Generalized Expanders

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Thanks

Thank you, John and Susan

Getting the most out of your telephone network

- A classical expander (Bassalygo, Pinsker at AT&T Bell Labs) is a family of *sparse*, *highly connected* graphs: for n vertices, want $O(n)$ edges, $O(\log n)$ diameter, $O(\log n)$-connected.
- Common definition: uniformly bounded isoperimetric number (Cheeger constant, spectral gap)

$$h(G) := \min_{1 \leq |S| \leq \frac{|G|}{2}} \frac{|\partial S|}{|S|}$$

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Examples

- The 3-regular graphs G'_p, p prime, with:
  vertex set $\mathbb{Z}_p$
  edge set $\{(x, x \pm 1), (x, x^{-1}) | x \in \mathbb{Z}_p\}$
  (uses Selberg $3/16$ from Number Theory)

Theorem (Margulis; see Lubotsky)

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- Networks: cheap, efficient, robust
- Pseudorandomness: Random walks on the graph approach the uniform distribution as quickly as possible.
- Error-correction: Simplest known efficient asymptotically good codes.
- many more (see for instance Hoory, Linial, Wigderson)

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One of the Definitions

Definition (Jerrum, Sinclair)

A sequence of finite connected graphs (X_n) is a *classical expander* if:

1. there's a uniform bound k on the degrees of vertices,
2. $|X_n| \to \infty$,
3. $\exists C > 0$ such that, for all n and all $f_n \in \ell^2(X_n)$,

$$\frac{1}{|X_n|^2} \sum_{x,y \in X_n} |f(x) - f(y)|^2 \leq \frac{C}{|X_n|} \sum_{\substack{x,y \in X_n \\ x,y \text{ adjacent}}} |f(x) - f(y)|^2.$$

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$$(\textit{some variance condition})$$

Coarse Embeddings

Definition (QI embedding)

A map $\phi : X \to Y$ is a *quasi-isometric embedding* if there exist constants K and C such that, for all $x, y \in X$,

$$\frac{1}{K}d(x,y) - C \leq d(\phi(x), \phi(y)) \leq Kd(x,y) + C.$$

Coarse Embeddings

Definition (QI embedding)

A map $\phi : X \to Y$ is a *quasi-isometric embedding* if there exist linear polynomials $\rho_-(z) = \frac{z}{K} - C$ and $\rho_+(z) = Kz + C$ such that, for all $x, y \in X$,

$$\rho_-(d(x,y)) \leq d(\phi(x), \phi(y)) \leq \rho_+(d(x,y)).$$

Coarse Embeddings

Definition (coarse embedding)

A map $\phi : X \to Y$ is a *coarse embedding* if there exist increasing unbounded functions ρ_- and ρ_+ such that, for all $x, y \in \mathcal{X}$,

$$\rho_-(d(x,y)) \leq d(\phi(x), \phi(y)) \leq \rho_+(d(x,y)).$$

A family (X_i) of metric spaces *coarsely embeds (uniformly embeds)* into Y if there is a family of coarse embeddings with a uniform choice of ρ_- and ρ_+.

- very weak on the small scale, but existence of coarse maps can impose conditions on asymptotic geometry/boundaries

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Results

Theorem (Gromov)

Let $\mathcal{H}$ denote a separable infinite-dimensional Hilbert space. A classical expander (X_n) does not coarsely embed into $\mathcal{H}$.

Theorem (Higson, Gromov)

Let M denote a Hadamard manifold (i.e. finite-dimensional CAT(0)). A classical expander (X_n) does not coarsely embed into M.

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Groups

- For a finitely generated group G, if G coarsely embeds into a Hilbert space, then G satisfies the Novikov conjecture
- An amenable group (or with finite asymptotic dimension, or with Property A) coarsely embeds into $\mathcal{H}$.
- Kazhdan's Property T ("opposite" amenability) used to construct expanders.
- Gromov has a finitely presented (Olshanskii, Sapir) group which (weakly) contains an expander (!) $\implies$ does not uniformly embed into Hilbert space. However, it satisfies Novikov.

