Research Statement

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I am a geometric group theorist. Geometric group theory is a highly interdisciplinary field focusing on the study of groups via their actions on geometric spaces. Geometric group theory uses the tools and approaches of algebraic topology, commutative algebra, semigroup theory, hyperbolic geometry, geometric analysis, combinatorics, computational group theory, computational complexity theory, logic, dynamical systems, probability theory, and other areas. It is a young and fast-growing field, with much of the work in the area accomplished within the past 30 years.

My work has embraced the interdisciplinary nature of my field – I have published theorems which could be classified in each of: group theory, commutative algebra, algebraic topology, combinatorics, coding theory, mathematical robotics, computational complexity theory, and differential geometry. For example, I have used tools as diverse as: exterior face algebras and Stanley-Reisner rings, differential forms, Fox calculus, cohomology rings, discrete Morse theory, face polynomials of simplicial complexes, linear codes, configuration spaces, fundamental groups, and coarse curvature conditions. My work has appeared or been accepted in top journals in a number of fields, including: the International Journal of Algebra and Computation; the Journal of Combinatorial Theory Series A; the Journal of Pure and Applied Algebra; and Algebraic and Geometric Topology.

Graph Braid Groups

A *braid group on a graph* with n strands is the fundamental group of the space of all configurations of n distinct points on a graph. Graph braid groups are perhaps best known for their interpretation as the motions of robots on a 1-dimensional track, and have applications in robotics.

My work on graph braid groups uses many ingredients, including: exterior face algebras, right-angled Artin groups, the language of differential forms, configuration spaces, and Forman's discrete Morse theory. In a series of five papers, three of which are joint work with Daniel Farley, I proved a number of results on graph braid groups.

Exterior face algebras and differential forms

In my deepest work to date, in [24] (which appeared in *Groups, Geometry, and Dynamics*), I used commutative algebra and the language of differential forms to prove that the cohomology algebra of certain tree braid groups is in fact an *exterior face algebra*:

Theorem 1 [24]: *Let T be a finite tree. For $n = 4$ or 5, $H^*(B_nT; \mathbb{Z}/2\mathbb{Z})$ is an exterior face algebra $\Lambda(\Delta)$. The simplicial complex Δ is unique and 1-dimensional.*

Using this result, I proved that one may reconstruct from a tree braid group on 4 or 5 strands the the unique underlying tree, yielding a solution to the isomorphism problem for these groups.

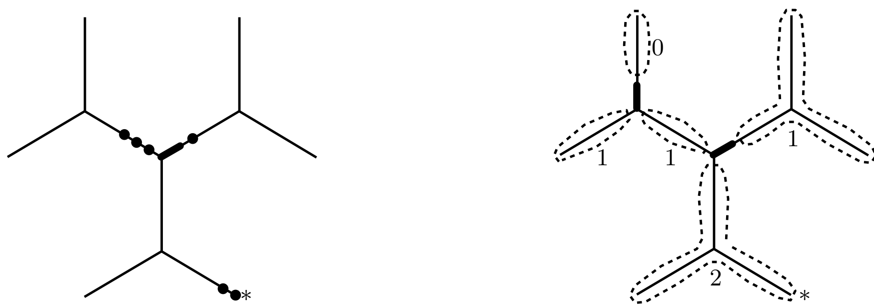
Cohomology

My work in [24] was motivated by my work with Daniel Farley in [13]. Farley [10, 11] computed presentations for both homology and cohomology rings of braid groups on trees. Then, jointly in [13], published in the

Journal of Pure and Applied Algebra, Farley and I were able to completely describe the cup product structure for these braid groups on trees.

Farley and I use this description to show that 'most' tree braid groups are not right-angled Artin groups. A *right-angled Artin group* (or RAAG) is a group with a finite presentation where each relation is a commutator of two generators. Right-angled Artin groups are often called *graph groups* because they can be encoded by a *defining graph*, whose vertices are generators and whose edges correspond to a commutation relation between the associated generators. This answered negatively a conjecture of Ghrist [16] that all graph braid groups are RAAGs.

