On the Geometry of $Out(F_n)$

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joint with Dima Savchuk

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Definitions

$$Inn(F_n) := \{conjugations\} \subset Aut(F_n)$$
$$Out(F_n) := Aut(F_n)/Inn(F_n)$$

- $Out(F_n)$ is related to:

$$Out(F_n) \twoheadrightarrow Out(\mathbb{Z}^n) = GL_n(\mathbb{Z})$$

$$MCG(S) \hookrightarrow Out(F_n),$$

where $\pi_1(S) \cong F_n$.

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Analogies

Leitmotiv (Vogtmann via Bestvina): $Out(F_n)$ satisfies a mix of properties, some inherited from MCGs, and others from arithmetic groups.

MCGs	$Out(F_n)$	$GL_n(\mathbb{Z})$	Algebraic properties
Teichmüller space	Culler, Vogtmann Outer Space	$GL_n(\mathbb{R})/O_n$ (symmetric spaces)	finiteness properties (co)homology calculations
Thurson normal form	Bestvina, Handel Train Tracks	Jordan normal form	growth rates subgroup fixed points
Harer bordification	Bestvina, Feighn bordification	Borel, Serre bordification	virtual duality group $H^i(G;\mathbb{Z}G)$ vanishes, $i \neq d$
measured laminations	$\mathbb{R}$-trees	flag manifold	Tits alternative
Harvey curve complex	???	Tits building	rigidity

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The Free Splitting Graphs

Free Splitting Graph FS_n and Free Splitting Dual Graph FS_n^*

- Vertices: (conjugacy classes of) nontrivial free splittings $F_n \cong A * B$
- Edges of FS_n: connect two splittings if they have a common refinement

$$(A * C) * B \overset{A*C*B}{\longleftrightarrow} A * (C * B)$$

- Edges of FS_n^*: connect two splittings $A_1 * B_1$ and $A_2 * B_2$ if $A_1 \cap A_2 \neq \emptyset$ or $A_1 \cap B_2 \neq \emptyset$ or $B_1 \cap A_2 \neq \emptyset$ or $B_1 \cap B_2 \neq \emptyset$.

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Our Goals

$$j : FS_n \to FS_n^*$$

is the map induced by identity on vertices, and is 1-Lipshitz.

Claim

There exists a set of infinite diameter in FS_n whose image under j has diameter 1.

Conjecture

The space FS_n contains quasiisometrically embedded quasi-flats. In particular, FS_n is not Gromov hyperbolic.

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i-length

- Motivating Example: $\langle a(b^2c^2d^2b)^k\rangle * \langle b, c, d\rangle$ *should* be far from $\langle a\rangle * \langle b, c, d\rangle$: it uses all the basis elements.

Definition (simple i-length of w)

The greatest number t such that w is of the form $w_1w_2\dots w_{t_j}$, where the Whitehead graph $\Gamma_{A-\{a_i\}}(w_i)$ has no cut vertex. Denoted $|w|_i^{simple}$.

- full i-length $|w|_i$ needs to take into account conjugation, and is a bit more complicated.

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Whitehead graphs and Separable sets

Definition (cut vertex)

A *cut vertex* v of Γ: $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1, \Gamma_2 \neq \emptyset$ and $\Gamma_1 \cap \Gamma_2 = \{v\}$.

Definition (Whitehead graph $\Gamma_A(\{x_1, \ldots, x_k\})$)

- Vertices are $A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$
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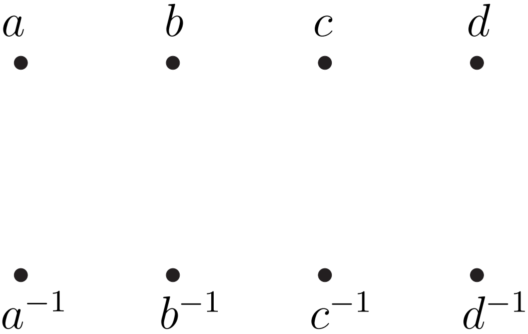

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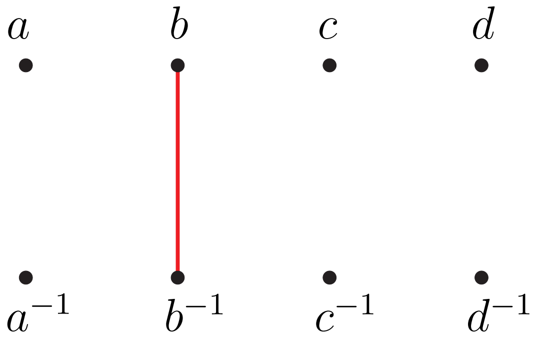

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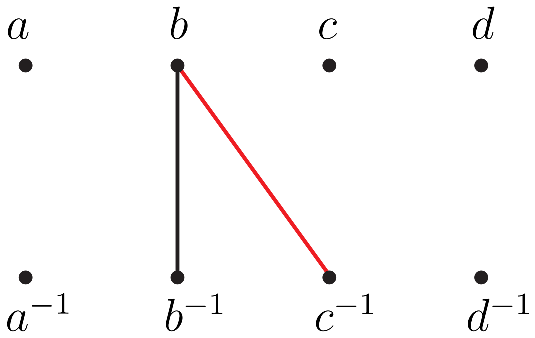

