

# Generalized Expanders

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# Outline

- 1 Classical Expanders
- 2 Generalized Expanders
- 3 Bounded Generalized Expanders
- 4 Next Steps

# Coarse Embeddings

## Definition (coarse embedding)

A family  $(X_i)_{i \in I}$  *coarsely embeds* into a metric space  $Y$  if there exists embeddings  $(F_i : X_i \rightarrow Y)$  which are *uniformly coarse*: there exist increasing unbounded  $\rho_-, \rho_+$  with, for all  $i \in I$  and  $x, y \in X_i$ ,

$$\rho_-(d(x, y)) \leq d(F_i(x), F_i(y)) \leq \rho_+(d(x, y)).$$

- impose conditions on asymptotic geometry - boundaries
- into linear spaces is strong - for f.g. gps, implies Novikov
- into Hilbert spaces related to amenability, exactness, Gromov's  $\alpha$ -T-menability/Haagerup, finite asymptotic dimension, coarse Baum-Connes, Novikov, ...

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# Madoff's Theorem

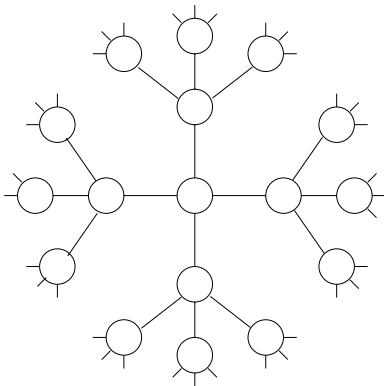
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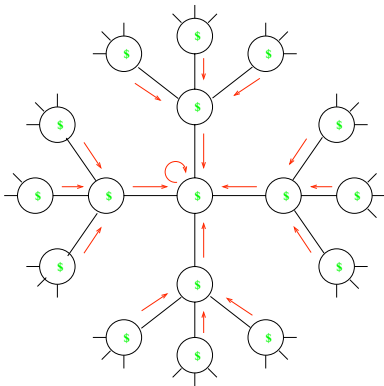
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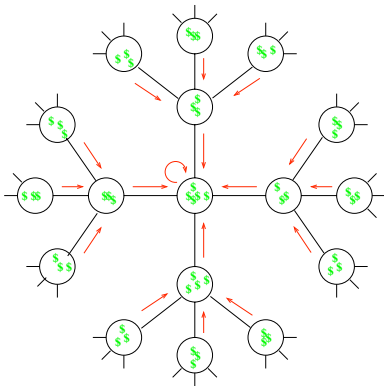
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# Getting the most out of your telephone network

- A classical expander (Bassalygo, Pinsker) is a family of *sparse, highly connected* graphs.
- Usually defined in terms of uniformly bounded isoperimetric number (Cheeger number, spectral gap)

$$h(G) := \min_{1 \leq |S| \leq \frac{|G|}{2}} \frac{|\partial S|}{|S|}$$

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## One of the Definitions

### Definition (Jerrum, Sinclair)

A sequence of finite connected graphs  $(X_n)$  is a *classical expander* if:

- 1 there's a uniform bound  $k$  on the degrees of vertices,
- 2  $|X_n| \rightarrow \infty$ ,
- 3  $\exists C > 0$  such that, for all  $n$  and all  $f_n \in \ell^2(X_n)$ ,

$$\frac{1}{|X_n|^2} \sum_{x,y \in X_n} |f(x) - f(y)|^2 \leq \frac{C}{|X_n|} \sum_{\substack{x,y \in X_n \\ x,y \text{ adjacent}}} |f(x) - f(y)|^2.$$

## Results

- The 3-regular graphs  $G'_p$ ,  $p$  prime, with vertex set  $\mathbb{Z}_p$  and edge set  $\{(x, x \pm 1), (x, x^{-1}) \mid x \in \mathbb{Z}_p\}$   
(uses Selberg 3/16 from Number Theory)

Theorem (Margulis; see Lubotsky)

*Families of Schreier graphs of finite-index subgroups of groups with Property T form an expander.*

Theorem (Gromov)

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## Preliminaries

- For  $r > 0$ , metric space  $X$ , the  $r$ -diagonal complement is

$$\Omega_r(X) := \{(x, y) \in X^2 \mid d(x, y) \geq r\}.$$

- Let  $(\mathcal{X}, \mu)$  be a sequence  $X_n$  with measures  $\mu_n$  defined on  $X_n^2$ , such that:
  - $\mu_n$  either zero or probability measure,
  - $\mu_n$  a probability measure infinitely often,
  - There is  $r_n > 0$  such that  $r_n \rightarrow \infty$ ,  $\mu_n$  supported on  $\Omega_{r_n}(X_n)$ .