Theorem 2 [13]: *The 'nice' Morse matchings of Theorem 3 in the case of tree braid groups yield a 'visual' interpretation of the cup product of two cohomology classes.*



(a) (b)

Figure 1: Figure 1(a) shows an example critical 1-cell as described in [12]. Each thick vertex and edge represents a strand. Figure 1(b) shows a 'cloud diagram' representing a cohomology class. There is a 1-cohomology class such that its cup product with the dual of the critical 1-cell in Figure 1(a) is the class in Figure 1(b).

Discrete Morse theory and presentations

My work on graph braid groups began in a paper with Farley [12], published in *Algebraic and Geometric Topology*, in which we applied Forman's discrete Morse theory [15] to graph braid groups. Discrete Morse theory is a method of simplifying a simplicial complex within its homotopy type by 'flowing' cells to a (typically much smaller) subset of *critical cells*. Farley and I found a *Morse flow* on a discretized version of the configuration space of points on a graph. Farley and I used this Morse flow to compute presentations of braid groups on trees. Figure 1(a) depicts pictorially one of the generators we describe. In a subsequent paper [14], published in *Forum Mathematicum*, we extended our results to presentations of braid groups on arbitrary graphs.

Theorem 3 [12, 14]: *Let Γ be any finite graph. There are many 'nice' Morse matchings on the discretized configuration space $U\mathcal{D}^n\Gamma$. Each Morse matching gives rise to a 'visual' presentation of the fundamental group of Γ, where generators are critical 1-cells and relations correspond to critical 2-cells.*

The discretized version of the configuration space we use was introduced by Abrams [1], who proved that this discretized version is usually homotopy equivalent to the original configuration space. More recently, I supervised two undergraduate students who have submitted a paper [21] characterizing exactly what 'usually' means. An additional paper based on this work characterizes which graph braid groups are isomorphic to classical braid groups [27].

Graph braid groups and right-angled Artin groups

Despite the fact that most tree braid groups are not RAAGs, the connection between graph braid groups and RAAGs is still strong: Crisp and Wiest [8] showed that every graph braid group embeds into a RAAG. Subsequently, I showed [23] that every RAAG embeds into a graph braid group (this paper appeared in *Geometriae Dedicata*).

Theorem 4 [23]: *Every right-angled Artin group G embeds into a graph braid group on n strands, where n is the chromatic number of the defining graph of G.*

The proof of this theorem introduces a construction which I call the graph halo. Given an arbitrary RAAG with defining graph Δ, the *graph halo* of Δ is the graph defined as having one simple loop for every vertex of Δ, where two such loops are amalgamated at a point if and only if the corresponding generators of the RAAG group do not commute. This embedding provides examples of hyperbolic surface subgroups embedded into (planar) graph braid groups. There were previously no known examples of hyperbolic surface subgroups in planar graph braid groups.

Theorem 5 [23]: *There exists a two-strand planar graph braid group which contains as a subgroup the fundamental group of a closed hyperbolic surface.*

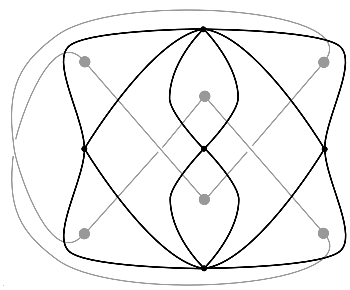


Figure 2: The 6-cycle, C_6, is shown in gray. The halo of C_6 is shown in black. The 2-strand braid group on this halo contains a hyperbolic surface subgroup.

Outer automorphisms of the free group

One key tool to understand the structure of a group is to understand the structure of its group of automorphisms, which often turns out to have a surprisingly rich and deep structure of its own. Three important examples of automorphism groups are: the general linear group for a free abelian group, mapping class group for a surface group, and the outer automorphism group $Out(F_n)$ of a free group F_n. These three classes of objects are deeply related, sharing many results and proof techniques, though each has its own idiosyncrasies. The study of $Out(F_n)$ has seen great activity over the past few decades, but many open questions remain, especially about its geometry.

