On the Geometry of the Free Factorization Graph

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work joint with Dima Savchuk

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Definitions

$$Inn(F_n) := \{conjugations\} \subset Aut(F_n)$$
$$Out(F_n) := Aut(F_n)/Inn(F_n)$$

Example

If $F_3 = \langle a, b, c \rangle$, then

$$(a, b, c) \mapsto (ab, b, c), \qquad (a, b, c) \mapsto (b^{-1}a, b, b^{-1}cb)$$

represent the same outer automorphism.

- $Out(F_n)$ is related to:

$$Out(F_n) \twoheadrightarrow Out(\mathbb{Z}^n) = GL_n(\mathbb{Z})$$
$$MCG(S) \hookrightarrow Out(F_n),$$

where $\pi_1(S) \cong F_n$.

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Analogies

Leitmotiv (Vogtmann via Bestvina): $Out(F_n)$ satisfies a mix of properties, some inherited from the mapping class group, and others from arithmetic groups.

MCGs	$Out(F_n)$	$GL_n(\mathbb{Z})$	Algebraic properties
Teichmüller space	Culler, Vogtmann Outer Space	$GL_n(\mathbb{R})/O_n$ (symmetric spaces)	finiteness properties (co)homology calculations
Thurson normal form	Bestvina, Handel Train Tracks	Jordan normal form	growth rates subgroup fixed points
Harer bordification	Bestvina, Feighn bordification	Borel, Serre bordification	virtual duality group $H^i(G;\mathbb{Z}G)$ vanishes, $i \neq d$
measured laminations	$\mathbb{R}$-trees	flag manifold	Tits alternative
Harvey curve complex	**???**	Tits building	rigidity

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Free Factorization Graph FF_n

- Vertices: conjugacy classes of nontrivial free factorizations

$$F_n \cong A * B$$

- Edges of FF_n: common refinements

$$(A * B) * C \overset{A*B*C}{\longleftrightarrow} A * (B * C)$$

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The Main Theorem

Theorem (SS)

For $n > 2$, FF_n contains a quasi-isometrically embedded copy of $\mathbb{Z}^m$ for every $m \geq 1$.

1. The space FF_n is not Gromov hyperbolic. Not 'correct' curve complex analogue.
2. The natural maps to every other proposed curve complex analogue send the quasi-flats to bounded diameter.
3. $\operatorname{asdim} FF_n = \infty$, but $Out(F_n) \curvearrowright FF_n$ cocompactly (and $\operatorname{asdim} Out(F_n)$ 'should be' finite).
4. $dim(Cone_\omega(FF_n)) = \infty$

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The Proof: Motivation

Fix a basepoint $[\langle a\rangle * \langle b,c\rangle]$ in FF_n. Consider

$$[\langle aw\rangle * \langle b,c\rangle], \qquad w \in \langle b,c\rangle.$$

- If w is primitive ($\langle b,c\rangle = \langle w,w'\rangle$) then distance is 2:

$$\begin{aligned}
[\langle a\rangle * \langle b,c\rangle] &= [\langle a\rangle * \langle w,w'\rangle] \\
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Constructing a quasiflat

Idea: want lower bound on number of conjugates of powers of primitives in product defining w.

Fix basepoint $\langle a_1 \rangle * \langle a_2, \ldots, a_n \rangle$. Let

$$p_t := a_2^{t+1} \ldots a_n^{t+1} a_2^{t+1} a_3^{t+1} a_2^{t+1}.$$

For $m \geq 1$, define $\Psi : \mathbb{Z}^m \to FF_n$ by

$$(k_1, \ldots, k_m) \mapsto [\langle a_1\, p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m} \rangle * \langle a_2, \ldots, a_n \rangle]$$

Note if $w \in \langle a_2, \ldots, a_n \rangle$ then $\langle a_2^w, \ldots, a_n^w \rangle = \langle a_2, \ldots, a_n \rangle$.

