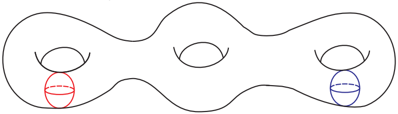
Definition

The curve complex $\mathcal{C}(S)$:

- **vertices:** homotopy classes of essential embedded 1-spheres in S.
- **simplices:** sets of 1-spheres which can be realized disjointly.

Theorem (Masur, Minsky)

$\mathcal{C}(S)$ is Gromov hyperbolic for $g > 2$.

$\mathcal{MCG}(S)$	$Out(F_n)$
$S = \#^n \mathbb{S}^1 \times \mathbb{S}^1$	$M = \#^n \mathbb{S}^2 \times \mathbb{S}^1$
surface S: 	doubled (solid) handlebody 
simple closed curves	spheres
splittings of $\pi_1(S)$ over $\mathbb{Z}$	free splittings of $\pi_1(M) = F_n$
separating curves	separating spheres
disjoint curves	disjoint spheres
curve complex $\mathcal{C}(S)$	sphere complex $\mathcal{S}(M)$ free splitting $\mathcal{FS}(M)$

The Splitting Complexes

Definition

The free splitting complex $\mathcal{FS}$, aka the sphere complex $\mathcal{S} = \mathcal{S}(M)$:

- **vertices:** free splittings; homotopy classes of essential embedded 2-spheres in M.
- **simplices:** common refinements; sets of 2-spheres which can be realized disjointly.

Definition

The edge splitting complex $\mathcal{ES}$, aka the separating sphere complex $\mathcal{SS}$: the subcomplex induced by only edge splittings/separating spheres.

$$[(A * B) * C] \overset{[A*B*C]}{\longleftrightarrow} [A * (B * C)]$$

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The Main Theorem

Theorem (SS)

For $n > 2$, $\mathcal{ES}$ contains a quasi-isometrically embedded copy of $\mathbb{Z}^m$ for every $m \geq 1$.

1. $\mathcal{ES}$ is not Gromov hyperbolic. Not 'correct' curve complex analogue
2. Quasiflats $\mapsto$ bounded sets in $\mathcal{S}$, $\mathcal{FF}$
3. This supports the analogy between $Out(F_n)$ and $\mathcal{MCG}$
4. asdim $\mathcal{ES} = \infty$, but $Out(F_n) \curvearrowright \mathcal{ES}$ cocompactly (and asdim $Out(F_n)$ 'should be' finite)
5. $dim(Cone_\omega(\mathcal{ES})) = \infty$

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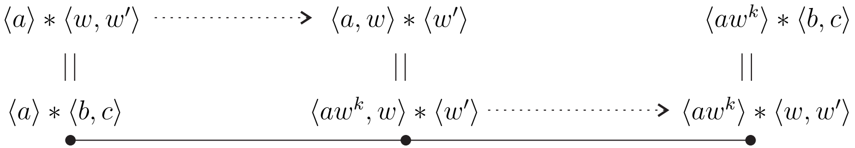
The Proof: Motivation

Fix a basepoint $[\langle a\rangle * \langle b, c\rangle]$ in ES_3.
If $\langle w, w'\rangle = \langle b, c\rangle$:



Whitehead

- Hope: if $w = u_1^{k_1} \dots u_l^{k_l}$, then

$$d([\langle a \rangle * \langle b, c \rangle], [\langle aw \rangle * \langle b, c \rangle]) \sim 2l.$$

- How to detect the u_i?

Theorem (Whitehead)

If x is a power of a primitive element in F_n, then there is a cut vertex in its Whitehead graph.

Definition (cut vertex)

A cut vertex v of Γ:
$\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \{v\}$.

Definition (Whitehead graph $\Gamma_A(x)$)

- Vertices are $A \sqcup A^{-1}$
- Edges: for every $a_i a_j$ in x, have edge a_i to a_j^{-1}.

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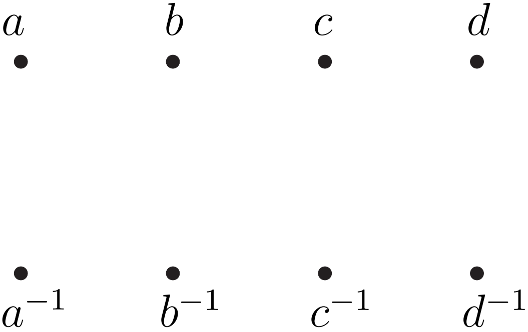
Whitehead Graph Example

bbccddb



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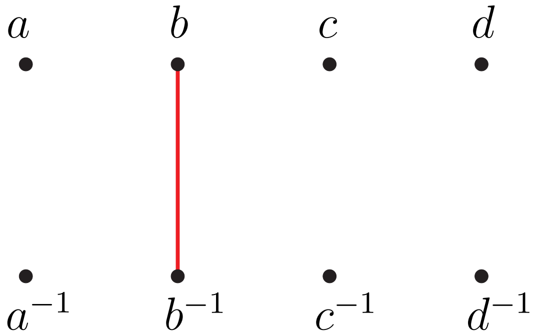