One of the most useful tools in understanding the geometry of the mapping class group is the *hierarchy* decomposition of geodesics with respect to the Harvey *curve complex* [20]. The curve complex $\mathcal{C}(S)$ of a surface S is the space whose vertices are isotopy classes of simple closed curves on S, and whose simplices correspond to vertices which have representatives that can be realized simultaneously disjointly. This complex carries a natural action of the mapping class group, but is not locally finite. It is however Gromov hyperbolic [19], which is one key ingredient in the hierarchy machinery.

Because of the deep analogy between $Out(F_n)$ and the mapping class group, it is conjectured that there will be an analogue of the curve complex with an action of $Out(F_n)$ which supports a hierarchy-type machinery.

My research has been on studying these potential analogues. In joint work with Dmytro Savchuk [25], I looked at the coarse geometry of what was then considered a potential analogue of the curve complex, the edge splitting graph. The *edge splitting graph* $\mathcal{ES}$ is the graph whose vertices are conjugacy classes of free splittings $\langle x_1, \dots, x_k \rangle * \langle x_{k+1}, \dots, x_n \rangle$ of F_n into two nontrivial free factors, and edges in $\mathcal{ES}$ correspond to common refinement of the associated splittings. The name comes from viewing such splittings as graph of groups decompositions of F_n with a single non-loop edge.

Theorem 6 [25]: *The edge splitting graph is not Gromov hyperbolic.*

The above theorem shows that the edge splitting graph is in fact not the 'correct' curve complex analogue, addressing questions posed in [18] and [2]. Our theorem itself is a consequence of a more general result, whose proof relies on an algebraic characterization of distance in $\mathcal{ES}$ which is based on work of Whitehead:

Theorem 7 [25]: *For $n > 2$, $\mathcal{ES}$ contains a quasiisometrically embedded copy of $\mathbb{R}^m$ for every $m \geq 1$.*

Corollary 8 [25]: *The space $\mathcal{ES}$ has infinite asymptotic dimension. The dimension of every asymptotic cone of $\mathcal{ES}$ is infinite.*

It is not known whether $Out(F_n)$ has finite asymptotic dimension, though the analogy with the general linear group and the mapping class group suggests that it will. To my knowledge, Corollary 8 makes the edge splitting complex the only naturally defined space with these properties and with a natural cocompact group action of a group which is not known to have infinite asymptotic dimension.

Theorem 7 has a further consequence. There is another proposed curve complex analogue called the *factor complex*, and a natural map $\mathcal{ES} \to \mathcal{F}$. We show this map is not a quasi-isometry, and that moreover there can be no coarsely equivariant quasi-isometry between $\mathcal{ES}$ and $\mathcal{F}$. This provides a negative answer to a question of Bestvina and Feighn [3].

Coding Theory

My first article with Joshua Brown Kramer concerns a problem related to cryptology and coding theory. It describes algorithms for determining when certain linear functions over finite vector spaces must be projections. The paper, [6], appears in the *Journal of Combinatorial Theory, Series A*, arguably the most highly respected journal for this type of combinatorics. The result itself is a generalization of the celebrated MacWilliams Extension Theorem. Let $\mathbb{F}$ be a finite field of dimension q. A multiset S of integers is *projection-forcing* if, for every $n, m \in \mathbb{Z}$ and every linear function $\phi : \mathbb{F}^n \to \mathbb{F}^m$ whose multiset of changes in Hamming weight is S, ϕ is a coordinate projection up to permutation and scaling of entries. The MacWilliams Extension Theorem says that $S = \{0, \dots, 0\}$ is projection-forcing for any size S. Brown Kramer and I give a (super-polynomial) algorithm to determine whether or not any given S is projection-forcing, as well as a polynomial algorithm for certain types of S.

Theorem 9 [6]: *Let k be a nonnegative integer, and q a prime power. There exists a matrix $M_{k,q}$ such that a multiset S of size $\frac{q^k-1}{q-1}$ is projection-forcing if and only if for each vector π, a permutation of S, either $M_{k,q}^{-1}\pi$ is nonnegative or it contains a non-integer entry.*

We define a function $\delta_q(S)$ for arbitrary q where S is a multiset of size $\frac{q^k-1}{q-1}$ which is computationally easy to calculate and yields the following result.

