Generalized Expanders

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21 March, 2009

Outline

1. Classical Expanders
2. Generalized Expanders
3. Bounded Generalized Expanders
4. Next Steps

Coarse Embeddings

Definition (coarse embedding)

A family $(X_i)_{i\in I}$ *coarsely embeds* into a metric space Y if there exists embeddings $(F_i : X_i \to Y)$ which are *uniformly coarse*: there exist increasing unbounded ρ_-, ρ_+ with, for all $i \in I$ and $x, y \in X_i$,

$$\rho_-(d(x,y)) \leq d(F_i(x), F_i(y)) \leq \rho_+(d(x,y)).$$

- impose conditions on asymptotic geometry - boundaries
- into linear spaces is strong - for f.g. gps, implies Novikov
- into Hilbert spaces related to amenability, exactness, Gromov's a-T-menability/Haagerup, finite asymptotic dimension, coarse Baum-Connes, Novikov, ...

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Theorem (Gromov)

If a graph has a Ponzi flow, it is not amenable.

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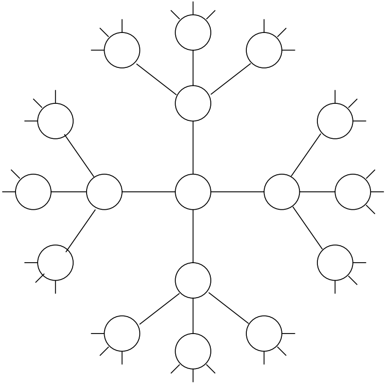

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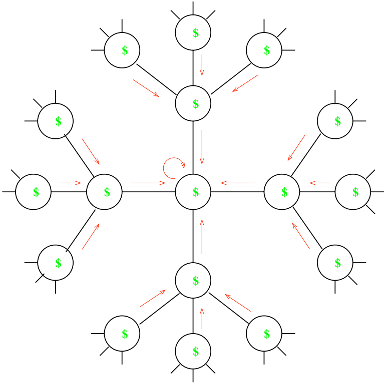

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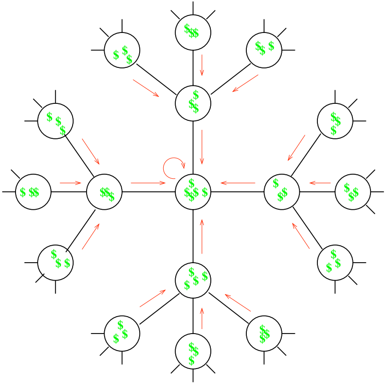

Getting the most out of your telephone network

- A classical expander (Bassalygo, Pinsker) is a family of *sparse*, *highly connected* graphs.
- Usually defined in terms of uniformly bounded isoperimetric number (Cheeger number, spectral gap)

$$h(G) := \min_{1 \leq |S| \leq \frac{|G|}{2}} \frac{|\partial S|}{|S|}$$

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One of the Definitions

Definition (Jerrum, Sinclair)

A sequence of finite connected graphs (X_n) is a *classical expander* if:

1. there's a uniform bound k on the degrees of vertices,
2. $|X_n| \to \infty$,
3. $\exists C > 0$ such that, for all n and all $f_n \in \ell^2(X_n)$,

$$\frac{1}{|X_n|^2} \sum_{x,y \in X_n} |f(x) - f(y)|^2 \leq \frac{C}{|X_n|} \sum_{\substack{x,y \in X_n \\ x,y \text{ adjacent}}} |f(x) - f(y)|^2.$$

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(Good) Examples

- (Used to be?) hard to come by.
- The 8-regular graphs G_m, $m \in \mathbb{N}$, with vertex set $\mathbb{Z}_m \times \mathbb{Z}_m$ and edge set where the neighbors of (x, y) are: $(x \pm y, y), (x \pm y + 1), (x, y \pm x), (x, y \pm x + 1)$. (Margulis; uses representation theory, harmonic analysis)
- The 3-regular graphs G'_p, p prime, with vertex set $\mathbb{Z}_p$ and edge set $\{(x, x \pm 1), (x, x^{-1}) | x \in \mathbb{Z}_p\}$ (uses Selberg $3/16$ from Number Theory)

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Expanders and Coarse Embeddings

- Relationship between expanders and Folner sequence/Ponzi flow/etc.?

Theorem (Gromov)

Let $\mathcal{H}$ denote a separable infinite-dimensional Hilbert space. A classical expander (X_n) does not coarsely embed into $\mathcal{H}$.

Proof: Consider $F_n : X_n \to \mathcal{H}$ with $|F_n(x) - F_n(y)| \leq \rho_+(d(x,y))$. WLOG, $\rho_+(t) = t$. By definition, $\exists C > 0$:

$$\frac{1}{|X_n|^2} \sum_{x,y \in X_n} |F(x) - F(y)|^2 \leq \frac{C}{|X_n|} \sum_{\substack{x,y \in X_n \\ d(x,y)=1}} |F(x) - F(y)|^2 \leq kC.$$

That $|X_n| \to \infty$ but X_n has degree at most k means points arbitrarily far apart are mapped withing kC of each other. No ρ_- exists.