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Proof of upper bound

Claim

$$d(\Psi(\vec{k}),\Psi(\vec{l})) \leq 2 * d(\vec{k},\vec{l}) + m.$$

Proof.

$$\begin{aligned}
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\end{aligned}$$

Takes $\sum_{i=1}^{m}(2|l_i - k_i| + 1) = 2d + m$ steps, where $d = d(\vec{k},\vec{l})$. □

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Takes $\sum_{i=1}^{m} (2|l_i - k_i| + 1) = 2d + m$ steps, where $d = d(\vec{k}, \vec{l})$. □

Proof of upper bound

Claim

$$d(\Psi(\vec{k}), \Psi(\vec{l})) \leq 2 * d(\vec{k}, \vec{l}) + m.$$

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Interpreting FF_n as automorphisms

- Fix $F_n = \langle a_1, \ldots, a_n \rangle$.
- For a tuple $\vec{x} = (x_1, \ldots, x_n)$ and an *index set* $S \subset [n]$, define $\vec{x}_S := \langle x_s | s \in S \rangle$.
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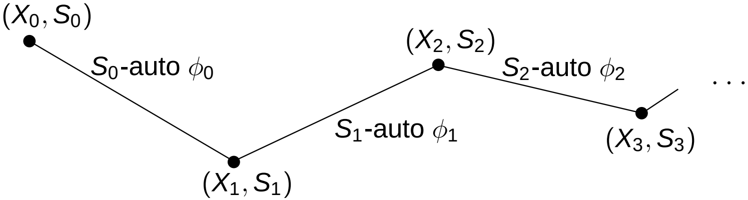
Geodesics in FF_n

- Edge = apply an S-auto, then change index set.



$$X_1 = \phi_0 X_0$$

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$$X_3 = \phi_2 X_2 = \phi_2 \phi_1 \phi_0 X_0$$

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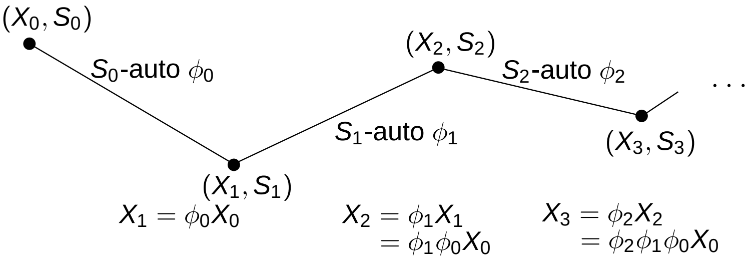

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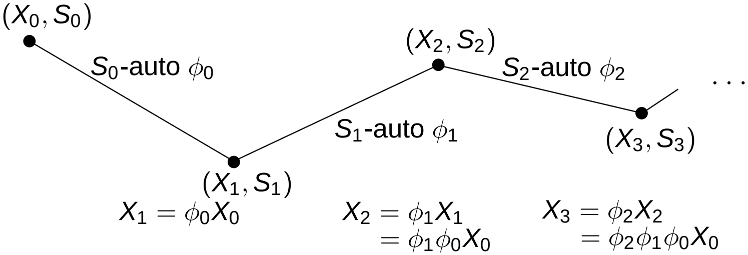

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Definitions

- Recall motivation $[\langle aw\rangle * \langle b,c\rangle], w \in \langle b,c\rangle$.
- Need w to 'force' index set changes. Use decomp of w into product of primitives. But primitives are too difficult to find, so use....

Definition (cut vertex)

A *cut vertex* v of Γ:
$\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \{v\}$.

Definition (Whitehead graph $\Gamma_A(x)$)

- Vertices are $A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$
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If x is a conjugate of a power of a primitive element in F_n, then there is a cut vertex in $\Gamma(\{x\})$.

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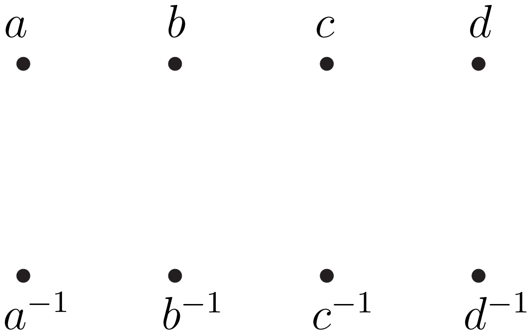
bbccddb



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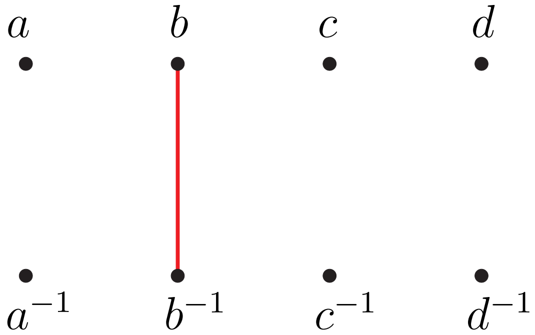

