Geometry of curve complex analogues for $Out(F_n)$, II

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11 November, 2010

Notation

F_n	–	free group of rank n
$Aut(F_n)$	–	automorphism group of F_n
$Inn(F_n)$	–	group of inner automorphisms of F_n
$Out(F_n)$	=	$Aut(F_n)/Inn(F_n)$

- $Out(F_n)$ is related to:

$$Out(F_n) \twoheadrightarrow Out(\mathbb{Z}^n) = GL_n(\mathbb{Z})$$

$$MCG(S) \hookrightarrow Out(F_n),$$

where $\pi_1(S) \cong F_n$.

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Vogtmann's Leitmotiv and Bestvina's table

$Out(F_n)$ satisfies a mix of properties, some inherited from the mapping class group, and others from arithmetic groups.

MCGs	$Out(F_n)$	$GL_n(\mathbb{Z})$	Algebraic properties
Teichmüller space	Culler, Vogtmann Outer Space	$GL_n(\mathbb{R})/O_n$ (symmetric spaces)	finiteness properties (co)homology calculations
Thurson normal form	Bestvina, Handel Train Tracks	Jordan normal form	growth rates subgroup fixed points
Harer bordification	Bestvina, Feighn bordification	Borel, Serre bordification	virtual duality group $H^i(G; \mathbb{Z}G)$ vanishes, $i \neq d$
measured laminations	$\mathbb{R}$-trees	flag manifold	Tits alternative
Harvey curve complex	**???**	Tits building	rigidity

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For example, curve complex was used to:

- compute asymptotic dimension of $\mathcal{MCG}$
- prove quasiisometric rigidity of $\mathcal{MCG}$:

Theorem (Mosher-Whyte)

If S_g^1 is an oriented, once-punctured surface of genus $g \geq 2$ with mapping class group $\mathcal{MCG}(S_g^1)$, and if K is a finitely generated group quasi-isometric to $\mathcal{MCG}(S_g^1)$, then there exists a homomorphism $K \to \mathcal{MCG}(S_g^1)$ with finite kernel and finite index image.

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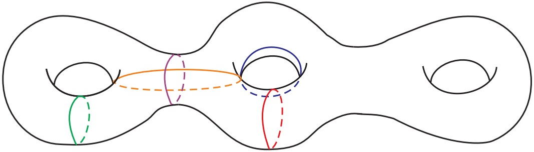
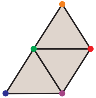
Curve Complex

surface S:



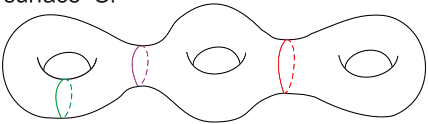
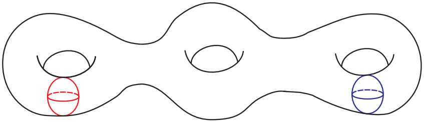
part of the curve complex $\mathcal{C}(S)$:



Definition

A curve complex $\mathcal{C}(S)$ is a simplicial complex defined as follows:

- **vertices:** isotopy classes of essential non-peripheral simple closed curves in S.
- **simplices:** a collection of $k+1$ vertices $\{\alpha_i\}$ forms a k-simplex whenever the α_i can be realized by disjoint curves in S.

$\mathcal{MCG}(S)$	$Out(F_n)$
surface S: 	doubled (solid) handlebody $H = \bigvee_{i=1}^n S_1 \times S_2$ 
simple closed curves	spheres
splittings of $\pi_1(S)$ over $\mathbb{Z}$	free splittings of $\pi_1(H) = F_n$
separating curves	separating spheres
adjacent curves in $\mathcal{C}(S)$ correspond to splittings of $\pi_1(S)$ over $\mathbb{Z}$ with common refinement	adjacent spheres in should correspond to free splittings of F_n over $\mathbb{Z}$ with common refinement
curve complex $\mathcal{C}(S)$	sphere complex (or free splitting graph FS_n)

The Free Splitting Graph

Free Splitting Graph FS_n

- Vertices: conjugacy classes of nontrivial free splittings (as free products or HNN extensions)

$$F_n \cong A * B \qquad \text{or} \qquad F_n \cong A*$$

- Edges of FS_n: common refinements

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The Main Theorem

Theorem (SS)

For $n > 2$, FF_n contains a quasi-isometrically embedded copy of $\mathbb{Z}^m$ for every $m \geq 1$.

