THE s/h-COBORDISM THEOREM

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1. Whitehead torsion

Let $R$ be a (unital associative) ring. The stable general linear group

$$GL(R) := \colim_{n \to \infty} GL_n(R)$$

is the direct limit given by the stabilization homomorphisms

$$GL_n(R) \longrightarrow GL_{n+1}(R) : A \longmapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}.$$ 

The $n$-th elementary subgroup $E_n(R) < GL_n(R)$ is generated by those matrices with 1’s along the diagonal and any element $r \in R$ at any $(i, j)$-th entry with $i \neq j$.

**Lemma 1** (Whitehead). The elementary subgroup $E(R) = \colim_{n \to \infty} E_n(R)$ equals the commutator subgroup of $GL(R)$.

The ‘generalized determinant’ $[A]$ is an abelian invariant defined as the stable class of an invertible matrix $A \in GL_n(R)$ under these row and column operations:

$$[A] \in K_1(R) := GL(R)^{ab} = \frac{GL(R)}{[GL(R), GL(R)]} = \frac{GL(R)}{E(R)}.$$ 

**Proposition 2.** The following two facts are easily verified. If $R$ is commutative, then the determinant $\det : K_1(R) \longrightarrow R^\times$ is defined and a split epimorphism. Furthermore, if $R$ is euclidean (in particular, a field), then $\det$ is an isomorphism.

Let $C_* = (C_*, d_*)$ be a contractible finite chain complex of based left $R$-modules. Here based means free with a chosen finite basis. Select a chain contraction $s_* : C_* \longrightarrow C_*+1$, which is a chain homotopy from id to 0; that is: $d \circ s + s \circ d = id - 0$. The the algebraic torsion is well-defined by the formula

$$\tau(C_*) := [d + s : C_{even} \longrightarrow C_{odd}] \in K_1(R),$$

with $C_{even} := C_0 \oplus C_2 \oplus \cdots + C_{2N}$ and $C_{odd} := C_1 \oplus C_3 \oplus \cdots$ finite based modules.

**Exercise 3.** Verify that $(d+s)^{-1} = (d+s)(1-s^2+\cdots+(-1)^N s^{2N}) : C_{odd} \longrightarrow C_{even}$.

Let $G$ be a group. Divide by trivial units in group ring for the Whitehead group

$$\text{Wh}(G) := K_1(\mathbb{Z}G)/\langle \mathbb{Z}^\times, G \rangle.$$ 

**Conjecture 4** (Hsiang). $\text{Wh}(G) = 0$ if $G$ is torsion-free.

Let $f : Y \longrightarrow X$ be a cellular homotopy equivalence of connected finite CW complexes. Write $\tilde{f} : \tilde{Y} \longrightarrow \tilde{X}$ for the induced $\pi_1 X$-equivariant homotopy equivalence of universal covers. Select a lift and orientation in $\tilde{X}$ of each cell in $X$. This gives a finite basis to the free $\mathbb{Z}[\pi_1 X]$-module complex $C_*\text{(*)}$. Do the same for $\tilde{Y}$.

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Dividing by these two sets of choices, the Whitehead torsion of $f$ is well-defined in terms of the algebraic mapping cone of the cellular map induced by $f$:

$$\tau(f) := [\tau(\text{Cone}(C, \bar{f}))[) \in \text{Wh}(\pi_1 X).$$

If the homotopy equivalence $f$ is not cellular, then $\tau(f) := \tau(f')$ is well-defined for any cellular approximation $f'$ to $f$. The homotopy equivalence $f : Y \to X$ is simple means that $\tau(f) = 0$. Clearly, any cellular homeomorphism is simple.

**Theorem 5** (Chapman). Any homeomorphism of finite CW complexes is simple.

This fundamental result is proven by showing that: $\tau(f) = 0$ if and only if $f \times \text{id}_Q$ is homotopic to a homeomorphism, where $Q := \prod_{n=1}^\infty [0, \frac{1}{n}]$ is the Hilbert cube. Here, one uses a geometric characterization of ‘simple’ in terms of a finite sequence of elementary expansions and elementary collapses of cancelling cell-pairs.

2. Statement of the s-cobordism theorem

A cobordism $(W^{n+1}; M^n, M')$ is a homotopy cobordism (shortly, $h$-cobordism) means that the inclusions $M \hookrightarrow W$ and $M' \hookrightarrow W$ are homotopy equivalences; that is, $M$ and $M'$ are deformation retracts of $W$. A smooth $h$-cobordism $(W; M, M')$ is simple (shortly, $s$-cobordism) means that these inclusions are simple. Here, we use the Whitehead triangulations induced by their smooth structures, in which each simplex has maximal rank, to parse the formulas $\tau(M \hookrightarrow W) = 0 = \tau(M' \hookrightarrow W)$.

