1. Rings with involution

An involution on a unital associative ring \( R \) is an order-two ring map \(- : R \rightarrow R^{op}\):

\[
\bar{r} = r \quad \text{and} \quad \bar{r + s} = \bar{r} + \bar{s} \quad \text{and} \quad \bar{rs} = \bar{s} \bar{r}.
\]

In particular, note \( \bar{0} = 0 \) and \( \bar{1} = 1 \), since

\[
\bar{0} = \bar{0 + 0} = \bar{0} - \bar{0} = 0 \\
\bar{1} = \bar{1} = \bar{1} = \bar{1} = 1.
\]

**Example 1.** Below are some frequently occurring rings with involution \((R, -)\).

1. (commutative ring, identity map)
2. (complex numbers \( \mathbb{C} \), complex conjugation: \( \overline{x + iy} = x - iy \))
3. \((n \times n \text{ matrices } M_n(\mathbb{C}), \text{ conjugate transpose } [a_{ij}]^* = [\overline{a_{ji}}]): (AB)^* = B^* A^*\)
4. \(\mathbb{Z}G\sigma = (\text{group ring } \mathbb{Z}G, \text{ geometric involution: } \bar{g} = \sigma(g)g^{-1}), \text{ where the given homomorphism } \sigma : G \rightarrow \{\pm 1\}\) is called an orientation character.

2. Symmetric & quadratic forms

Let \( M \) be a based left \( R \)-module. A sesquilinear form is a bi-additive function

\[
\lambda : M \times M \rightarrow R \quad \text{satisfying} \quad \lambda(rx, sy) = r \lambda(x, y) \bar{s}.
\]

It is \((-1)^k\)-symmetric means \(\lambda(y, x) = (-1)^k \lambda(x, y)\). It is nonsingular means

\[
M \rightarrow M^* := \text{Hom}_R(M, R) : y \mapsto \lambda(-, y)
\]

is an isomorphism with zero torsion, with respect to the dual basis of \( M^* \), in the reduced \( K \)-group \( \tilde{K}_1(R) := K_1(R)/K_1(\mathbb{Z}) \).

**Exercise 2.** Turn the right \( R \)-module structure on \( M^* \) into a left one, using \(-\).

**Exercise 3.** Show \( \varphi : M \rightarrow M^{**}; x \mapsto (f \mapsto \overline{\lambda(f(x))}) \) is an isomorphism \((M \text{ f.g. free}).

A quadratic refinement of \((M, \lambda)\) is a function that is ‘quadratic’ and ‘refinement’:

\[
\mu : M \rightarrow \frac{R}{\{r - (-1)^k \bar{r}\}} \quad \text{such that} \quad \begin{cases} 
\mu(rx) = r \mu(x) \bar{r} \\
\mu(x + y) = \mu(x) + \mu(y) + [\lambda(x, y)] \\
\lambda(x, x) = \mu(x) + (-1)^k \mu(x) \in R.
\end{cases}
\]

**Exercise 4.** \((M, \lambda)\) admits a unique quadratic refinement if \(2\) is a unit in \( R \).

**Example 5.** A hyperbolic form is the triple \( \mathcal{H}(M) = (M \oplus M^*; \left(\begin{smallmatrix} 0 & I \\ (-1)^k I & 0 \end{smallmatrix}\right); \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right))\).

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Example 6 (quadratic forms in classic linear algebra). Let $A$ be an $n \times n$ matrix over $\mathbb{C}$ such that the symmetrization $A + A^*$ is invertible. Take $k = 0$ and $M = \mathbb{C}^n$ and $\lambda(x, y) = x^t(A + A^*)y$ and $\mu(x) = [x^tA\overline{x}] \in \mathbb{C}/\{z - \overline{z}\}$.

3. Definition of $L_{2k}(R)$

A sublagrangian $F$ in a $(-1)^k$-quadratic form $Q = (M, \lambda, \mu)$ is a free submodule $F \subset M$ with a basis that extends to $M$ which is simple-isomorphic to the preferred one such that $\lambda$ and $\mu$ vanish on $F$. It is a lagrangian if it is maximal such object.

Example 7. $\Delta := \{(x, x) \mid x \in M\}$ is a lagrangian in $(M \oplus M, \lambda \oplus -\lambda, \mu \oplus -\mu)$.

Exercise 8. A nonsingular $Q$ admits a lagrangian $F$ if and only if $Q$ is isomorphic to the hyperbolic form $H$. The image of $F^*$ in $M$ is a complementary lagrangian.

The abelian monoid $L_{2k}(R)$ consists of the stable isomorphism classes of (simple) nonsingular $(-1)^k$-quadratic forms over $(R, \cdot)$. Here, the sum operation is $Q + Q' = (M \oplus M', (\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda' \end{smallmatrix}), (\begin{smallmatrix} \mu \\ \mu' \end{smallmatrix}))$, and stably isomorphic means $Q + \mathcal{H}(R^n) \cong Q' + \mathcal{H}(R^n)$.

Exercise 9. It is an abelian group. (Hint: Use the diagonal $\Delta$.)

Note $L_0 = L_4 = L_8 = \ldots$ and $L_2 = L_6 = L_{10} = \ldots$. Write $L_0^*(G^\omega) = L_0^*(\mathbb{Z}[G]^\omega)$.