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Preliminaries

- For $r > 0$, metric space X, the *r-diagonal complement* is
$$\Omega_r(X) := \{(x,y) \in X^2 | d(x,y) \geq r\}.$$
- Let $(\mathcal{X}, \mu)$ be a sequence X_n with measures μ_n defined on X_n^2, such that:
  - μ_n either zero or probability measure,
  - μ_n a probability measure infinitely often,
  - There is $r_n > 0$ such that $r_n \to \infty$, μ_n supported on $\Omega_{r_n}(X_n)$.

  This is a *measured family*.
- Measured family $(\mathcal{X}, \mu)$ has *Poincaré inequality* for metric space Z if, for all 1-Lipschitz map $f : \mathcal{X} \to Z$:
$$\mathbf{Var}_{\mu_n}(f) := \sum_{(a,b)\in X_n^2} d(f(a), f(b))^2 \mu_n(a,b) \leq K^2.$$

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Definition

Definition (Ostrovskii, Tessera)

For any family of metric spaces $\mathcal{C}$, $(\mathcal{X}, \mu)$ is a $\mathcal{C}$*-expander* if $(\mathcal{X}, \mu)$ has a Poincaré inequality with uniform constant K for every $Z \in \mathcal{C}$. We say $(\mathcal{X}, \mu)$ is a *generalized expander* if $(\mathcal{X}, \mu)$ is a generalized $\mathcal{C}$-expander for some family $\mathcal{C}$.

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Results

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A classical expander is a $\{\mathcal{H}\}$-expander.

Theorem (Tessera, (Ostrovskii))

Let $\mathcal{C}$ be a class of metric spaces satisfying some conditions*. A $\mathcal{C}$-expander does not coarsely embed into any $Z \in \mathcal{C}$.

A metric space X does not coarsely embed into any element $Z \in \mathcal{C}$ if and only if some $\mathcal{C}$-expander coarsely embeds into X.

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Theorem (Tessera, (Ostrovskii))

Let $\mathcal{C}$ be a class of metric spaces satisfying some conditions*. A $\mathcal{C}$-expander does not coarsely embed into any $Z \in \mathcal{C}$.

A metric space X does not coarsely embed into any element $Z \in \mathcal{C}$ if and only if some $\mathcal{C}$-expander coarsely embeds into X.

*: associated sheaf is *p-admissible*. Satisfied by: separable Hilbert, L^p, $CAT(0)$, . . .

Motivation

For $\{X_n, \mu_n\}$, define:

$$\alpha_n := \frac{1}{|X_n|^2 \min_{(a,b)\in supp(\mu_n)} (\mu_n(a,b))}$$

and

$$\beta_n := \left(r_n^2 \frac{|X_n|^2 - |\Omega_{r_n}(X_n)|}{|X_n|^2} \right).$$

Theorem

For ν_n the uniform measure on X_n^2, Z a space where $\mathcal{X}$ has a Poincaré inequality, and 1-Lipschitz $F_n : X_n \to Z$,

$$\textbf{Var}_{\nu_n}(F_n) \leq \alpha_n K^2 + \beta_n.$$

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Proof

Split up $\mathbf{Var}_{\nu_n}(F_n)$ over Ω_n and Ω_n^c:

$$\begin{aligned}
\mathbf{Var}_{\nu_n}(F_n) &= \sum_{(x,y)\in X_n^2} |F(x)-F(y)|^2 \nu_n(x,y) \\
&= \frac{1}{|X_n|^2} \sum_{(x,y)\in\Omega_n} |F(x)-F(y)|^2 + \frac{1}{|X_n|^2} \sum_{(x,y)\in\Omega_n^c} |F(x)-F(y)|^2 \\
&\leq \alpha_n \sum_{(x,y)\in\Omega_n} |F(x)-F(y)|^2 (\mu_n(x,y)) + \frac{1}{|X_n|^2} \sum_{(x,y)\in\Omega_n^c} r_n^2 \\
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A generalized expander is *bounded* if $\alpha, \beta < \infty$.

- All classical expanders are bounded ($\alpha_n, \beta_n \to 0$)
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Coarse Embeddings

Lemma

If $\mathcal{X}$ is bounded and has a Poincaré inequality for Z, there exists k and k' so for any n and 1-Lipschitz $F_n : X_n \to Z$,

$$\mathbf{Var}_{\nu_n}(F_n) \leq \alpha K^2 + \beta + k' = k.$$

Theorem

Let $\mathcal{M}$ be a Hadamard manifold, and let $\mathcal{X}$ be a bounded Hilbert-expander. Then $\mathcal{X}$ does not coarsely embed into $\mathcal{M}$.

Proof similar to standard argument for classical expanders.

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Modifications

Play with definition.

- make $supp(\mu_n) = \Omega$?
- assume coarsely connected?
- assume *nested*?
  - All obstruction theorems still hold

Analyze "Moduli space" of expanders

- (Good) examples of non-classical generalized expanders?
- Examples of non-bounded classical expanders?
- Good notion of equivalence of expanders?
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Boundaries

- What can be said about "boundaries" of expanders?
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  - "Claim": So do bounded generalized expanders
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