BRAID GROUPS ON GRAPHS

Dissertation Defense
for

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Configuration Spaces

Let X be an arbitrary space. A *configuration* of n strands on X is an unordered n-tuple of n distinct points in X. The *configuration space* of n points on X, UC^nX, is the set of all possible n-strand configurations.

$$UC^nX := (\Pi^n X \backslash Diag)/S_n$$

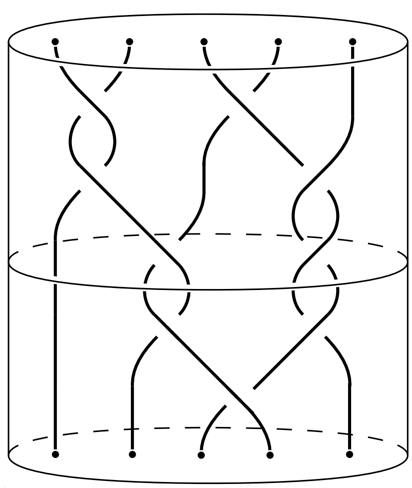
A loop in a configuration space corresponds to identical objects running around on X without colliding and returning to the initial configuration. An equivalence class of loops is a *braid*.

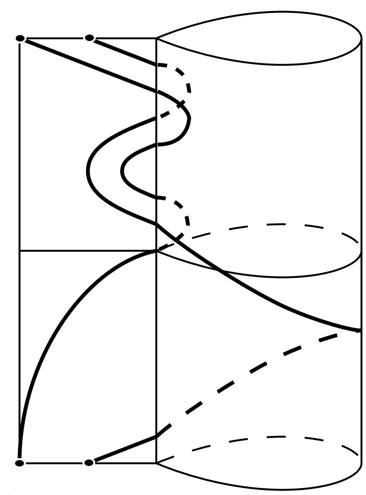
$$B_nX := \pi_1 UC^nX$$

Classical and Graph Braid Groups

A classical Artin braid group is on the unit disc:

$$B_n = \pi_1 UC^n D^2$$





A graph braid group is on a graph:

$$B_n\Gamma = \pi_1 UC^n \Gamma$$

Graph braid groups can model motions of robots on factory floors.

Discretizing Configuration Spaces

A graph Γ has a CW-complex structure, so $\Pi^n\Gamma$ has a product CW-complex structure.

An open k-cell is an ordered collection of k edges and $n-k$ vertices of Γ.

Define

$$Diag' = \{c \in \Pi^n\Gamma | c \cap Diag \neq \emptyset\}.$$

Definition. The *unlabelled discretized configuration space* $U\mathcal{D}^n\Gamma$ of n strands on Γ is the quotient

$$U\mathcal{D}^n\Gamma := (\Pi^n\Gamma \backslash Diag')/S_n.$$

Under most circumstances, the unlabelled configuration space of Γ is homotopy equivalent to $U\mathcal{D}^n\Gamma$. Specifically:

Theorem (Abrams). *For any $n > 1$ and any finite graph Γ with at least n vertices, the unlabelled configuration space of n points on Γ strong deformation retracts onto $UD^n\Gamma$ if*

1. *each path between distinct vertices of degree not equal to 2 passes through at least $n-1$ edges; and*

2. *each path from a vertex to itself that cannot be shrunk to a point in Γ passes through at least $n+1$ edges.*

Such a graph is *sufficiently subdivided* for n. We always assume sufficient subdivision.

Some Known Results

1. (Ghrist) $UC^n\Gamma$ is a $K(B_n\Gamma, 1)$

2. (Ghrist) $UC^n\Gamma \stackrel{sdr}{\to} X$ where $\dim X \leq \#\{v \in \Gamma | deg(v) \geq 3\}$

3. (Abrams) $UD^n\Gamma$ is is locally CAT(0), so graph braid groups have solvable word and conjugacy problems

4. (Crisp, Wiest) Graph braid groups embed nicely in right-angled Artin groups, and so are linear, biorderable, residually finite, and residually nilpotent

Discrete Gradient Vector Fields

Let X be a finite regular CW complex, with K_i the set of open i-cells. A *discrete vector field* W on X is a sequence of partial functions $W_i : K_i \to K_{i+1}$ such that:

1. Each W_i is injective;

2. if $W_i(\sigma) = \tau$, then $\sigma < \tau$;

3. $\text{im } (W_i) \cap \text{dom } \left(W_{i+1}\right) = \emptyset$.

