On the Geometry of the Free Factorization Graph

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work joint with Dima Savchuk

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The Free Factorization Graph

Free Factorization Graph FF_n

- Vertices: conjugacy classes of nontrivial free factorizations

$$F_n \cong A * B$$

- Edges of FF_n: common refinements

$$(A * B) * C \stackrel{A*B*C}{\longleftrightarrow} A * (B * C)$$

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The Main Theorem

Theorem (SS)

For $n > 2$, FF_n contains a quasi-isometrically embedded copy of $\mathbb{Z}^m$ for every $m \geq 1$.

1. The space FF_n is not Gromov hyperbolic. Not 'correct' curve complex analogue.
2. The natural maps to other proposed curve complex analogues (including free splitting graph/sphere graph, free factor graph) send the quasi-flats to bounded diameter.
3. asdim $FF_n = \infty$, but $Out(F_n) \curvearrowright FF_n$ cocompactly (and asdim $Out(F_n)$ 'should be' finite).
4. $dim(Cone_\omega(FF_n)) = \infty$

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Interpreting FF_n as automorphisms

- Fix $F_n = \langle a_1, \ldots, a_n \rangle$.
- Note for $\phi \in Aut(F_n)$, $\phi \leftrightarrow (\phi(a_1), \ldots, \phi(a_n))$
- Define $\mathcal{S} :=$ set of proper nonempty subsets of $\{1, \ldots, n\}$.
- There is a surjective map $Aut(F_n) \times \mathcal{S} \to V(FF_n)$ given by $(\vec{x}, S) \mapsto [\langle \vec{x}_S \rangle * \langle \vec{x}_{\overline{S}} \rangle]$, where $\vec{x}_S := \{x_s | s \in S\}$.
- Representation of vertices as $(\vec{x}, S)$ not unique: map factors through $Out(F_n) \times \mathcal{S}$, or we can permute indices, exchange S and $\overline{S}$, or apply an automorphism ϕ to $\vec{x}$ fixing $\langle \vec{x}_S \rangle$ and $\langle \vec{x}_{\overline{S}} \rangle$ setwise.
- ϕ is an S-*automorphism* if, for any basis $\vec{x}$,

$$\phi(\langle \vec{x}_S \rangle) = \langle \vec{x}_S \rangle, \qquad \phi(\langle \vec{x}_{\overline{S}} \rangle) = \langle \vec{x}_{\overline{S}} \rangle$$

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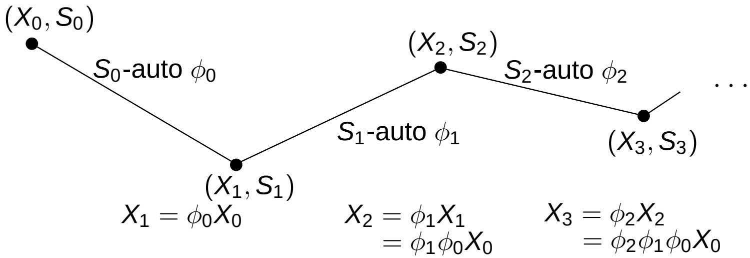
Geodesics in FF_n

- Edge = apply an S-auto, then compatibly change index set.



- Up to distance 2, S_i is determined by ϕ_i.
- A geodesic corresponds to *minimizing the number of index set changes* - i.e. minimizing l such that $\phi = \phi_l \ldots \phi_1 \phi_0$.

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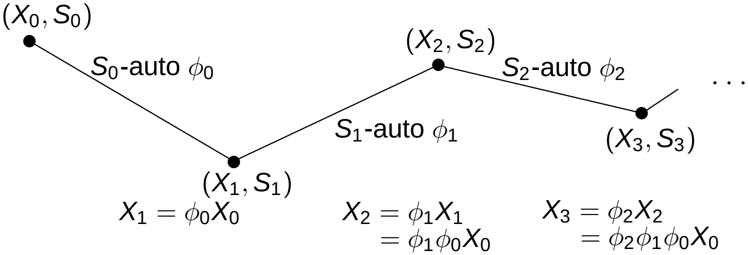

