

On the Geometry of the Free Splitting Graph

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work joint with Dima Savchuk

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Definitions

$$\text{Inn}(F_n) := \{\text{conjugations}\} \subset \text{Aut}(F_n)$$

$$\text{Out}(F_n) := \text{Aut}(F_n) / \text{Inn}(F_n)$$

Example

If $F_3 = \langle a, b, c \rangle$, then

$$(a, b, c) \mapsto (ab, b, c), \quad (a, b, c) \mapsto (b^{-1}a, b, b^{-1}cb)$$

represent the same outer automorphism.

- $\text{Out}(F_n)$ is related to:

$$\text{Out}(F_n) \twoheadrightarrow \text{Out}(\mathbb{Z}^n) = \text{GL}_n(\mathbb{Z})$$

$$\text{MCG}(S) \hookrightarrow \text{Out}(F_n),$$

where $\pi_1(S) \cong F_n$.

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Analogies

Leitmotiv (Vogtmann via Bestvina): $Out(F_n)$ satisfies a mix of properties, some inherited from the mapping class group, and others from arithmetic groups.

MCGs	$Out(F_n)$	$GL_n(\mathbb{Z})$	Algebraic properties
Teichmüller space	Culler, Vogtmann Outer Space	$GL_n(\mathbb{R})/O_n$ (symmetric spaces)	finiteness properties (co)homology calculations
Thurston normal form	Bestvina, Handel Train Tracks	Jordan normal form	growth rates subgroup fixed points
Harer bordification	Bestvina, Feighn bordification	Borel, Serre bordification	virtual duality group $H^i(G; \mathbb{Z}G)$ vanishes, $i \neq d$
measured laminations	\mathbb{R} -trees	flag manifold	Tits alternative
Harvey curve complex	???	Tits building	rigidity

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Free Splitting Graph FS_n

- Vertices: conjugacy classes of nontrivial free splittings

$$F_n \cong A * B$$

- Edges of FS_n : common refinements

$$(A * B) * C \overset{A * B * C}{\longleftrightarrow} A * (B * C)$$

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The Main Theorem

Theorem (SS)

For $n > 2$, FS_n contains a quasi-isometrically embedded copy of \mathbb{Z}^m for every $m \geq 1$.

- 1 The space FS_n is not Gromov hyperbolic. Not 'correct' curve complex analogue.
- 2 The natural maps to every other proposed curve complex analogue send the quasi-flats to bounded diameter.
- 3 $\text{asdim } FS_n = \infty$, but $\text{Out}(F_n) \curvearrowright FS_n$ cocompactly (and $\text{asdim } \text{Out}(F_n)$ 'should be' finite).
- 4 $\dim(\text{Cone}_\omega(FS_n)) = \infty$

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The Proof: Motivation

Fix a basepoint $[\langle a \rangle * \langle b, c \rangle]$ in FS_n . Consider

$$[\langle aw \rangle * \langle b, c \rangle], \quad w \in \langle b, c \rangle.$$

- If w is primitive ($\langle b, c \rangle = \langle w, w' \rangle$) then distance is 2:

$$\begin{aligned} [\langle a \rangle * \langle b, c \rangle] &= [\langle a \rangle * \langle w, w' \rangle] \\ &\rightarrow [\langle a, w \rangle * \langle w' \rangle] \\ &= [\langle aw, w \rangle * \langle w' \rangle] \\ &\rightarrow [\langle aw \rangle * \langle w, w' \rangle] \\ &= [\langle aw \rangle * \langle b, c \rangle] \end{aligned}$$

- If w is a power of a primitive, same.
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Constructing a quasiflat

Idea: want lower bound on number of conjugates of powers of primitives in product defining w .

Fix basepoint $\langle a_1 \rangle * \langle a_2, \dots, a_n \rangle$. Let

$$p_t := a_2^{t+1} \dots a_n^{t+1} a_2^{t+1} a_3^{t+1} a_2^{t+1}.$$

For $m \geq 1$, define $\Psi : \mathbb{Z}^m \rightarrow FS_n$ by

$$(k_1, \dots, k_m) \mapsto [\langle a_1 p_1^{k_1} p_2^{k_2} \dots p_m^{k_m} \rangle * \langle a_2, \dots, a_n \rangle]$$

Note if $w \in \langle a_2, \dots, a_n \rangle$ then $\langle a_2^w, \dots, a_n^w \rangle = \langle a_2, \dots, a_n \rangle$.