A W-*path* is a sequence of p-cells $\sigma_0, \sigma_1, \ldots, \sigma_r$ such that $\sigma_{i+1} \neq \sigma_i$ and $\sigma_{i+1} < W(\sigma_i)$.

Definition. A *discrete gradient vector field* is a discrete vector field W such that W has no non-stationary closed W-paths.

A cell $\sigma \in$ im W is *collapsible*. A cell $\sigma \in$ dom W is *redundant*. A cell $\sigma \in X \backslash$(im $W \cup$ dom W) is *critical*.

Theorem (F, S; cf. Brown; cf. Forman). *Let X be a finite regular connected CW complex with a discrete gradient vector field W. Then:*

$$\pi_1(X) \cong \langle \Sigma \mid \mathcal{R} \rangle,$$

where Σ is the set of positive critical 1*-cells that aren't contained in T, and*

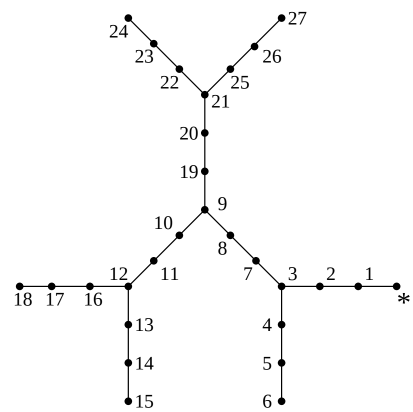
$$\mathcal{R} = \{r(w) | w \text{ a boundary of a crit. 2-cell}\}.$$

If there is a unique critical 0-cell (and there always will be), this presentation is unique given W and is called the *Morse presentation*.

Graphs

Let Γ be a finite simplicial graph. If $v \in V\Gamma$, the degree of v is $d(v)$. If $d(v) \geq 3$, v is *essential*. Let T be a maximal tree.

Pick $* \in V\Gamma$ with $d(*) = 1$. Embed T in $\mathbb{R}^2$. Define $\leq$ on $VT = V\Gamma$ by numbering VT with a clockwise traversal of T from $*$.



Number edges adjacent to v from 0 in order of traversal. Each edge numbering is a *direction* from v. For $v \neq *$, the edge 0 from v is $e(v)$.

If $e \in E\Gamma$, the endpoints of e are $\iota(e)$ and $\tau(e)$, where $\iota(e) > \tau(e)$. If $e \in \Gamma \backslash T$, e is a *deleted edge*.

Defining W for a Graph

Let $c \in UD^n\Gamma$. If $v \in c$, v is *blocked* in c if $v = *$ or $e(v) \cap c_i \neq \emptyset$ for some i. Otherwise, v is unblocked and $(c - \{v\} + \{e(v)\})$ is a *reduction*.

If $e \in c$, e is *not order-respecting* if e is a deleted edge or v is adjacent to $\tau(e)$ and $\tau(e) < v < \iota(e)$ for some $v \in c$.