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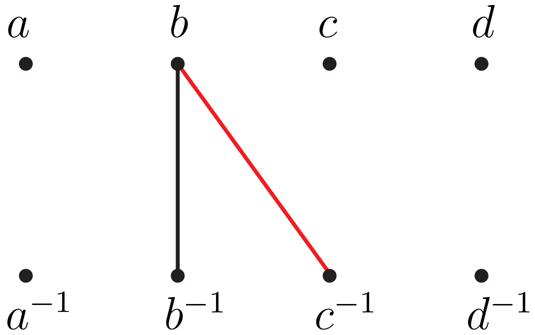

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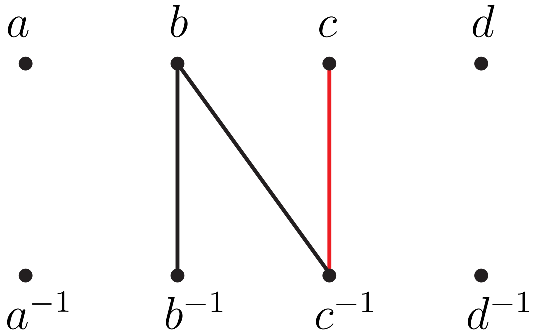

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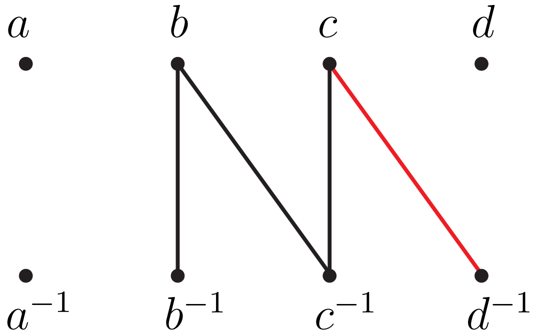

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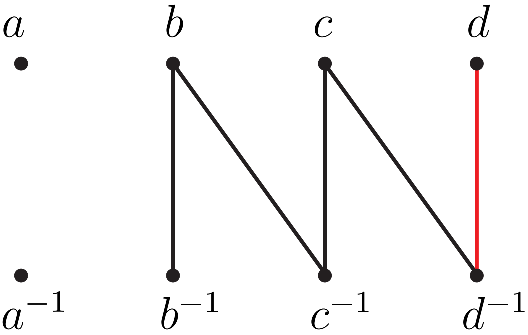

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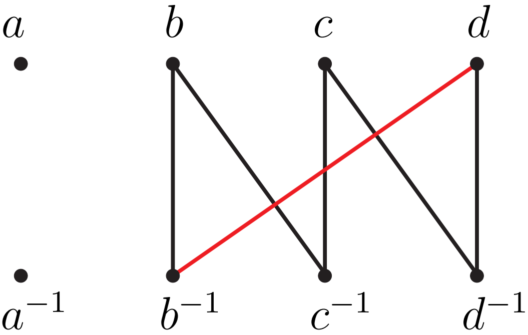

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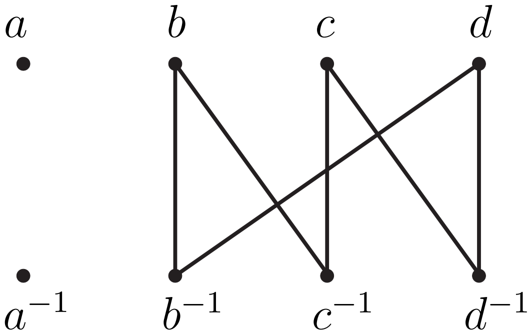

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Definition (simple i-length of w)

For w not involving a_i, $|w|_i^{simple} := \max t$ such that $w = w_1 \dots w_t$ where the Whitehead graph $\Gamma_{A-\{a_i\}}(w_i)$ has no cut vertex.

- could be 0.
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Main Computational Theorems

Theorem (SS)

For any i,

$$d_{FF_n}\left([(\vec{a}, S_a)], [(\vec{x}, S_x)]\right) \geq \frac{|\vec{x}|_i}{24} - 2.$$

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