Geodesic Growth of the Braid Group on Three Strands and a Related Group

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Definitions

Let G be a group having generating set S and Cayley graph $\Gamma = \Gamma_{G,S}$.

- Geodesic Growth Series (GGS):

$$\mathcal{G}_{(G,S)}(x) = \sum_{n=0}^{\infty} a_n x^n$$

$a_n :=$ # geodesic words of length n in S^*.

- Spherical Growth Series (SGS):

$$\mathcal{S}_{(G,S)}(x) = \sum_{n=0}^{\infty} b_n x^n$$

$b_n :=$ # el'ts of G at d_S-distance n from 1.

- Rationality: A power series is rational if it may be expressed in the form $p(x)/q(x)$, where $p(x), q(x) \in \mathbb{Z}[[x]]$.

For which groups are these rational?

Proof Methods

Growth series of a regular language is rational.

- SGS:

  - Show set of length-minimal normal forms are a regular language

- GGS:

  - Show Falsification by Fellow Traveller

  - Show geodesics are a regular language, i.e. show finitely many cone types

    - *depends on generating set* (Neumann, Shapiro; Cannon; Stoll), suggesting GGS does as well

- Steps for GGS: Cayley Graph, Geodesics, FSA, Adjacency Matrix

Where Regularity is Known

Regularity of the set of geodesics:

1. hyperbolic groups (Cannon; Neumann, Shapiro; ECHLPT, attributed to Gromov and others)

2. Right-angled Artin and Coxeter Groups (Loeffler-Meier-Worthington)

3. Irreducible Affine Coxeter Groups (Noskov)

B_3 and G_3

- B_3: $< a, b | aba = bab >$
  - Tits' conjecture: $< a^2, b^2 > \cong F_2$ (Droms, Lewin, Servatius)
- G_3: $< a, b | aba = bab, a^2 b^2 = b^2 a^2 >$
  - Normal subgroup: $< a^2, b^2, ab^2 a^{-1} > \cong \mathbb{Z}^3$
  - Quotient: S_3
- Thm(S): The set of geodesics for both B_3 and G_3 are regular.
  - Griffing, Patlovany, Talley; McCammond
  - Charney, Meier

The Cayley Graph of G_3

Making the Cayley Graph of B_3

The Cayley Graph of B_3

Geodesics of G_3

- Since G_3 is virtually abelian (see $\mathbb{Z}^3$ in Cayley graph), intuitively geodesics are paths which always go away from the origin.

- This is mostly correct - for a given direction (x, y, or z), a geodesic must consistently go either in the positive direction or the negative direction.

- Exception: $ab^{2n}a^{-1}$.

FSA for G_3

Geodesics of B_3

- Consequence of $(aba)a = b(aba)$

- * condition: $w \in B_3$ does not contain as subwords elements of both *+ $:= \{ab, ba\}$ and *- $:= \{(ab)^{-1}, (ba)^{-1}\}$.

- ** condition: $w \in B_3$:
  - w does not contain as subwords both (aba) and either a^{-1} or b^{-1}, and
  - w does not contain as subwords both $(aba)^{-1}$ and either a or b.

FSA for B_3

Growth Series

- Thm(S) - The GGS for:
  - B_3 is $\frac{z^4+3z^3+z+1}{(z^2+z-1)(z^2+2z-1)}$
  - G_3 is $-\frac{32z^7-16z^6-20z^5+10z^4+6z^3+z^2+2z+1}{(2z^3-z^2-2z+1)(2z^2-1)}$

  (computed using Maple)

- For comparison...
  Thm(Brazil, KBMAG) - The SGS for:
  - B_3 is $\frac{2z^4+z^3-1}{(2z^3+z^2-3z+1)(z-1)}$