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Proof of upper bound

Claim

$$d(\Psi(\vec{k}), \Psi(\vec{l})) \leq 2 * d(\vec{k}, \vec{l}) + m.$$

Proof.

$$\begin{aligned} \Psi(\vec{k}) &= [\langle a_1 p_1^{k_1} p_2^{k_2} \dots p_m^{k_m} \rangle * \langle a_2, \dots, a_n \rangle] \\ &= [\langle p_1^{k_1} p_2^{k_2} \dots p_m^{k_m} a_1 \rangle * \langle a_2, \dots, a_n \rangle] \\ &\rightsquigarrow [\langle p_1^{k_1} p_2^{k_2} \dots p_m^{k_m} a_1 p_1^{l_1 - k_1} \rangle * \langle a_2, \dots, a_n \rangle] \\ &= [\langle p_2^{k_2} \dots p_m^{k_m} a_1 p_1^{l_1} \rangle * \langle a_2, \dots, a_n \rangle] \\ &\rightsquigarrow \dots \\ &\rightsquigarrow [\langle p_m^{k_m} a_1 p_1^{l_1} p_2^{l_2} \dots p_m^{l_m - k_m} \rangle * \langle a_2, \dots, a_n \rangle] \\ &= [\langle a_1 p_1^{l_1} p_2^{l_2} \dots p_m^{l_m} \rangle * \langle a_2, \dots, a_n \rangle] = \Psi(\vec{l}). \end{aligned}$$

Takes $\sum_{i=1}^m (2|l_i - k_i| + 1) = 2d + m$ steps, where $d = d(\vec{k}, \vec{l})$. □

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Interpreting FS_n as automorphisms

- Fix $F_n = \langle \mathbf{a}_1, \dots, \mathbf{a}_n \rangle$.
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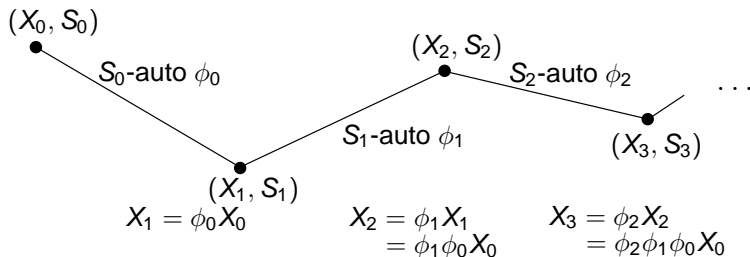
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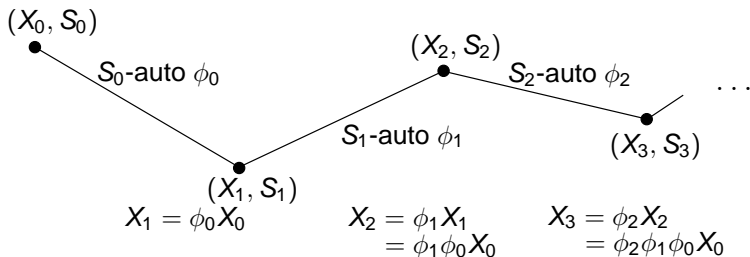
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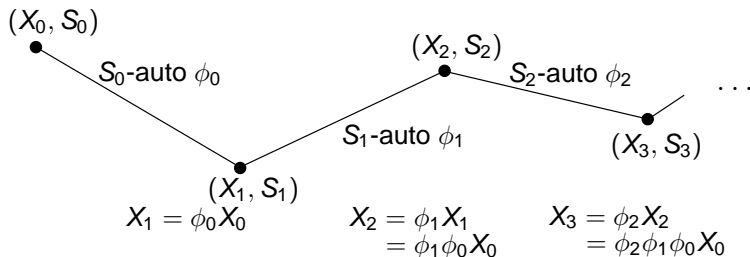
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Definitions

- Recall motivation [$\langle aw \rangle * \langle b, c \rangle$], $w \in \langle b, c \rangle$.

- Need w to 'force' index set changes. Use decomp of w into product of primitives. But primitives are too difficult to find, so use....