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Definition

Definition (Full Expander; c.f. Tessera, Ostrovskii)

Let X be a metric space equipped with a Borel measure μ. Define the *diagonal complement* Ω_r as $\{(x, y) \in X^2 | d(x, y) \geq r\}$. We say (X, μ) is a *full expander* if:

1. **Widely supported:** $\mu(\Omega_r) \geq 1$ for all r.
2. **Exhaustion by distance thresholds:** there exists $x_0 \in X$ and an unbounded sequence $0 < x_1 < x_2 < \dots$ such that, for any $x, y \in X^2$, $\mu(x, y) > 0$ iff there is $n \in N$ with $x, y \in B(x_0, r_{n+1})$ and $d(x, y) > r_n$.
3. **Bounded variance:** for any Lipschitz $f : X \to \mathcal{H}$, there exists K so

$$Var_\mu f := \sum_{x,y \in X} |f(x) - f(y)|^2 \mu(x, y) \leq K^2.$$

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Theorem (Ostrovskii; Tessera; KS)

A classical expander induces (is) a full expander.

Theorem (Tessera; (Ostrovskii); KS(?))

*A full expander does not coarsely embed into $\mathcal{H}$ *.*

*A metric space X does not coarsely embed into $\mathcal{H}$ * if and only if some full expander coarsely embeds into X.*

*: can be generalized to: any class of metric spaces such that the associated sheaf is *p-admissible*. Satisfied by: separable Hilbert, L^p, $CAT(0)$, ...

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A *bounded expander* is a full expander (X, μ) such that $\alpha < \infty$ and $\beta < \infty$, where $\alpha := \sup_{n\to\infty} \alpha_n$ and $\beta := \sup_{n\to\infty} \beta_n$, and

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Theorem (KS)

Classical expanders induce (are) bounded expanders.

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Let $\mathcal{M}$ be a Hadamard manifold, and let $\mathcal{X}$ be a bounded expander. Then $\mathcal{X}$ does not coarsely embed into $\mathcal{M}$.

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Open Questions

- Are there any full expanders which are not bounded?
- Find a good notion of equivalence of expanders
- Are there any examples of full expanders which do not arise from classical expanders?
- Can one construct a classical expander from a full expander?Is there an infinite classical expander (as opposed to a family)?
- What can be said about "boundaries" of expanders?
- Classical expanders obstruct: all cohomology Lipschitz (of Connes, Gromov, Moscovici).
  - "Claim": So do bounded expanders.
  - What about other homology/cohomology properties? H_{uf} of Block and Weinberger should be trivial $\implies$ "Ponzi flow" showing space is not amenable. What is the flow?
  - Understand how classes in cohomology of a group might satisfy Novikov, but not come from a "boundary".

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Madoff's Theorem

Theorem (Gromov)

If a graph has a Ponzi flow, it is not amenable.

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Theorem (Gromov)

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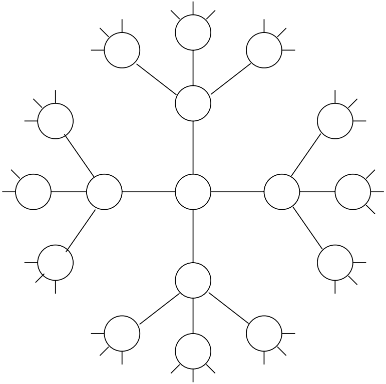

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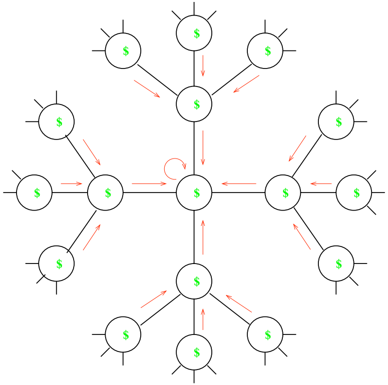

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