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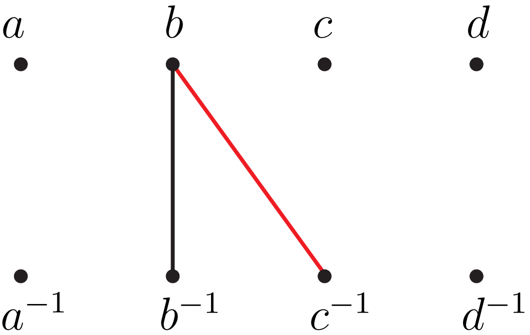

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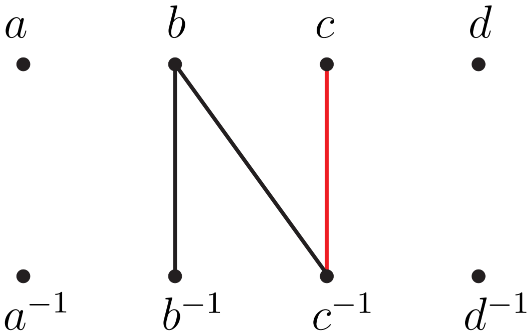

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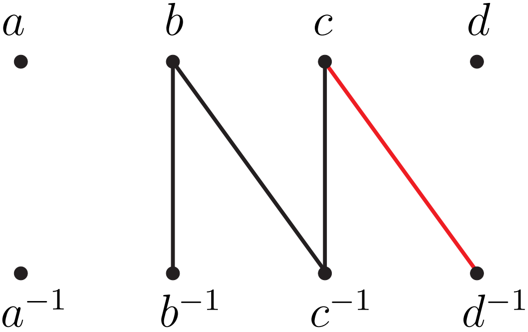

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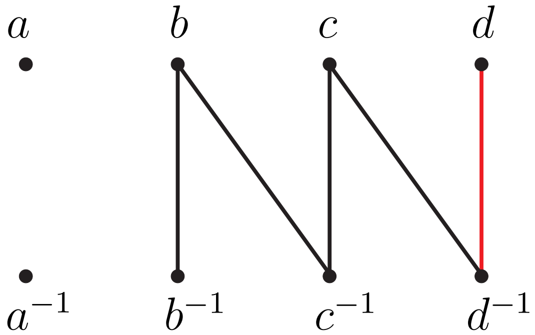

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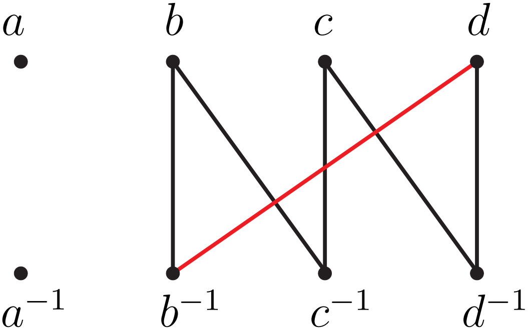

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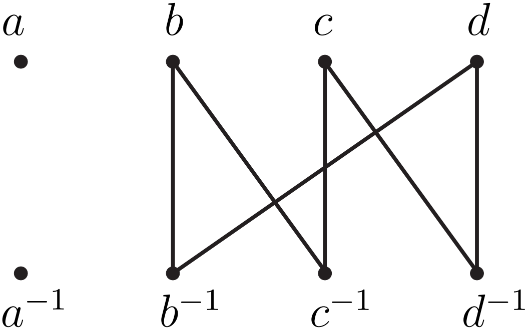

Simple i-Length

Look at Whitehead graphs of subwords:

Definition (simple i-length of w)

For w not involving a_i, $|w|_i^{simple} := \; max \; t$ such that $w = w_1 \ldots w_t$ where the Whitehead graph $\Gamma_{A-\{a_i\}}(w_j)$ has no cut vertex.

- could be 0.
- No cut vertex $\implies$ not a power of a primitive.

Conjugate reduced i-Length

But

$$|u|_i^{simple} \gg 0; |v|_i^{simple} = 0; w = v^u = u^{-1}vu \implies |w|_i^{simple} \gg 0.$$

If v is primitive, $|w|_i^{simple} \not\leq d([\langle a\rangle * \langle b, c\rangle], [\langle aw\rangle * \langle b, c\rangle]) \leq 2$.