Theorem 10 [6]: *If $\delta_q(S) > -q^{k-1}$ then S is projection-forcing.*

Face polynomials and simplicial subdivision

My second combinatorial result, with Emanuele Delucchi and Aaron Pixton, looks at the effect on the face polynomial of subdividing a simplicial complex [9], and is published in *Discrete Mathematics*. Brenti and Welker [5] have shown that, for any simplicial complex X, the roots of the face polynomials $f^{X^{(n)}}(t)$ of the n^{th} barycentric subdivision $X^{(n)}$ of X in fact converge to limit roots, depending only on the dimension of X. We reprove and improve this result:

Theorem 11 [9]: *Let X be a $(d-1)$-dimensional simplicial complex. The $d-1$ largest roots of $f^{X^{(n)}}(t)$ converge to $d-1$ values which are the roots of a polynomial $p_d(t)$, depending only on d, whose coefficients are listed in the last row of the inverse of a particular matrix, P_d. Moreover, for any dimension $(d-1)$, the $d-1$ 'limit roots' are invariant under the map $x \mapsto \frac{-x}{x+1}$.*

This paper involves a combination of algebraic, geometric, and combinatorial techniques. A key ingredient is an algebraic characterization of the effect of barycentric subdivision on face polynomials. Barycentric subdivision induces a function $b\colon \mathbb{Z}[t] \to \mathbb{Z}[t]$. We list the values of b on monomials as coefficients in the formal power series in the variable x over $\mathbb{Z}[t]$, by defining $B\colon \mathbb{Z}[t][[x]] \to \mathbb{Z}[t][[x]]$ to be given by $B(\sum_{k\geq 0} g_k(t)x^k) = \sum_{k\geq 0} b\big(g_k(t)\big)x^k$.

Theorem 12 [9]: *In $\mathbb{Z}[t][[x]]$, barycentric subdivision satisfies the identity*

$$B(e^{tx}) = \frac{1}{1-(e^x-1)t}.$$

Subsets of bases in relatively free groups

While working on the paper [25], Savchuk and I came across an interesting question, whose simplest formulation is as follows. Given a free group $F := F(\{a_1,\dots,a_n\})$ with a fixed basis $A = \{a_1,\dots,a_n\}$, suppose that a subset S of F is primitive in F (i.e., is a part of some basis of F) and is such that its elements expressed in terms of basis A do not involve generators $a_l,\dots,a_n$. Is it always true that S will also be primitive in $\hat{F} := F(\{a_1,\dots,a_{l-1}\})$? In [26], accepted in the *International Journal of Algebra and Computation*, we answered this question and generalized it to several relatively free groups.

Theorem 13 [26]: *Let $S \subset \hat{F}$ be primitive in F mod V, where V is a fully invariant subgroup of F. Let $\hat{V} := V \cap \hat{F}$. Then S is primitive in $\hat{F}$ mod $\hat{V}$ if F/V is free, free nilpotent, or free solvable.*

The proof in the free case is based on a primitivity criterion for subsets of free groups that uses Fox calculus. We are now generalizing this result using versions of Fox calculus in some abelian varieties.

Classical Braid Groups

The classical Artin braid group can be defined as the fundamental group of the space of configurations of a set of distinct points on a disk. There is a very large body of literature on classical braid groups, as they are important in a number of fields, including knot theory, low-dimensional manifold theory, cryptography, and quantum theory. My first published paper [22], published in *Contemporary Mathematics*, concerned the braid group on three strands, B_3. There, I found a characterization for geodesic words with respect to the standard generating set.