Definition (cut vertex)

A cut vertex v of Γ :

$$\Gamma = \Gamma_1 \cup \Gamma_2 \text{ and } \Gamma_1 \cap \Gamma_2 = \{v\}.$$

Definition (Whitehead graph $\Gamma_A(x)$)

- Vertices are $A \cup A^{-1} = \{a_1, a_1^{-1}, \dots, a_n, a_n^{-1}\}$
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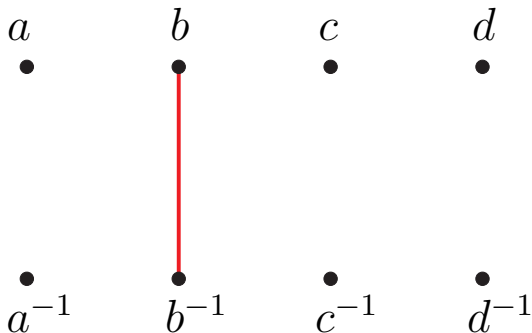
bbccddb

a *b* *c* *d*
● ● ● ●

● ● ● ●
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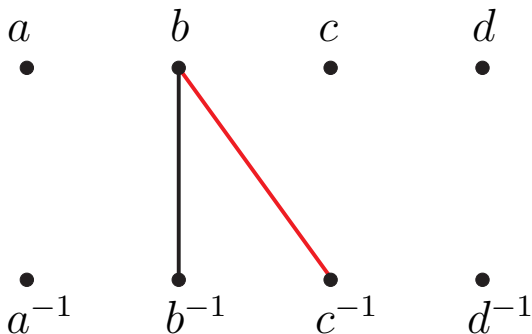
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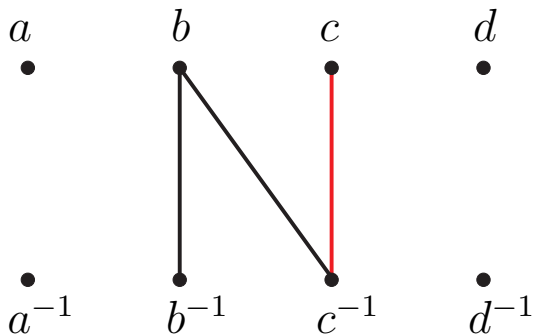
Whitehead Graph Example

b **bc** $cddb$



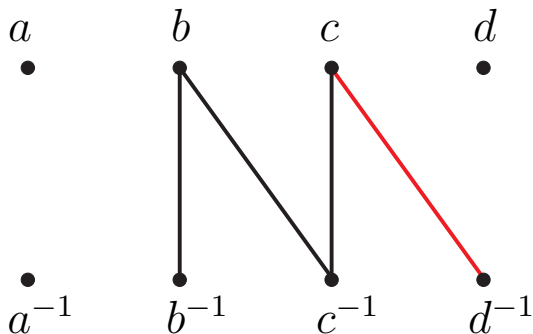
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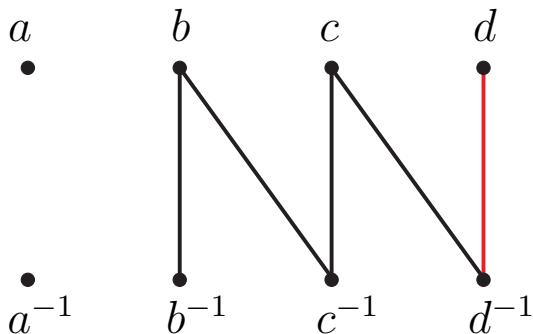
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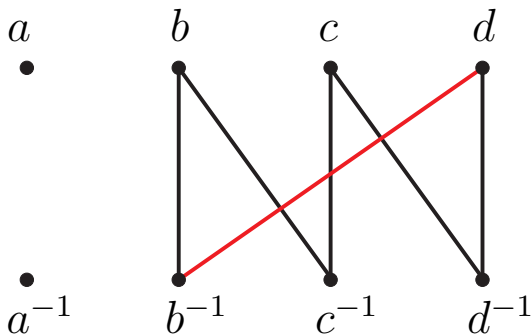
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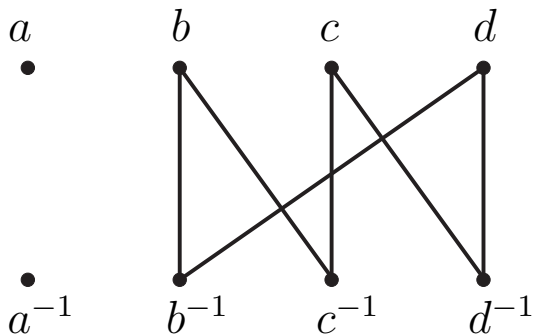
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$bbccd$ **db**



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i -Length

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For w not involving a_i , $|w|_i^{simple} := \max t$ such that $w = w_1 \dots w_t$ where the Whitehead graph $\Gamma_{A-\{a_i\}}(w_i)$ has no cut vertex.

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Main Computational Theorems

Theorem (SS)

For any i ,

$$d_{FS_n}([\vec{a}, S_a], [(\vec{x}, S_x)]) \geq \frac{|\vec{x}|_i}{24} - 2.$$

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