THE MODULI SET OF $\mathbb{RP}^n \# \mathbb{RP}^n$

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Let $X$ be a finite simplicial complex. The simple moduli set $\mathcal{M}_{\text{TOP}}(X)$ consists of the homeomorphism classes $[M]$ of closed topological manifolds $M$ in the simple homotopy type of $X$. Recall that the simple structure set $\mathcal{S}_{\text{TOP}}(X)$ consists of the $s$-bordism classes $[M, h]$ of simple homotopy equivalences $h : M \to X$ where $M$ is a closed topological manifold. Assuming the validity of the $s$-cobordism theorem, this equivalence relation becomes less esoteric: $[M, h] = [M', h']$ if and only if there exists a homeomorphism $\phi : M \to M'$ such that $h$ is homotopic to $h' \circ \phi$.

Write $\text{hAut}(X)$ for the group of homotopy classes of simple homotopy equivalences $X \to X$. This has a canonical left-action on $\mathcal{S}_{\text{TOP}}(X)$ given by post-composition. Therefore, the moduli set is the quotient of the structure set:

$$\mathcal{M}_{\text{TOP}}(X) = \text{hAut}(X) \backslash \mathcal{S}_{\text{TOP}}(X).$$

Let $X$ be a closed connected topological manifold of dimension $n$. Let $Y$ be a connected codimension-one submanifold of $X$ that is separating: $X - Y$ is disconnected. Assume $Y$ is incompressible in $X$: $\pi_1 Y \to \pi_1 X$ is injective. So $X = X_1 \cup_Y X_2$ and there is an injective amalgamated product of fundamental groups: $G = G_1 *_H G_2$.

A homotopy equivalence $h : M \to X$ from a closed manifold $M$ is split along $Y$ if $h$ is transverse to $Y$ and the restriction $h : h^{-1} Y \to Y$ is also a homotopy equivalence.

Below we make use of the orientation character, a homomorphism $w : G \to \{\pm 1\}$.

The simple split structure set $\mathcal{S}_{\text{TOP}}^\text{split}(X; Y)$ consists of the split homotopy classes of simple split homotopy equivalences $h : M \to X$. Note the forgetful map

$$\mathcal{S}_{\text{TOP}}^\text{split}(X; Y) \to \mathcal{S}_{\text{TOP}}(X).$$

Sylvain Cappell [5] defined the surgery obstruction to splitting a simple structure:

$$\mathcal{S}_{\text{TOP}}(X) \to \text{UNil}_{n+1}^s(\mathbb{Z}[G]; \mathbb{Z}[G_1 - H]^{w_1}, \mathbb{Z}[G_2 - H]^{w_2}).$$

The target of his function is an algebraically defined abelian group, out of which he provides an algebraically defined monomorphism into Wall’s group [4]:

$$\text{UNil}_{n+1}^s \to L_{n+1}^s(\mathbb{Z}[G]^{w}).$$

Thus we may consider the composite function

$$\mathcal{S}_{\text{TOP}} \times \text{UNil}_{n+1}^s \to \mathcal{S}_{\text{TOP}} \times L_{n+1}^s \to \mathcal{S}_{\text{TOP}},$$

where the last function is the right-action given by Wall realization ($n \geq 5$) [9]. Cappell’s nilpotent normal cobordism construction [5] shows that this composite is surjective ($n \geq 6$). A relative form of it establishes injectivity, as announced in [2].

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1We only consider homotopy equivalences that are simple: they have zero Whitehead torsion.

2Using the $h$-cobordism theorem ($n \geq 4$), one can show that the connected sum is a bijection:

$$\# : \mathcal{S}_{\text{TOP}}(X_1) \times \mathcal{S}_{\text{TOP}}(X_2) \to \mathcal{S}_{\text{TOP}}^\text{split}(X_1 \# X_2; S^{n-1}).$$
Next, we describe the calculation of $\mathcal{S}_{\text{TOP}}(\mathbb{R}P^n \# \mathbb{R}P^n)$. Stallings showed that the Whitehead group of $\mathbb{Z}/2 \ast \mathbb{Z}/2$ (and all its subgroups) is trivial [8]. Therefore, from above with $\varepsilon = (-1)^{n+1}$, we obtain a decomposition of the structure set ($n \geq 4$):

(1) $\mathcal{S}_{\text{TOP}}(\mathbb{R}P^n) \times \mathcal{S}_{\text{TOP}}(\mathbb{R}P^n) \times \text{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \xrightarrow{\sim} \mathcal{S}_{\text{TOP}}(\mathbb{R}P^n \# \mathbb{R}P^n)$.