1. The space FF_n is not Gromov hyperbolic. Not ‘correct’ curve complex analogue.
2. This supports the analogy between $Out(F_n)$ and $\mathcal{MCG}$.
3. The natural maps to other proposed curve complex analogues (including free splitting graph/sphere graph and the free factor graph) send the quasi-flats to bounded diameter.
4. $\operatorname{asdim} FF_n = \infty$, but $Out(F_n) \curvearrowright FF_n$ cocompactly (and $\operatorname{asdim} Out(F_n)$ ‘should be’ finite).
5. $dim(Cone_\omega(FF_n)) = \infty$

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Vertices of FF_n

How do we work with FF_n?

- Fix a basis $(a_1, \dots, a_n)$ of F_n.
- Note for $\phi \in Aut(F_n)$, $\phi \leftrightarrow (\phi(a_1), \dots, \phi(a_n))$
- **Vertex in FF_n:**

$$[\langle x_1, \dots, x_k \rangle * \langle x_{k+1}, \dots, x_n \rangle]$$

can be identified by

$$\big((x_1, x_2, \dots, x_n), \{1, \dots, k\}\big)$$

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Vertices of FF_n

More generally:

- $\mathcal{S} :=$ set of proper nonempty subsets of $\{1, \ldots, n\}$.
- There is a surjective map $Aut(F_n) \times \mathcal{S} \to V(FF_n)$ given by $(X, S) \mapsto [\langle X_S \rangle * \langle X_{\overline{S}} \rangle]$, where $X_S := \{x_s | s \in S\}$.

Representation of vertices as (X, S) is not unique:

- $\phi \in Aut(F_n)$ is an *S-automorphism* if for some (hence any) basis $\vec{x}$:

$$\phi(\vec{x}_S) = \vec{x}_S, \qquad \phi(\vec{x}_{S^c}) = \vec{x}_{S^c}$$

- We don't change the image of (X, S) if we:
  - shuffle the factors by S-automorphisms (fixing factors setwise)
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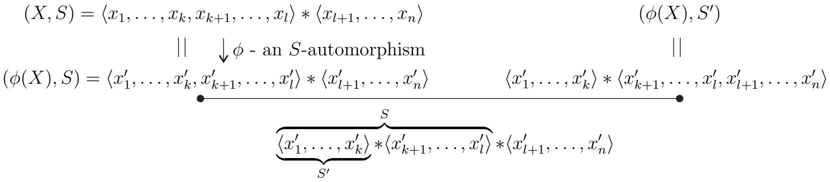
Edges of FF_n

- **Edge in FF_n:**



$$(X,S) = \langle x_1,\dots,x_k,x_{k+1},\dots,x_l\rangle * \langle x_{l+1},\dots,x_n\rangle$$

$$||\quad \downarrow \phi \text{ - an } S\text{-automorphism}$$

$$(\phi(X),S) = \langle x'_1,\dots,x'_k,x'_{k+1},\dots,x'_l\rangle * \langle x'_{l+1},\dots,x'_n\rangle$$

$$(\phi(X),S')$$

$$||$$

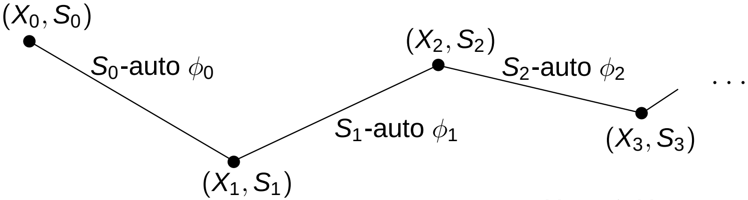
$$\langle x'_1,\dots,x'_k\rangle * \langle x'_{k+1},\dots,x'_l,x'_{l+1},\dots,x'_n\rangle$$

$$\overbrace{\underbrace{\langle x'_1,\dots,x'_k\rangle}_{S'} * \langle x'_{k+1},\dots,x'_l\rangle}^{S} * \langle x'_{l+1},\dots,x'_n\rangle$$

- Edge = apply an S-auto, then compatibly change index set.