4. Signature

Any $(+1)$-symmetric invertible matrix over $\mathbb{Z}$ has all eigenvalues real and nonzero. Moreover, by the spectral theorem and taking square roots, this matrix is congruent $(\cong C : C^tAC)$ over $\mathbb{R}$ to a diagonal matrix of $+1$'s (the number of which is the positive inertia $i_+$) and $-1$'s (the number of which is the negative inertia $i_-$).

Theorem 10 (Sylvester’s Law of Inertia, 1852). Congruent matrices over $\mathbb{R}$ have equal inertia $(i_+, i_-)$. So the signature $i_+ - i_-$ is an invariant of congruence classes.

Thus the signature function $\text{sign} : L_0(\mathbb{Z}) \rightarrow \mathbb{Z}$ is a well-defined homomorphism. Because of the existence of the quadratic refinement, all the diagonal entries of the symmetric matrix are even. In fact, sign is injective with image $8\mathbb{Z}$ via Gauss sums.

Remark 11. Signature injects from $L_0(\mathbb{R})$ or $L_0(\mathbb{C})$ onto $4\mathbb{Z}$, from $L_0(\mathbb{H})$ onto $2\mathbb{Z}$.

5. Some computations of $L_{2k}(R)$

Remark 12. An isomorphism $L_2(\mathbb{Z}) \rightarrow L_2(\mathbb{F}_2) \xrightarrow{\text{Art}} \mathbb{Z}/2$ exists; see its minilecture.

Let $G$ be a finite group. Via irreducible representations of $G$, by the theorems of Maschke and Artin–Wedderburn, there is an isomorphism of rings with involution:

$$\mathbb{R}G \cong M_{r_1}(\mathbb{R}) \times \cdots \times M_{r_n}(\mathbb{C}) \times \cdots \times M_{s_1}(\mathbb{H}) \times \cdots .$$

The complex numbers $\mathbb{C}$ and quaternions $\mathbb{H}$ are equipped with their conjugations.

Example 13. Here are decompositions with representations indicated in subscript.

1. $\mathbb{R}C_2 = \mathbb{R}_+ \times \mathbb{R}_-^{(p-1)/2}$
2. $\mathbb{R}C_p = \mathbb{R}_+ \times \prod_{a=1}^{\phi(p)^2} \mathbb{C}_{\zeta^a}$ where $p$ is an odd prime and $\zeta := e^{2\pi i/p} \in \mathbb{C}$
3. $\mathbb{R}Q_8 = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}_- \times \mathbb{H}$ with $1 \rightarrow \{\pm 1\} \rightarrow Q_8 \rightarrow C_2 \times C_2 \rightarrow 1$.

Proposition 14 (Morita equivalence). Any $L_*(M_n(R))$ is isomorphic to $L_*(R)$. 

Therefore, the *multisignature* of $G$ is the induced homomorphism

$$\text{msign} : L_*\left(ZG\right) \rightarrow L_*\left(RG\right) \rightarrow \cdots L_*\left(\mathbb{C}\right) \oplus \cdots \oplus L_*\left(H\right) \oplus \cdots.$$ 

**Theorem 15** (Wall). On $L_{2k}(ZG)$, $\text{msign}$ has kernel and cokernel finite 2-groups.

The reduced $L$-groups $\tilde{L}_*(R) := L_*(R)/L_*(Z)$ satisfy $L_*(ZG) = L_*(Z) \oplus L_*(ZG)$.

**Theorem 16** (Bak). Suppose $G$ has odd order. On $\tilde{L}_{2k}(ZG)$, $\text{msign}$ is injective.

### 6. Symmetric & Quadratic Formations

A $(-1)^k$-quadratic formation is a triple $(Q;F,G)$, consisting of a nonsingular $(-1)^k$-quadratic form $Q = (M,\lambda,\mu)$ over $(R,\cdot)$, along with a lagrangian $F$ and a sublagrangian $G$ in $Q$. The formation is nonsingular means that $G$ is a lagrangian.

**Example 17.** A hyperbolic formation is the triple $(\mathcal{H}(M);M \oplus 0,0 \oplus M^*)$.

**Example 18.** A boundary formation is a triple $(\mathcal{H}(M);M \oplus 0,\Gamma_f = \{(x,g(x))\})$, where $f : M \rightarrow M^*$ is any homomorphism of left $R$-modules and $g = f - (-1)^k f^*$.

**Exercise 19.** The previous two examples are isomorphic if $g$ is an isomorphism. So, if $M$ has even rank, then boundary formations generalize hyperbolic formations.

### 7. Definition of $L_{2k+1}(R)$

The abelian monoid $L_{2k+1}(R)$ under component-wise sum consists of stabilized boundary isomorphism classes of nonsingular $(-1)^k$-quadratic formations over $R$. Boundary isomorphic is isomorphic after adding boundary formations to both sides. Stably isomorphic is isomorphic after adding hyperbolic formations to both sides.

**Exercise 20.** It is an abelian group. (Hint: Choose complements, as in Exercise 8.)

As functors there is 4-periodicity, $L_1 = L_5 = L_9 = \ldots$ and $L_3 = L_7 = L_{11} = \ldots$.

### 8. Some Computations of $L_{2k+1}(R)$

**Remark 21.** An isomorphism $L_3(C_2) \xrightarrow{\partial} L_4(F_2) \xrightarrow{\text{Arf}} \mathbb{Z}/2$ exists, by Rim’s square.

**Theorem 22** (Connolly–Hausmann). Let $G$ be finite. Then $16 \cdot L_{2k+1}^*(G) = 0$.

**Theorem 23** (Bak). Suppose $G$ has odd order. Then $L_{2k+1}^*(G) = 0$. 