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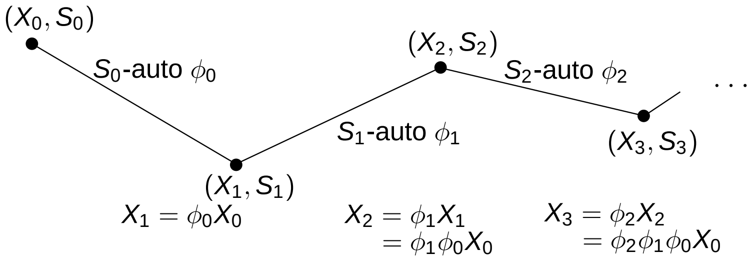

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The Proof: Motivation

Fix a basepoint $[\langle a, b\rangle * \langle c\rangle]$ in FF_n. Consider

$$[\langle a, b\rangle * \langle cw\rangle], \qquad w \in \langle a, b\rangle.$$

- If w is primitive ($\langle a, b\rangle = \langle w, w'\rangle$) then distance is 2:

$$\begin{aligned}
[\langle a, b\rangle * \langle c\rangle] &= [\langle w', w\rangle * \langle c\rangle] \\
&\rightarrow [\langle w'\rangle * \langle w, c\rangle] \\
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Whitehead

Want w, expressed minimally as a product of powers of primitives, to 'force' index set changes. But primitives are too difficult to find, so use....

Definition (cut vertex)

A *cut vertex* v of Γ:
$\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \{v\}$.

Definition (Whitehead graph $\Gamma_A(x)$)

- Vertices are $A \sqcup A^{-1}$
- Edges: for every $a_i a_j$ in x, have edge a_i to a_j^{-1}.

Theorem (Whitehead)

If x is a power of a primitive element in F_n, then there is a cut vertex in $\Gamma(\{x\})$.

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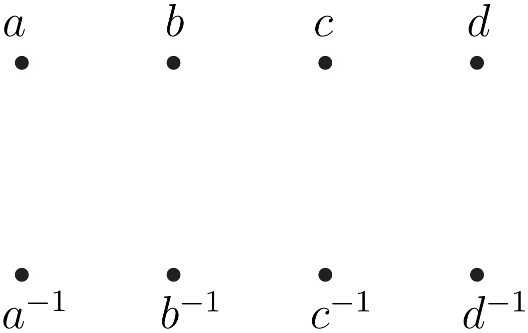

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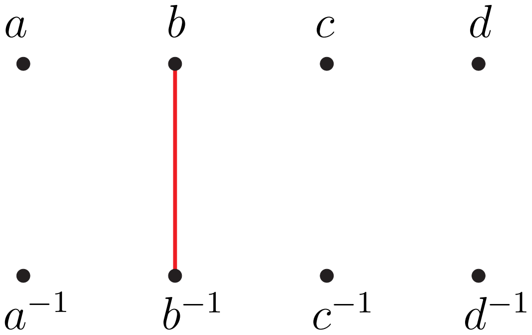

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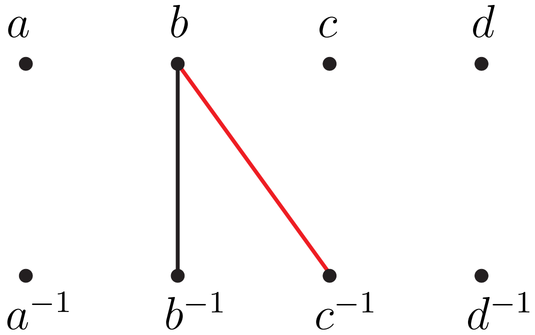

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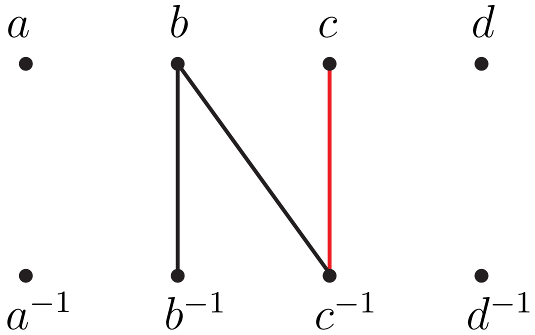