Edges of FF_n

- **Edge path in FF_n:**



(X_0, S_0) S_0-auto ϕ_0 (X_1, S_1) S_1-auto ϕ_1 (X_2, S_2) S_2-auto ϕ_2 (X_3, S_3) $\dots$

$$X_1 = \phi_0 X_0$$

$$\begin{aligned} X_2 &= \phi_1 X_1 \\ &= \phi_1 \phi_0 X_0 \end{aligned}$$

$$\begin{aligned} X_3 &= \phi_2 X_2 \\ &= \phi_2 \phi_1 \phi_0 X_0 \end{aligned}$$

- Up to distance 2, (X_i, S_i) is determined by X_i.
- A geodesic corresponds to *minimizing the number of index set changes* - i.e. minimizing l such that $\phi = \phi_l \dots \phi_1 \phi_0$.

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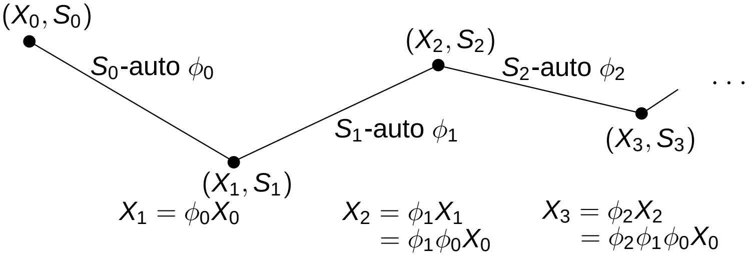

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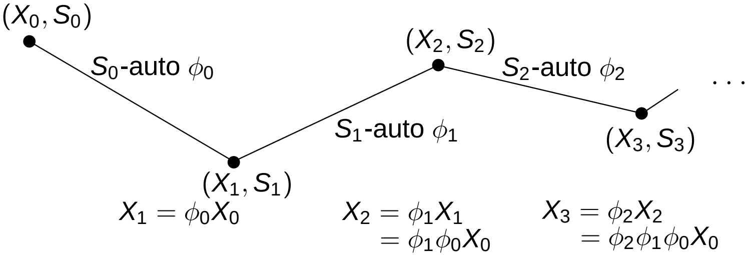

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The Proof: Motivation

Fix a basepoint $[\langle a\rangle * \langle b, c\rangle]$ in FF_3.

If $\langle w, w'\rangle = \langle b, c\rangle$ (so w is primitive in $\langle b, c\rangle$) then distance between $[\langle a\rangle * \langle b, c\rangle]$ and $[\langle aw^k\rangle * \langle b, c\rangle]$ is 2:

$$\begin{tikzcd}
\langle a\rangle * \langle w, w'\rangle \arrow[r, dashed] \arrow[d, equal] & \langle a, w\rangle * \langle w'\rangle \arrow[d, equal] & \langle aw^k\rangle * \langle b, c\rangle \arrow[d, equal] \\
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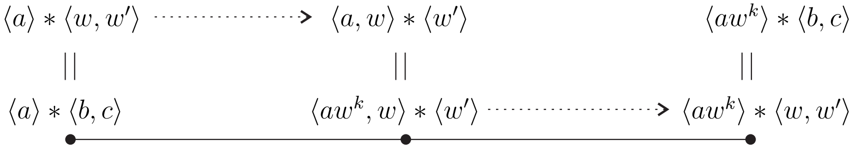

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Whitehead

Want w, expressed minimally as a product of powers of primitives, to 'force' index set changes. But primitives are too difficult to find, so use....

Definition (cut vertex)

A *cut vertex* v of Γ:
$\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \{v\}$.

Definition (Whitehead graph $\Gamma_A(x)$)

- Vertices are $A \sqcup A^{-1}$
- Edges: for every $a_i a_j$ in x, have edge a_i to a_j^{-1}.

Theorem (Whitehead)

If x is a power of a primitive element in F_n, then there is a cut vertex in $\Gamma(\{x\})$.