This is a *measured family*.

- Measured family  $(\mathcal{X}, \mu)$  has *Poincaré inequality* for metric space  $Z$  if, for all 1-Lipschitz map  $f : \mathcal{X} \rightarrow Z$ :

$$\mathbf{Var}_{\mu_n}(f) := \sum_{(a,b) \in X_n^2} d(f(a), f(b))^2 \mu_n(a, b) \leq K^2.$$



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# Definition

## Definition (Ostrovskii, Tessera)

For any family of metric spaces  $\mathcal{C}$ ,  $(\mathcal{X}, \mu)$  is a  $\mathcal{C}$ -*expander*, or *generalized expander*, if  $(\mathcal{X}, \mu)$  has a Poincaré inequality with uniform constant  $K$  for every  $Z \in \mathcal{C}$ .

(slightly modified from Tessera: does not reference space obstructing; only need subsequences)

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## Theorem (Ostrovskii, Tessera)

*A classical expander is a  $\{\mathcal{H}\}$ -expander.*

## Theorem (Tessera, (Ostrovskii))

*Let  $\mathcal{C}$  be a class of metric spaces satisfying some conditions\*.  
A  $\mathcal{C}$ -expander does not coarsely embed into any  $Z \in \mathcal{C}$ .*

*A metric space  $X$  does not coarsely embed into any element  
 $Z \in \mathcal{C}$  if and only if some  $\mathcal{C}$ -expander coarsely embeds into  $X$ .*

\*: associated sheaf is  $p$ -admissible. Satisfied by: separable Hilbert,  $L^p$ ,  $CAT(0)$ , ...

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# Motivation

For  $\{X_n, \mu_n\}$ , define:

$$\alpha_n := \frac{1}{|X_n|^2 \min_{(a,b) \in \text{supp}(\mu_n)} (\mu_n(a, b))}$$

and

$$\beta_n := \left( r_n^2 \frac{|X_n|^2 - |\Omega_{r_n}(X_n)|}{|X_n|^2} \right).$$

## Theorem

For  $\nu_n$  the uniform measure on  $X_n^2$ ,  $Z$  a space where  $\mathcal{X}$  has a Poincaré inequality, and 1-Lipschitz  $F_n : X_n \rightarrow Z$ ,

$$\mathbf{Var}_{\nu_n}(F_n) \leq \alpha_n K^2 + \beta_n.$$

Idea: Split up  $\mathbf{Var}_{\nu_n}(F_n)$  over  $\Omega_n$  and  $\Omega_n^c$ :

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A generalized expander is *bounded* if  $\alpha, \beta < \infty$ .

- All classical expanders are bounded ( $\alpha_n, \beta_n \rightarrow 0$ )
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# Coarse Embeddings

## Lemma

*If  $\mathcal{X}$  is bounded and has a Poincaré inequality for  $Z$ , there exists  $k$  and  $k'$  so for any  $n$  and 1-Lipschitz  $F_n : X_n \rightarrow Z$ ,*

$$\mathbf{Var}_{\nu_n}(F_n) \leq \alpha K^2 + \beta + k' = k.$$

## Theorem

*Let  $\mathcal{M}$  be a Hadamard manifold, and let  $\mathcal{X}$  be a bounded Hilbert-expander. Then  $\mathcal{X}$  does not coarsely embed into  $\mathcal{M}$ .*

Proof similar to standard argument for classical expanders.

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# Modifications

Play with definition.

- make  $\text{supp}(\mu_n) = \Omega$ ?
- assume coarsely connected?
- assume *nested*?
  - All obstruction theorems still hold

Analyze "Moduli space" of expanders

- Good notion of equivalence of expanders?
- Good topology or metric on "Moduli space" of expanders?
- Examples of non-classical generalized expanders?
- Examples of non-bounded classical expanders?

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- Classical expanders obstruct: all cohomology Lipschitz (of Connes, Gromov, Moscovici).
  - "Claim": So do bounded generalized expanders
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