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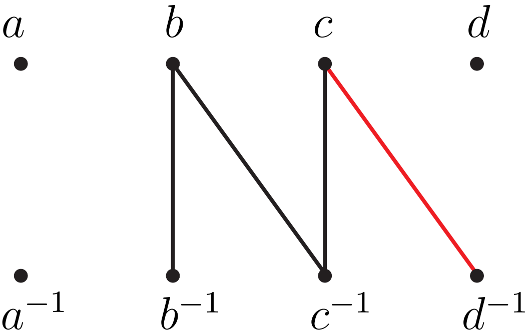

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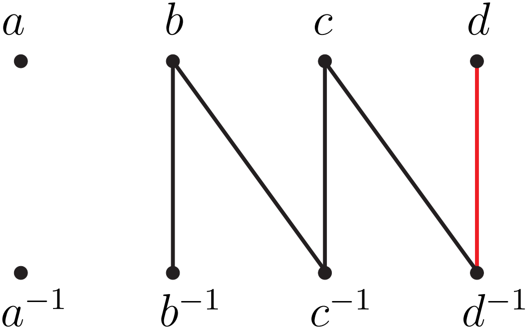

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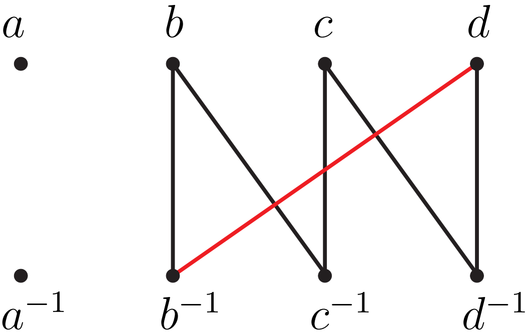

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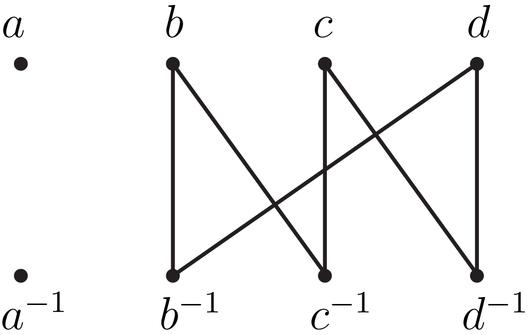

Simple i-Length

Look at Whitehead graphs of subwords:

Definition (simple i-length of w)

For w not involving a_i, $|w|_i^{simple} :=$ *max* t such that $w = w_1 \ldots w_t$ where the Whitehead graph $\Gamma_{A-\{a_i\}}(w_i)$ has no cut vertex.

- could be 0.
- No cut vertex in Whitehead graph - $\{a_i^{\pm 1}\}$ 'uses' all letters but a_i, forcing an index change.

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Conjugate reduced i-Length

But what about $|u|_i^{simple} = 1000000$, $|v|_i^{simple} = 0$, and $w = v^u = u^{-1}vu$? Then $|w|_i^{simple} \sim 2000000$, but if v is primitive then $d([\langle a,b\rangle * \langle c\rangle], [\langle a,b\rangle * \langle cw\rangle]) \leq 2$.

Definition (CR i-length of w)

For w not involving a_i, $|w|_i^{cr} :=$ *min* k such that

$$w =_{reduced} v_1^{u_1} \ldots v_l^{u_l}$$

and

$$k = (l-1) + \sum_{j=1}^{l} |v_j|_i^{simple}.$$

- Full i-length $|\cdot|_i$ measures complexity of an arbitrary basis, by measuring the maximal CR i-length of subwords not involving a_i.