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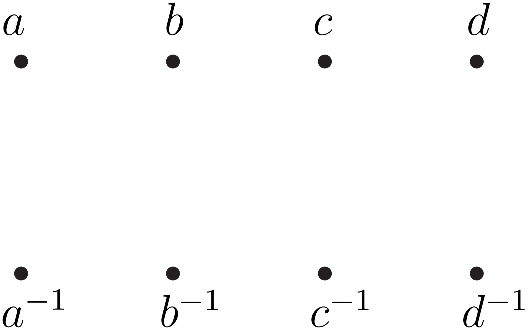
Whitehead Graph Example

bbccddb



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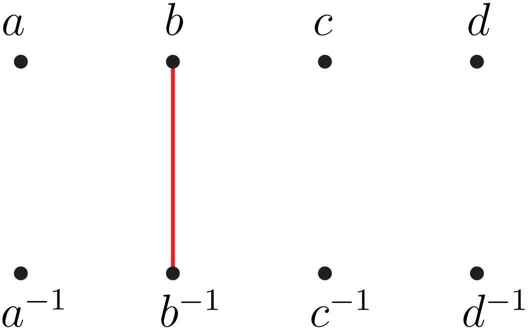
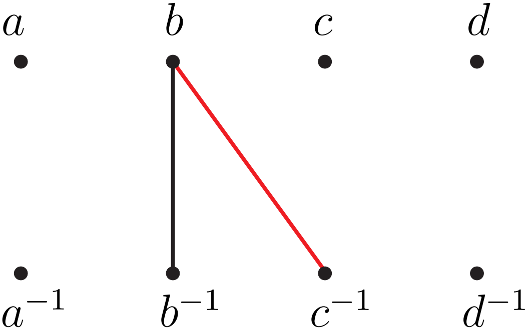


Whitehead Graph Example

$b\color{red}{bc}\color{black}{cddb}$



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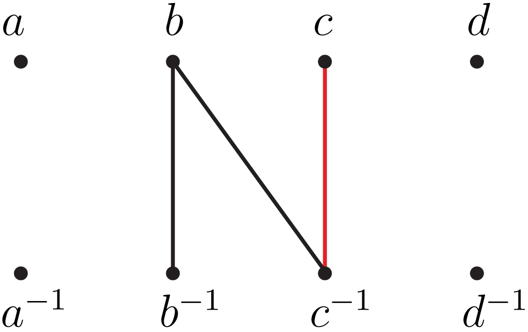

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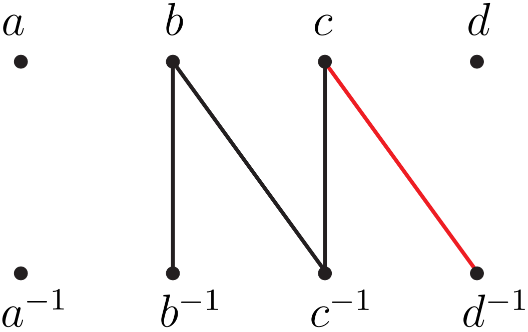

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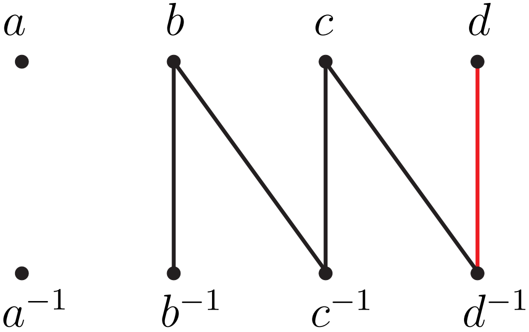

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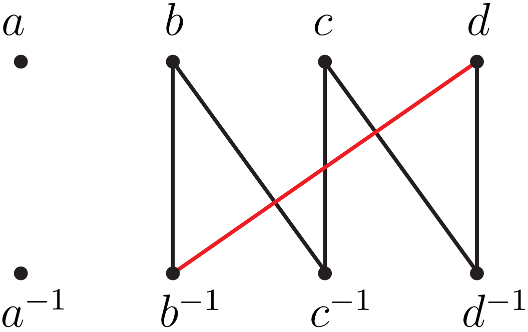

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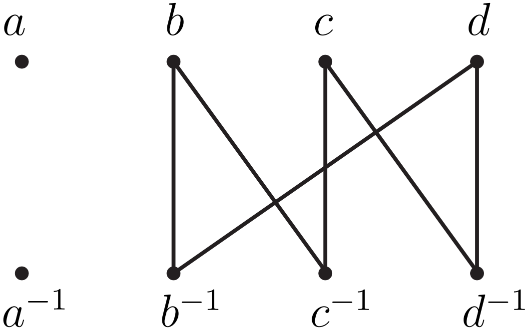

Simple i-Length

Look at Whitehead graphs of subwords:

Definition (simple i-length of w)

For w not involving a_i, $|w|_i^{simple} := \; max\ t$ such that $w = w_1 \dots w_t$ where the Whitehead graph $\Gamma_{A-\{a_i\}}(w_i)$ has no cut vertex.