Definition (F,S). For c a cell of $UD^n\Gamma$,

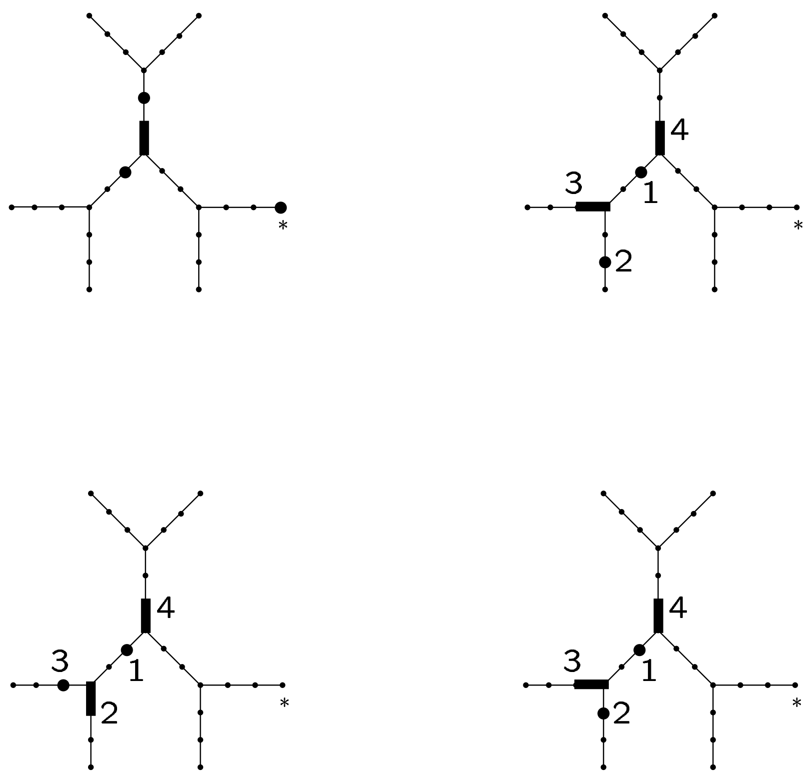
1. Traversing T from $*$, if an unblocked vertex is hit before an order-respecting edge, $W(c) = (c - \{v\} + \{e(v)\})$; c is redundant.

2. If an order-respecting edge is first, $c \in \text{im } W$; c is collapsible.

3. Otherwise, $W(c)$ is undefined; c is critical.

Theorem (F,S). *W is a discrete gradient vector field on $UD^n\Gamma$.*

Examples: The tree T_{min}



Observations

Theorem (F, S). *Let* Γ, T, W, *be as before. Let* D *be the number of deleted edges. With minor assumptions, the Morse presentation for* $B_n\Gamma$ *has*

$$D + \sum_{\substack{v \in V(T) \\ essential}} \sum_{i=2}^{d(v)-1} \left[\binom{n+d(v)-2}{n-1} - \binom{n+d(v)-i-1}{n-1} \right]$$

generators.

Corollary (Ghrist). *If* Γ *is a* radial *tree - i.e. has exactly one essential vertex,* v *- then* $B_n\Gamma$ *is free of rank*

$$\sum_{i=2}^{d(v)-1} \left[\binom{n+d(v)-2}{n-1} - \binom{n+d(v)-i-1}{n-1} \right].$$

Theorem (F,S). *Let* Γ *be a tree and* c *a critical cell of* $UD^n\Gamma$. *Let*

$$k := \min\left\{ \left\lfloor \frac{n}{2} \right\rfloor, \#\{v \in \Gamma^0 \mid v \text{ is essential}\} \right\}.$$

Then $\dim c \leq k$. *In particular,* $UD^n\Gamma \stackrel{sdr}{\to} X$ *where* $\dim X \leq k$.

Theorem (F, S; cf. Mautner). *Let*

$$k := \min\left\{ \left\lfloor \frac{n+1-\chi(\Gamma)}{2} \right\rfloor, \#\{v \in \Gamma^0 \mid v \text{ is essential}\} \right\}.$$

$UD^n\Gamma \stackrel{sdr}{\to} X$ *where* $\dim X \leq k$.