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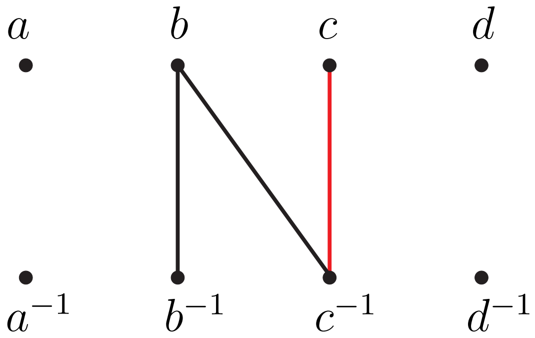

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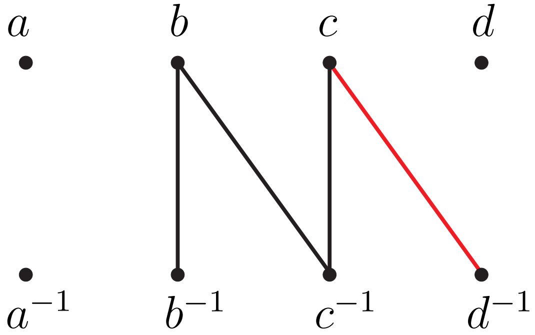

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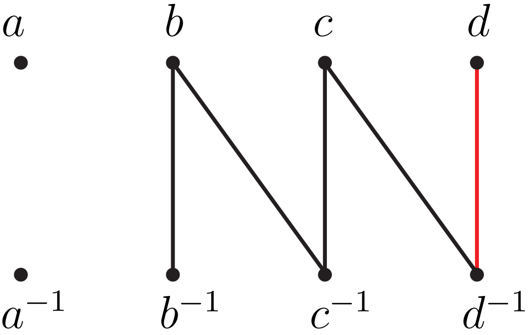

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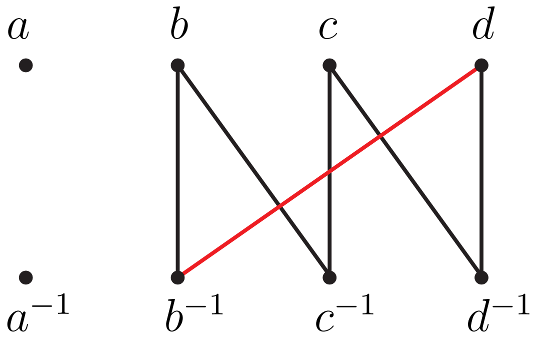

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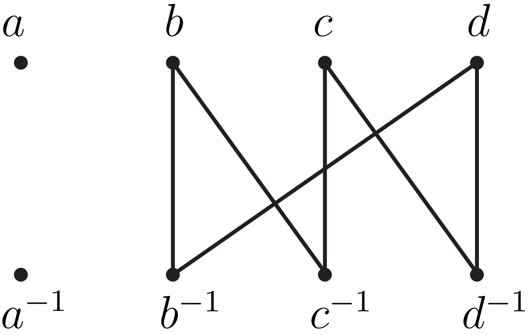

Theorem (Stallings)

If X is a separable set in F_n, then there is a cut vertex in $\Gamma(X)$.

Definition (separable set)

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Applications of i-length

Lemma

For a basis X of F_n, $|X|_i > 0$ for at most 1 index i.

Theorem

For a basis X of F_n, if $|X|_i > 0$, there exist words w_L and w_R such that, for $\alpha_X : a_i \mapsto w_L^{-1} a_i w_R^{-1}$, $|\alpha_X X|_i = 0$.

Lemma

A vertex in FS_n is characterized by a basis and an index set, (X, S). Following an edge from (X, S) in FS_n is characterized by: applying an automorphism which preserves $\langle x_s \in X | s \in S \rangle$ and $\langle x_s \in X | s \notin S \rangle$, then changing index set S.

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Applications Continued

Theorem

For $S \subset \{1, \ldots, n\}$ proper nonempty, any basis X, and any automorphism $\phi \in Aut(F_n)$ which is the identity on $\langle x_j \in X | j \notin S \rangle$,

$$|X|_i - 18 \leq |\phi X|_i \leq |X|_i + 18.$$

Corollary

A path from A to X in FS_n must take at least $|X|_i / 18$ edges.

Claim

There exist bases with arbitrarily large i-length.

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Thank You

Interpreting FS_n as automorphisms

Recall:

$$(A * C) * B \overset{A*C*B}{\longleftrightarrow} A * (C * B)$$

- Fix a standard basis $(a_1, \ldots, a_n)$ of F_n.
- We can identify free splitting $A * B$ of F_n with: a basis $(x_1, \ldots, x_n)$ of F_n and a choice of index set S so $A \cong \langle x_s | s \in S \rangle$ and $B \cong \langle x_s | s \notin S \rangle$, *up to* conjugation, permutation of indices, exchanging S and $\overline{S}$, and an S-automorphism - i.e. an automorphism preserving A and B.
- A basis of $(x_1, \ldots, x_n)$ corresponds to an automorphism $(a_1, \ldots, a_n) \mapsto (x_1, \ldots, x_n)$.

Geodesics in FS_n

- An edge corresponds to applying an S-automorphism, then changing index set S.
- An edge path from $(\vec{x}, S_0)$ corresponds to: an S_0-auto ϕ_0, new index set S_1, an S_1-auto ϕ_1, new index set S_2, etc.
- Up to distance 1, S_i is determined by ϕ_i.
- A geodesic corresponds to minimizing the number of index set changes - i.e. minimizing l such that $\phi = \phi_0\phi_1 \ldots \phi_l$.