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Main Computational Theorems

Theorem (SS)

For any $x_j \in \vec{x}$ containing a_i,

$$|\vec{x}|_i - 2 \leq |x_j|_i \leq |\vec{x}|_i + 2.$$

Theorem (SS)

For any vertex $[(\vec{x}, S_x)]$ and any S-auto ϕ with $\phi|_{\langle \vec{x}_{\overline{S}} \rangle} = id$,

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Idea: Find $x_j \in \vec{x}_{\overline{S}}$ containing a_i (but may have to alter $\vec{x}_{\overline{S}}$). Then

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Constructing a quasiflat

Fix basepoint $\langle a_1, \ldots, a_{n-1}\rangle * \langle a_n\rangle$. Let

$$p_t := a_1^{t+1} \ldots a_{n-1}^{t+1} a_1^{t+1} a_2^{t+1} a_1^{t+1}.$$

Claim*: p_t is a product of no fewer than 2 powers of primitives.

For $m \geq 1$ and $\vec{k} \in \mathbb{Z}^m$, let $w_{\vec{k}} := p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$.
Define $\Psi : \mathbb{Z}^m \to FF_n$ by

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Lower Bound

Theorem (SS)

For

$$\omega = w_{\vec{k}}^{-1} w_{\vec{l}} = p_m^{-k_m} \dots p_2^{-k_2} p_1^{-k_1} p_1^{l_1} p_2^{l_2} \dots p_m^{l_m}$$

and $d = d(\vec{k}, \vec{l}) = \sum_{i=1}^{m} |l_i - k_i|$, *we have*

$$|p_t|_n \geq 1; \qquad |\omega|_n \geq \frac{d}{11} - \frac{21}{11}.$$

Relating Simple and CR i-length

Definition (canceling pairs)

- *Canceling pair*: an occurrence of two subwords u and u^{-1} of w.
- *Family*: a finite set of disjoint canceling pairs of w.
- *Nested Family*: A family $\mathcal{F}$ where w is *not* $w = ...u...v...u^{-1}...v^{-1}...$.
- $w - \mathcal{F}$: the set of maximal subwords of w outside of $\mathcal{F}$.
- $|w - \mathcal{F}|_i^{simple} := |\mathcal{F}| + \sum_{w' \in w - \mathcal{F}} |w'|_i^{simple}$.

Example: $w = baabbcb^{-1}a^{-1}b^{-1}c^{-1}bc$, and $|w - \mathcal{F}|_d^{simple} = 3 + |a|_d^{simple} + |bc|_d^{simple} + |b|_d^{simple} = 3$.

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Relating Simple and CR i-length

Lemma

For w reduced,

$$|w|_i^{cr} \geq \min_{\mathcal{F}} \left(\max \left\{ \frac{1}{2}|\mathcal{F}| - 1, \frac{1}{5}|w - \mathcal{F}|_i^{simple} - 3 \right\} \right).$$

Proof: van Kampen diagram.

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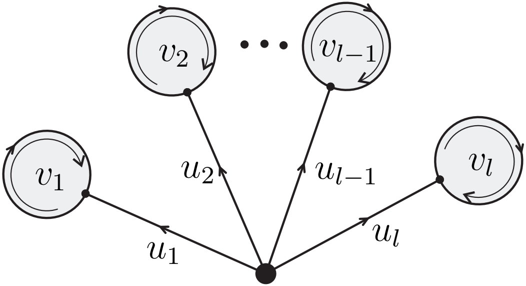
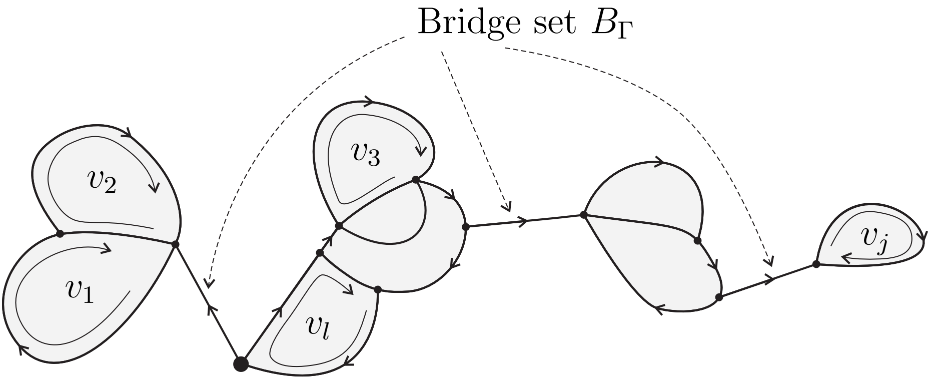
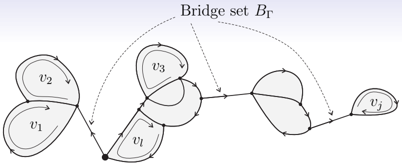
Proof: van Kampen diagram.