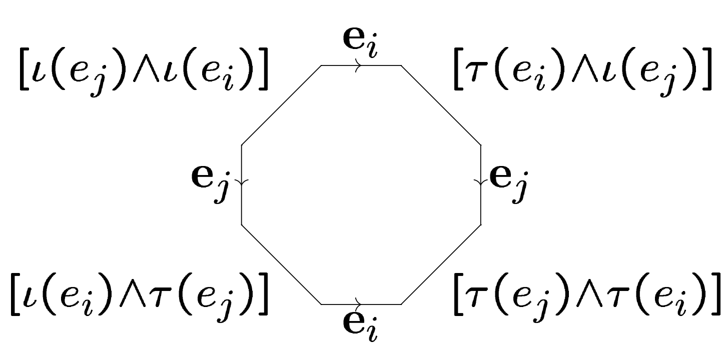
Presentations

Theorem (F,S). *For Γ a tree, $B_n\Gamma$ has a Morse presentation where generators are critical* 1-*cells and relators are reduced forms $w(c)$ of boundaries of critical* 2-*cells c, which are explicitly given....*

A *corner* is an essential vertex A and $0 < i < j < d(A)$.

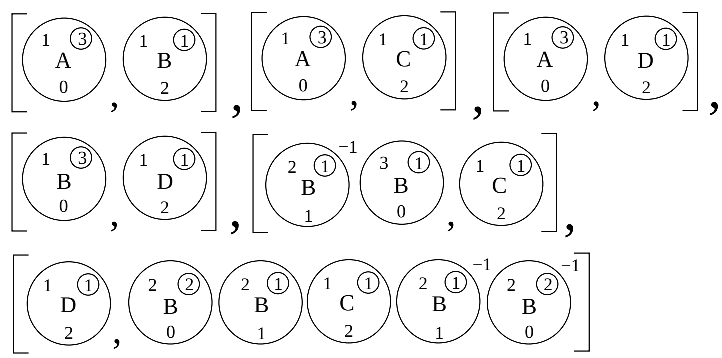
Corners $\leftrightarrow$ critical 1-cells at essential vertices.

Theorem (F,S). *For any Γ, $B_2\Gamma$ has a Morse presentation P, where the generators of P consist of all corners plus all deleted edges, and the relations of P are*

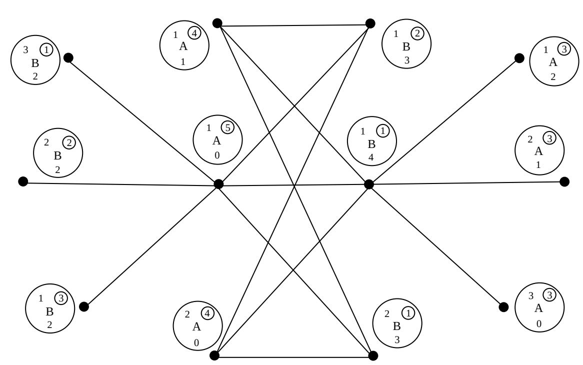


Examples

- B_4T_{min} has 24 generators and relators:



- B_6T_H, where T_H is the H graph, is F_{18} free product the RAAG on:



- For Γ the 4-spoked wheel, $B_2\Gamma \cong F_5$ (after rewriting to eliminate existing 6 relators).

Right-Angled Artin Groups

Definition. Let Δ be a finite simple graph. The *right-angled Artin group* associated to Δ is $G(\Delta)$ generated by $V\Delta$, with relations commutators corresponding to edges in Δ.

Theorem (Droms). *Two graphs Δ and Δ' are graph isomorphic if and only if the groups $G(\Delta)$ and $G(\Delta')$ are group isomorphic.*

Theorem (Crisp, Wiest). *For any finite graph Γ and any n, there exists a graph Δ such that $B_n\Gamma$ embeds in $G(\Delta)$.*

Theorem (S). *For every finite graph Δ and any coloring C of Δ with n colors, there exists a graph Γ such that the right-angled Artin group $G(\Delta)$ embeds into the graph braid group $B_n\Gamma$.*

Δ-halos

Definition (Halo). A connected graph Γ is a Δ-*halo* if for each $a_i \in V\Delta$, there is a simple edge loop γ_i in Γ (an *Artin loop*), such that:

- for each color $c \in \{1, \ldots, n\}$, there exists $x_c \in V\Gamma$ with

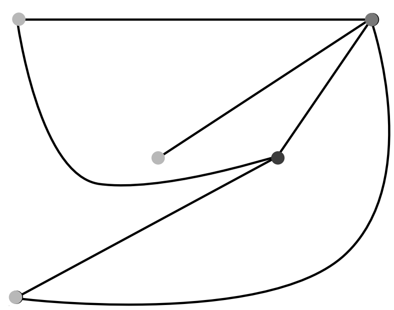
$$x_c = \bigcap_{iC(a_i)=c} \gamma_i \cap \bigcap_{iC(a_i)\neq c} (\Gamma \backslash \gamma_i).$$

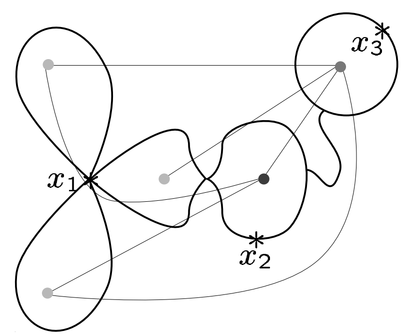
- if $(a_i, a_j) \notin E(\Delta)$, then $\gamma_i \cap \gamma_j = v$ for some vertex v. If $C(a_i) = C(a_j)$ then $v = x_{C(a_i)}$; otherwise, v is on no other Artin loop.

- if $(a_i, a_j) \in E(\Delta)$, $\gamma_i \cap \gamma_j = \emptyset$.

$\{x_1, \ldots, x_n\} \in UC^n\Gamma$ is the *Artin basepoint*.

Example and Sketch of Proof





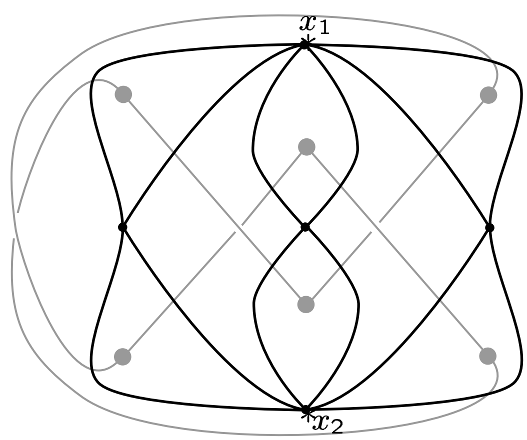
Sketch of proof of theorem: Take the element corresponding to $a_i \in V\Delta$ to γ_i^2.

Compose the induced map with Crisp and Wiest's embedding back into right-angled Artin groups. Show the composition is injective using HNN structure on right-angled Artin groups and Britton's Lemma.

Corollaries

Corollary (S). *Let n_0 be the chromatic number for Δ. Then there exists an embedding of $G(\Delta)$ into $B_{n_0}\Gamma$ for some Δ-halo Γ.*

Example. Let $A = G(C_6)$. A C_6-halo for the 2-coloring is:



Theorem (S). *There exists a* planar *graph braid group which contains a hyperbolic surface subgroup, with only $n = 2$ strands.*

Definitions for Cohomology on Trees

Define an equivalence relation $\sim$ on cells of $U\mathcal{D}^nT$: $c \sim c'$ if

1. $E(c) = E(c')$, and

2. for any connected component C of $T - \bigcup_{e \in E(c)} e$,

$$|C \cap (c \backslash E(c))| = \left| C \cap (c' \backslash E(c')) \right| .$$

Define a partial order $\leq$ on classes: $[c] \leq [c_1]$ if there exists $\hat{c} \in [c]$, $\hat{c}_1 \in [c_1]$ such that $\hat{c} \leq \hat{c}_1$.