- could be 0.
- No cut vertex in Whitehead graph - it is not a power of a primitive.

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Conjugate reduced i-Length

But

$$|u|_i^{simple} \gg 0; |v|_i^{simple} = 0; w = v^u = u^{-1}vu \implies |w|_i^{simple} \gg 0.$$

If v is primitive, $|w|_i^{simple} \not\lesssim d([\langle a,b\rangle * \langle c\rangle], [\langle a,b\rangle * \langle cw\rangle]) \leq 2$.

Definition (CR i-length of w)

For w not involving a_i, $|w|_i^{cr} :=$ $min\ k$ such that

$$w =_{reduced} v_1^{u_1} \dots v_l^{u_l}$$

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$$k = (l-1) + \sum_{j=1}^{l} |v_j|_i^{simple}.$$

- Full i-length $|\cdot|_i$ measures complexity of an arbitrary basis, by measuring the maximal CR i-length of subwords not involving a_i.

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Main Computational Theorems

Theorem (SS)

For any $x_j \in \vec{x}$ containing a_i,

$$|\vec{x}|_i - 2 \leq |x_j|_i \leq |\vec{x}|_i + 2.$$

Theorem (SS)

For any vertex $[(\vec{x}, S_x)]$ and any S-auto ϕ with $\phi|_{\langle \vec{x}_{\overline{S}} \rangle} = id$,

$$|\vec{x}|_i - 12 \leq |\phi\vec{x}|_i \leq |\vec{x}|_i + 12.$$

Idea: Find $x_j \in \vec{x}_{\overline{S}}$ containing a_i (but may have to alter $\vec{x}_{\overline{S}}$). Then

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Theorem (SS)

$$d\left([(\vec{a}, S_a)], [(\vec{x}, S_x)]\right) \geq \frac{|\vec{x}|_i}{24} - 4.$$

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Constructing a quasiflat

Fix basepoint $\langle a_1, \ldots, a_{n-1}\rangle * \langle a_n\rangle$. Let

$$p_t := a_1^{t+1} \ldots a_{n-1}^{t+1} a_1^{t+1} a_2^{t+1} a_1^{t+1}.$$

For $m \geq 1$ and $\vec{k} \in \mathbb{Z}^m$, let $w_{\vec{k}} := p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$.
Define $\Psi : \mathbb{Z}^m \to FF_n$ by

$$(k_1, \ldots, k_m) \mapsto [\langle a_1, \ldots, a_{n-1}\rangle * \langle a_n\, w_{\vec{k}}\rangle]$$

Theorem

$$\frac{1}{11} d(\vec{k}, \vec{l}) - \frac{21}{11} \leq d(\Psi(\vec{k}), \Psi(\vec{l})) \leq 2 * d(\vec{k}, \vec{l}) + m.$$

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Relating Simple and CR i-length

Definition (CR i-length of w)

$$|w|_i^{cr} := \min\left\{ (l-1) + \sum_{j=1}^{l} |v_j|_i^{simple} \mid w =_{reduced} v_1^{u_1} \dots v_l^{u_l} \right\}.$$

Definition (canceling pairs)

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- *Family*: a finite set of disjoint canceling pairs of w.
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Relating Simple and CR i-length

Lemma

For w reduced,

$$|w|_i^{cr} \geq \min_{\mathcal{F}} \left(\max \left\{ \frac{1}{2}|\mathcal{F}| - 1, \frac{1}{5}|w - \mathcal{F}|_i^{simple} - 3 \right\} \right).$$

Proof: van Kampen diagram.