Definition (CR i-length of w)

For w not involving a_i, $|w|_i^{cr} :=$ *min* k such that

$$w =_{reduced} v_1^{u_1} \dots v_l^{u_l}$$

and

$$k = (l-1) + \sum_{j=1}^{l} |v_j|_i^{simple}.$$

- considers *all* possible conjugations
- only need 'obvious' conjugations
- full i-length is for sets and bases

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Main Computational Theorems

Theorem (SS)

For any $x_j \in \vec{x}$ containing a_i,

$$|\vec{x}|_i - 2 \leq |x_j|_i \leq |\vec{x}|_i + 2.$$

Theorem (SS)

For any vertex $[(\vec{x}, S)]$ and any S-auto ϕ with $\phi|_{\langle \vec{x}_{\overline{S}} \rangle} = id$,

$$|\vec{x}|_i - 12 \leq |\phi\vec{x}|_i \leq |\vec{x}|_i + 12.$$

Theorem (SS)

$$d\left([(\vec{a}, S_a)], [(\vec{x}, S_x)]\right) \geq \frac{|\vec{x}|_i}{24} - 4.$$

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Constructing a quasiflat

Fix basepoint $\langle a_1, \dots, a_{n-1}\rangle * \langle a_n\rangle$. Let

$$p_t := a_1^{t+1} \dots a_{n-1}^{t+1} a_1^{t+1} a_2^{t+1} a_1^{t+1}.$$

For $m \geq 1$ and $\vec{k} \in \mathbb{Z}^m$, let $w_{\vec{k}} := p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$.
Define $\Psi : \mathbb{Z}^m \to \mathcal{ES}$ by

$$(k_1, \dots, k_m) \mapsto [\langle a_1, \dots, a_{n-1}\rangle * \langle a_n\, w_{\vec{k}}\rangle]$$

Theorem

$$\frac{1}{11} d(\vec{k}, \vec{l}) - \frac{21}{11} \leq d(\Psi(\vec{k}), \Psi(\vec{l})) \leq 2d(\vec{k}, \vec{l}) + m.$$

Lower Bound

Theorem (SS)

For

$$\omega = w_{\vec{k}}^{-1} w_{\vec{l}} = p_m^{-k_m} \dots p_2^{-k_2} p_1^{-k_1} p_1^{l_1} p_2^{l_2} \dots p_m^{l_m}$$

and $d = d(\vec{k}, \vec{l}) = \sum_{i=1}^{m} |l_i - k_i|$*, we have*

$$|p_t|_n \geq 1; \qquad |\omega|_n \geq \frac{d}{11} - \frac{21}{11}.$$

Proof: Characterize all ‘obvious’ families of canceling pairs.

Upper Bound

Theorem

$$\frac{1}{11}d(\vec{k},\vec{l}) - \frac{21}{11} \leq d(\Psi(\vec{k}),\Psi(\vec{l})) \leq 2d(\vec{k},\vec{l}) + m.$$

Proof.

Note if $w \in H := \langle a_1, \ldots, a_{n-1} \rangle$ then $H^w = H$.

$$\begin{aligned}
\Psi(\vec{k}) &= [H * \langle a_n p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m} \rangle] \\
&= [H * \langle p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m} a_n \rangle] \\
&\rightsquigarrow [H * \langle p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m} a_n p_1^{l_1 - k_1} \rangle] \\
&= [H * \langle p_2^{k_2} \ldots p_m^{k_m} a_n p_1^{l_1} \rangle] \\
&\rightsquigarrow \ldots (m \text{ times}) \\
&\rightsquigarrow [H * \langle a_n p_1^{l_1} \ldots p_m^{k_m} \rangle] = \Psi(\vec{l}).
\end{aligned}$$

Takes $\sum_{i=1}^{m} (2|l_i - k_i| + 1) = 2d(\vec{k},\vec{l}) + m$ steps. □

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Upper Bound

Theorem

$$\frac{1}{11}d(\vec{k},\vec{l}) - \frac{21}{11} \leq d(\Psi(\vec{k}),\Psi(\vec{l})) \leq 2d(\vec{k},\vec{l}) + m.$$

Proof.

Note if $w \in H := \langle a_1, \ldots, a_{n-1}\rangle$ then $H^w = H$.

$$\begin{aligned}
\Psi(\vec{k}) &= [H * \langle a_n p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}\rangle] \\
&= [H * \langle p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m} a_n\rangle] \\
&\rightsquigarrow [H * \langle p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m} a_n p_1^{l_1 - k_1}\rangle] \\
&= [H * \langle p_2^{k_2} \ldots p_m^{k_m} a_n p_1^{l_1}\rangle] \\
&\rightsquigarrow \ldots (m\ times) \\
&\rightsquigarrow [H * \langle a_n p_1^{l_1} \ldots p_m^{k_m}\rangle] = \Psi(\vec{l}).
\end{aligned}$$

Takes $\sum_{i=1}^{m}(2|l_i - k_i| + 1) = 2d(\vec{k},\vec{l}) + m$ steps. □