Define the cellular cocycle $\phi_{[c]}$ to be:

$$\begin{aligned} \phi_{[c]}(\tilde{c}) = 1 \qquad & if\ \tilde{c} \sim c \\ \phi_{[c]}(\tilde{c}) = 0 \qquad & otherwise\ . \end{aligned}$$

Homology and Cohomology on Trees

Theorem (F). $H_i(U\mathcal{D}^nT;\mathbb{Z}/2\mathbb{Z})$ *is free abelian with basis* $\{c|c$ *a critical* i*-cell*$\}$.

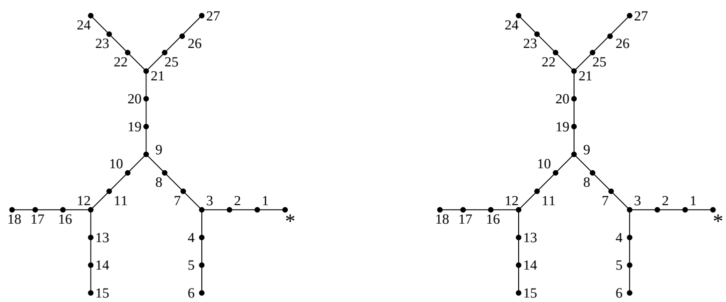
Theorem (F,S).

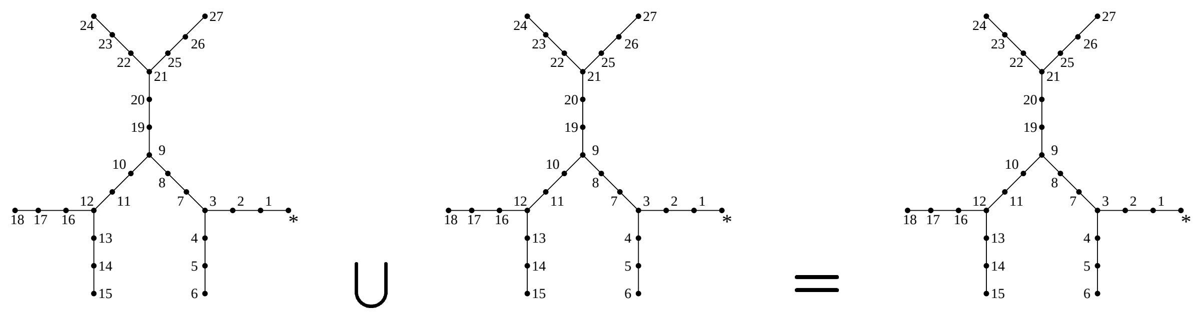
1. $H^i(U\mathcal{D}^nT;\mathbb{Z}/2\mathbb{Z})$ *is free abelian with basis* $\{c^*|c$ *a critical* i*-cell*$\}$.

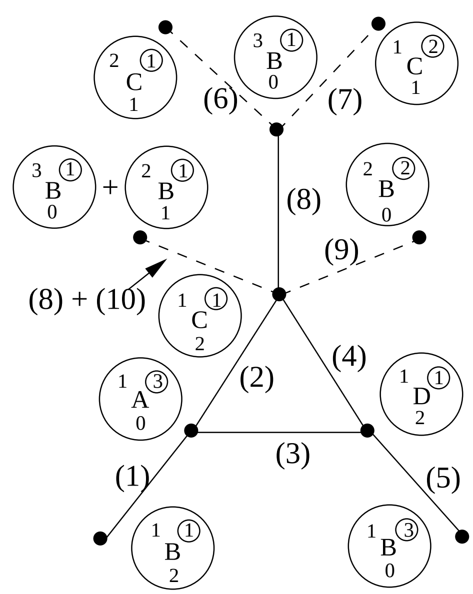
2. *For* c *critical, if* $[c]$ *is the l.u.b. of distinct classes* $[c_1],\ldots,[c_i]$ *of* 1*-cells, then, WLOG,* $c_1,\ldots,c_i$ *are critical and*
$$c_1^* \cup \ldots \cup c_i^* = c^*.$$