Relating Simple and CR i-length

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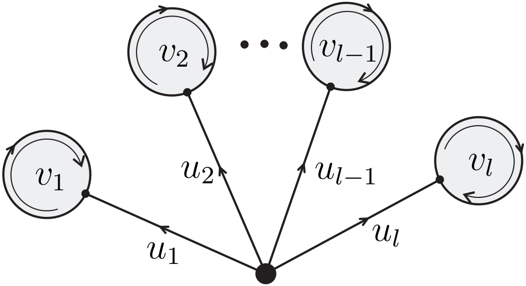
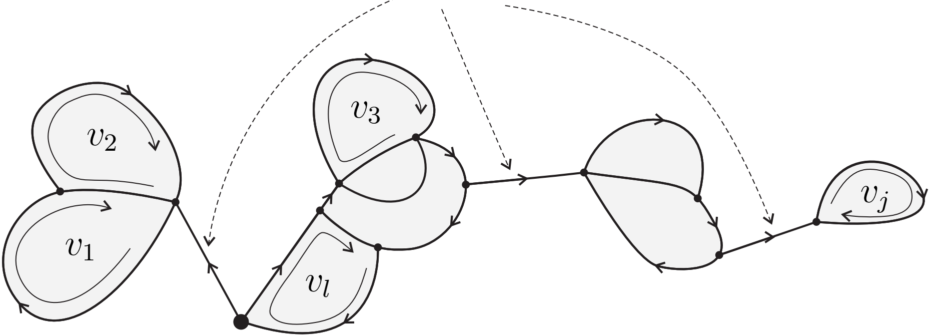
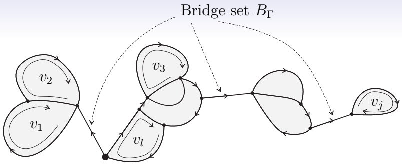
Proof: van Kampen diagram.

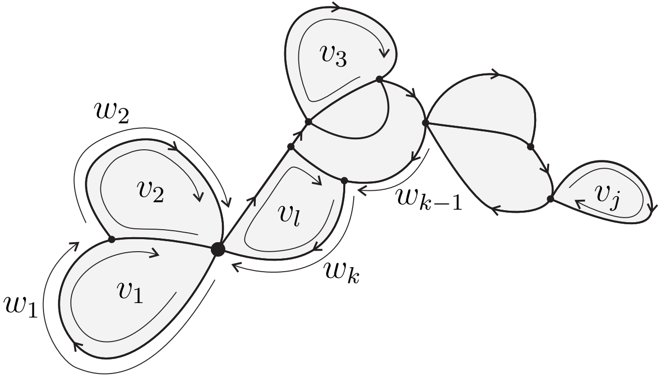
Take optimal $\gamma = v_1^{u_1} \ldots v_l^{u_l}$ so that $w =_{reduced} \gamma$. Note w is trivial in $\langle a_1, \ldots, a_n | v_1, \ldots, v_n \rangle$, so fold γ to w:





Bridge set B_Γ


Bridge set B_Γ
v_2
v_3
v_1
v_l
v_j


w_2
v_3
v_2
v_l
w_{k-1}
v_j
v_1
w_1
w_k

Lower Bound

Theorem (SS)

For

$$\omega = w_{\vec{k}}^{-1} w_{\vec{l}} = p_m^{-k_m} \dots p_2^{-k_2} p_1^{-k_1} p_1^{l_1} p_2^{l_2} \dots p_m^{l_m}$$

and $d = d(\vec{k}, \vec{l}) = \sum_{i=1}^{m} |l_i - k_i|$, *we have*

$$|p_t|_n \geq 1; \qquad |\omega|_n \geq \frac{d}{11} - \frac{21}{11}.$$

Proof: Characterize all ‘obvious’ families of canceling pairs.

Upper Bound

Theorem

$$\frac{1}{11}d(\vec{k},\vec{l}) - \frac{21}{11} \leq d(\Psi(\vec{k}),\Psi(\vec{l})) \leq 2 * d(\vec{k},\vec{l}) + m.$$

Proof.

Note if $w \in H := \langle a_1, \ldots, a_{n-1} \rangle$ then $H^w = H$.

$$\begin{aligned}
\Psi(\vec{k}) &= [H * \langle a_n p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m} \rangle] \\
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Takes $\sum_{i=1}^{m} (2|l_i - k_i| + 1) = 2d(\vec{k},\vec{l}) + m$ steps. ☐

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