Take optimal $\gamma = v_1^{u_1} \dots v_l^{u_l}$ so that $w =_{reduced} \gamma$. Note w is trivial in $\langle a_1, \dots, a_n | v_1, \dots, v_n \rangle$, so fold γ to w:




Bridge set B_Γ
v_2
v_1
v_3
v_l
v_j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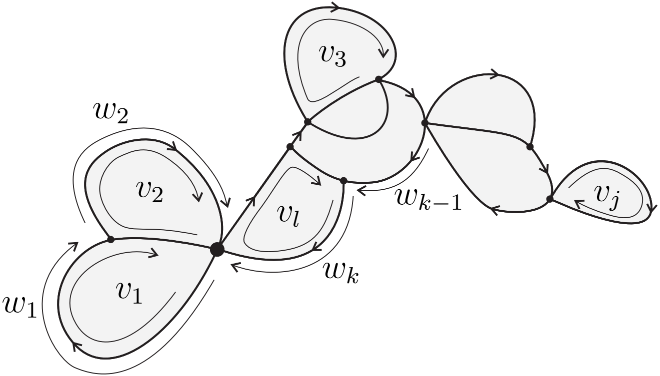
w_2
v_2
v_3
v_l
w_{k-1}
v_j
w_k
v_1
w_1

Upper Bound

Claim

$$d(\Psi(\vec{k}),\Psi(\vec{l})) \leq 2 * d(\vec{k},\vec{l}) + m.$$

Proof.

Note if $w \in H := \langle a_1, \ldots, a_{n-1}\rangle$ then $H^w = H$.

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Takes $\sum_{i=1}^{m}(2|l_i - k_i| + 1) = 2d(\vec{k},\vec{l}) + m$ steps. □

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Takes $\sum_{i=1}^{m} (2|l_i - k_i| + 1) = 2d(\vec{k}, \vec{l}) + m$ steps. □

Upper Bound

Claim

$$d(\Psi(\vec{k}), \Psi(\vec{l})) \leq 2 * d(\vec{k}, \vec{l}) + m.$$

Proof.

Note if $w \in H := \langle a_1, \ldots, a_{n-1} \rangle$ then $H^w = H$.

$$\begin{aligned} \Psi(\vec{k}) &= [H * \langle a_n p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m} \rangle] \\ &= [H * \langle p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m} a_n \rangle] \\ &\rightsquigarrow [H * \langle p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m} a_n p_1^{l_1 - k_1} \rangle] \\ &= [H * \langle p_2^{k_2} \ldots p_m^{k_m} a_n p_1^{l_1} \rangle] \\ &\rightsquigarrow \ldots (m \text{ times}) \\ &\rightsquigarrow [H * \langle a_n p_1^{l_1} \ldots p_m^{k_m} \rangle] = \Psi(\vec{l}). \end{aligned}$$

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$$\begin{array}{rcl} \Psi(\vec{k}) & = & [H * \langle a_n p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m} \rangle] \\ & = & [H * \langle p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m} a_n \rangle] \\ & \rightsquigarrow & [H * \langle p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m} a_n p_1^{l_1 - k_1} \rangle] \\ & = & [H * \langle p_2^{k_2} \ldots p_m^{k_m} a_n p_1^{l_1} \rangle] \\ & \rightsquigarrow & \ldots (m \text{ times}) \\ & \rightsquigarrow & [H * \langle a_n p_1^{l_1} \ldots p_m^{k_m} \rangle] = \Psi(\vec{l}). \end{array}$$

Takes $\sum_{i=1}^{m} (2|l_i - k_i| + 1) = 2d(\vec{k}, \vec{l}) + m$ steps. □