López de Medrano calculated the structure set of real projective groups were computed partially by Cappell [3], later fully by Connolly–Davis [6]:

$$\text{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) = \begin{cases} (\mathbb{Z}/2)^{2k} \oplus (\mathbb{Z}/4)^{\infty} & \text{if } n = 4k \\ (\mathbb{Z}/2)^{2k} & \text{if } n = 4k + 1, 4k + 2 \\ (\mathbb{Z}/2)^{2k} \oplus \mathbb{Z} & \text{if } n = 4k + 3. \end{cases}$$

This is detected by normal invariants in $\mathbb{Z}/2$ along the submanifolds $\mathbb{R}P^i$ and, if $n = 4k + 3$, the Browder–Livesay desuspension invariant in $\mathbb{Z}$. The above UNil-groups were computed partially by Cappell [3], later fully by Connolly–Davis [6]:

$$\text{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) = \begin{cases} (\mathbb{Z}/2)^{2k} \oplus (\mathbb{Z}/4)^{\infty} & \text{if } n = 4k \\ (\mathbb{Z}/2)^{2k} & \text{if } n = 4k + 1 \\ 0 & \text{if } n = 4k + 2, 4k + 3. \end{cases}$$

For $n = 4k + 1$, this is given by the Arf invariant for quadratic forms over the function field $\mathbb{F}_2(t)$. For $n = 4k$, this is given by a two-stage obstruction for quadratic linking forms. This completes the calculation of the structure set $\mathcal{S}_{\text{TOP}}(\mathbb{R}P^n \# \mathbb{R}P^n)$.

Finally, we calculate the moduli set $\mathcal{M}_{\text{TOP}}(\mathbb{R}P^n \# \mathbb{R}P^n)$. Cappell showed that the group $\text{hAut}(\mathbb{R}P^n \# \mathbb{R}P^n)$ is generated by three diffeomorphisms $\gamma_1, \gamma_2, \gamma_3$ [3, p. 397]. Subsequently, Brookman–Davis–Khan [1] determined that the action of $\text{hAut}$ on the left side of (1) induced by the bijection of (1) has quotient set

$$\text{Sym}^2 \mathcal{M}_{\text{TOP}}(\mathbb{R}P^n) \times U_n \xrightarrow{\sim} \mathcal{M}_{\text{TOP}}(\mathbb{R}P^n \# \mathbb{R}P^n).$$

Here, $\text{Sym}^2 \mathcal{M}_{\text{TOP}}(\mathbb{R}P^n)$ denotes the unordered pairs in the moduli set. Also, $U_n$ denotes the quotient set of $\text{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ by the interchange of the two $\mathbb{Z}$-bimodules $\mathbb{Z}$ (this switch map is induced by $\gamma_1$); this was computed in [1]. One easily shows that $\text{hAut}(\mathbb{R}P^n)$ is generated by the ‘reflection’ $\beta_n$ in $\mathbb{R}P^{n-1}$ so that $\mathcal{M}_{\text{TOP}}(\mathbb{R}P^n) = \mathcal{S}_{\text{TOP}}(\mathbb{R}P^n)$ for $n = 4k, 4k + 1, 4k + 2$ and $\mathcal{M}_{\text{TOP}}(\mathbb{R}P^{4k+3}) = (\mathbb{Z}/2)^{2k} \times \mathbb{Z}_{2^0}$. Thus we have computed the moduli set $\mathcal{M}_{\text{TOP}}(\mathbb{R}P^n \# \mathbb{R}P^n) (n \geq 4)$.

References