**Example 6.** The product $s$-cobordism on $M$ is $(M^n \times [0, 1]; M \times \{0\}, M \times \{1\})$.

**Theorem 7** (Mazur–Stallings–Barden, the $s$-cobordism theorem). Let $n > 4$. Any smooth $s$-cobordism $(W^{n+1}; M, M')$ is diffeomorphic to the product, relative to $M$.

**Corollary 8** (Smale, the $h$-cobordism theorem). Let $n > 4$. Any simply connected smooth $h$-cobordism $(W^{n+1}; M, M')$ is diffeomorphic to the product, relative to $M$.

(S Donaldson demonstrated this statement is false when $n = 4$.) More generally:

**Theorem 9** (realization). Let $M$ a connected closed smooth manifold of dimension $n > 4$. Under Whitehead torsion of the inclusion of $M$, the set of diffeomorphism classes $\text{rel} M$ of smooth $h$-cobordisms on $M$ corresponds bijectively to $\text{Wh}(\pi_1 M)$.

3. Application

**Corollary 10** (the generalized Poincaré conjecture). Let $m > 5$. Any closed smooth manifold in the homotopy type of the $m$-dimensional sphere is homeomorphic to it.

This is true without the ‘smooth’ hypothesis, and also such for dimensions $m \leq 5$.

**Proof.** Let $\Sigma^m$ be a smooth homotopy $m$-sphere. Consider the smooth cobordism $(W^m; M^{m-1}, M')$ where $W := \Sigma - D^m - \bar{D}^m_+$ and $M := \partial D_-$ and $M' := \partial D_+$. Since $m > 2$, by the Seifert–vanKampen theorem, $W$ is simply connected, as well as $M$ and $M'$. Using excision, the relative homology with integer coefficients is

$$H_*(W; M) \xrightarrow{\cong} H_*(\Sigma - \bar{D}_+ + D_-) = \tilde{H}_*(\Sigma - \text{point}) = 0.$$ 

Then, by the Whitehead theorem, the inclusion $M \hookrightarrow W$ is a homotopy equivalence, and similarly $M' \hookrightarrow W$ is also. So, since $n := m - 1 > 4$, by the $h$-cobordism theorem, $(W; S^n, S^n)$ is diffeomorphic to the product $(S^n \times [0, 1]; S^n \times \{0\}, S^n \times \{1\})$, relative to the canonical identification $S^n = S^n \times \{0\}$ extending smoothly to discs.
Hence $\Sigma - \hat{D}_+ = D_+ \cup W$ is diffeomorphic to the disc $D_m = D^m \times \{0\} \cup S^n \times [0, 1]$. By the Alexander trick, the restricted exotic diffeomorphism $S_+ \to S^n$ extends to a homeomorphism of cones. Thus, $\Sigma$ is homeomorphic to $S^m = D^m \cup_{homeo} D^m$. □

The proof shows more: $\Sigma$ is diffeomorphic to a twisted double $D^m \cup_{diffeo} D^m$.

4. **Proof outline of the $h$-cobordism theorem**

A good reference is page 87 of the monograph of C Rourke and B Sanderson.

(1) Consider a ‘nice’ handle decomposition of $W$ relative to $M$, say via a so-called nice Morse function: handles arranged in increasing index and different handles having different critical values. It exists for all dimensions.

(2) Since $\pi_0(M) \to \pi_0(W)$ is surjective (nonexample: $W = M \times I \sqcup S^{n+1}$), we can cancel each 0-handle with a corresponding 1-handle.

(3) Since $\pi_1(M) \to \pi_1(W)$ is surjective (nonexample: $W = m \times I \# S^1 \times S^n$), we can trade each remaining 1-handle for a new 3-handle. This part works for the non-simply connected case as well.

(4) Dually eliminate the $(n+1)$-handles and $n$-handles, working relative to $M'$.

(5) Similarly, since $\pi_k(M) \to \pi_k(W)$ is surjective, we can trade each $k$-handle for a new $(k+2)$-handle. Only $(n-1)$-handles and $(n-2)$-handles remain.

(6) Flip the resulting handle decomposition upside down: only 2-handles and 3-handles relative to $M'$. Since $\pi_1(M') = 1$ and $H_2(W, M'; Z) = 0$, we can cancel each such 2-handle with a 3-handle.

(7) Thus we obtain only 3-handles relative to $M'$. But $H_3(W, M'; Z) = 0$, so actually there are no 3-handles remaining! Therefore, we can conclude that $W$ is diffeomorphic to $M \times I$ relative to $M \times \{0\}$.

Above, the canceling and trading of handles necessitates the Whitney trick ($n > 4$).