Theorem 14 [22]: *A freely reduced word $w \in \{a,b,a^{-1},b^{-1}\}^*$ is a geodesic for $B_3 = \langle a,b | aba = bab\rangle$ if and* *only if w does not contain as subwords any of the following: elements of both $\{ab, ba\}$ and $\{a^{-1}b^{-1}, b^{-1}a^{-1}\}$; both aba and either a^{-1} or b^{-1}; both $a^{-1}b^{-1}a^{-1}$ and either a or b.*

I developed this characterization of geodesics by providing an explicit construction of the Cayley graph of B_3 for the standard generating set. I used this description of geodesics to analyze the behavior of the *geodesic growth series*, the series whose defining sequence is the number of geodesic words of a given length.

Theorem 15 [22]: *The set of geodesic words for B_3 with respect to the standard generating set is regular, and the geodesic growth series is*

$$\mathcal{G}_{(B_3,S)}(x) = \frac{x^4 + 3x^3 + x + 1}{(x^2 + x - 1)(x^2 + 2x - 1)}.$$

Robot Motion Planning

In work motivated by the relationship between graph braid groups and robotics, together with Joshua Brown Kramer, I studied certain robot motion planning algorithms [7]. This paper appeared in the *International Journal of Computational Geometry and Applications*, a top journal for mathematical robotics. We showed that in a certain sense it is mathematically impossible for a robot motion planning algorithm trying to navigate in an unknown environment from one point to another to work efficiently in every situation.

Theorem 16 [7]: *If $n \geq 3$ then every algorithm that solves the Navigation problem has no upper bound on competitiveness with respect to optimal length. The (slightly modified) Navigation problem has a universal lower bound on competitiveness with respect to optimal length l_{opt} given by $\frac{l_{opt}^n}{\kappa^{n-2}(r+\epsilon)}$, where $\kappa = 2\sqrt{2r\epsilon + \epsilon^2}$ and n is the dimension.*

Future directions

There are open questions pertaining to each of the topics I have studied, and I have ideas for approaching many of these questions. For sake of brevity, though, in this statement I will only describe some of my thoughts on open questions and future directions for my research in $Out(F_n)$.

Very recently, it has been shown that there are in fact hyperbolic complexes on which $Out(F_n)$ acts naturally, including the factor complex mentioned above [4] as well as the splitting complex [17]. The *splitting complex* is the complex whose vertices are conjugacy classes of splittings of the free group F_n as either an amalgamated free product of two subgroups or as a free HNN-extension of a subgroup. Edges correspond to two splittings having a common refinement. Even with hyperbolicity known, there are many further questions to answer. I have recently been working (together with Dmytro Savchuk and Matt Clay) on two projects related to understanding the geometry of the splitting complex. The first project involves understanding how vertices 'look' from given fixed vertex in the splitting complex. I have a combinatorial description of viewing one splitting with respect to another which I call a *sphere tree*. These sphere trees have connections to established notions in the study of $Out(F_n)$ (in particular, a very useful construction called the *Guirardel core*). I also have a description of how these sphere trees evolve along what should be geodesic paths in the splitting complex.

One application of my work on sphere trees is to my second project on $Out(F_n)$, for my work on 'projection to the link of a vertex'. Projection of a vertex in the curve complex to the link of another vertex is a key ingredient in the hierarchy machinery for mapping class groups. In part with Dmytro Savchuk and Matt

Clay, I have been working to translate this concept of subsurface projection to the $Out(F_n)$ world. I have a notion of projection to a link which generalizes the projection used in the curve complex. I can prove that my projection map is coarsely well-defined, coarsely restrictable to sub-submanifolds, coarsely surjective, coarsely Lipschitz, and satisfies a metric condition called a *Behrstock inequality*. These properties should be viewed as something like a list of minimum prerequisites for a projection map to be useful. My future research is to understand more about this projection map, including its relationship to a potential hierarchy-type machinery. My current 'goal theorem' should be called the *bounded geodesic image theorem*:

Project 17: *Prove the* Bounded geodesic image theorem*: for any vertex v and any geodesic γ in the splitting complex, if γ does not contain v, then the projection of γ to the link of v has uniformly bounded diameter.*

The bounded geodesic image theorem (properly restated) holds for the curve complex, and is closely related to the hyperbolicity of that complex.

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