3. *If* $[c_1],\ldots,[c_i]$ *are distinct classes of* 1*-cells having l.u.b.* $[c]$*, then*
$$\left[\phi_{[c_1]}\right] \cup \ldots \cup \left[\phi_{[c_i]}\right] = \left[\phi_{[c]}\right].$$
Otherwise,
$$\left[\phi_{[c_1]}\right] \cup \ldots \cup \left[\phi_{[c_i]}\right] = 0.$$

Example: T_{min}







Ghrist's Conjecture

Conjecture (Ghrist; Abrams). Ghrist's Conjecture. *The (pure) braid group of any planar graph is a right-angled Artin group.*

Theorem (Abrams). B_2K_5 *and* $B_2K_{3,3}$ *are hyperbolic surface groups, and are therefore not right-angled Artin.*

Theorem (Connolly, Doig). *Braid groups on* linear trees *are right-angled Artin.*

- There are known examples of tree braid groups which are not right-angled Artin (Mautner).

Exterior Face Algebras

Definition. Let K be a finite simplicial complex with vertices $\{v_1, \ldots, v_n\}$ The $(\mathbb{Z}/2\mathbb{Z})$ *exterior face algebra* $\Lambda(K)$ on K has:

- basis $\{v_{i_1} \ldots v_{i_j} | i_1 < \cdots < i_j\}$

- relations

$$\begin{array}{ll} v_i v_j = v_j v_i & \\ v_i^2 = 0 & \\ v_{i_1} \ldots v_{i_k} = 0 & \text{when } i_1 < \cdots < i_k \text{ and} \\ & \{v_{i_1}, \ldots, v_{i_k}\} \notin K. \end{array}$$

Theorem (Gubeladze). $\Lambda(K) \cong \Lambda(K')$ *if and only if* $K \cong K'$.

Counterexamples to Ghrist's Conjecture

Theorem (F,S). *Let T be a finite tree. The tree braid group B_nT is a right-angled Artin group if and only if T is linear or $n < 4$.*

Proof relies on:

- Cohomology for any RAAG is an exterior face algebra over a flag complex.
- $H^*(T_{min})$ is an exterior face algebra, but is not over a flag complex.
- Embedding T_{min} into a nonlinear tree induces nice maps on cohomology.

Cohomology for B_4T

Theorem (S). *Let T be a tree. For $n = 4$, $H^*(B_4T; \mathbb{Z}/2\mathbb{Z})$ is an exterior face algebra. The simplicial complex Δ defining the exterior face algebra structure is unique and 1-dimensional.*

Idea of proof: usually, if c_1 and c_2 are critical, then

$$c_1^* \cup c_2^* = c^*$$

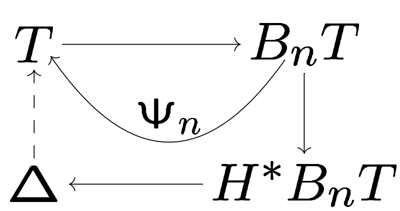
where c is critical.

Take care of the 'usually': rewrite Morse basis so exceptional cases no longer cup with anything, and leave everything else alone.

Rigidity for B_4T

Theorem (S). *Two finite trees are homeomorphic if and only if their* 4 *strand braid groups are isomorphic.*

Idea of proof: Reconstruct T from Δ using *neighborhood hierarchies*.



Neighborhood Hierarchies

Let $N_v := \{v' \in \Delta | (v, v') \in EV\}$ be the *neighborhood* of v. Subset inclusion defines a preorder: $v \leq_N v' \leftrightarrow 0 \subsetneq N_v \subseteq N_{v'}$.

For any $[v_0]$, form a $\leq_N$-hierarchy H_{v_0}: a graph with vertices $[v]$ for $[v] \leq_N [v_0]$ and edges for largest subsets.

The graph H_{v_0} is a tree, a subtree of T, and just needs more leaves.

The Isomorphism Problem for B_4T

Theorem (S). *There exists an algorithm which, given two four strand tree braid groups on finite trees, decides whether the two groups are isomorphic.*

Also true for $n = 5$ or $n \geq 4$?