BASIC COMMUTATORS IN WEIGHTS SIX AND SEVEN AS RELATORS

DAVID A. JACKSON

Abstract. Charles Sims has asked whether or not the lower central subgroup $\gamma_n(F)$ of a free group F coincides with the normal closure in F of the basic commutators of weight n. This question has a positive answer in weights at most 5, but remains an open question in general. In earlier work with Gaglione and Spellman, it was shown that $\gamma_n(F)$ is the normal closure in F of the basic commutators of weights n through $2n-4$. Here, we specialize to the case where F has rank 2 and show that $\gamma_6(F)$ is the normal closure in F of the basic commutators of weights 6 and 7.

1. Introduction and Notation

We write F_r for the free group on r generators, y^x for $x^{-1}yx$, $[y, x]$ for the commutator $y^{-1}x^{-1}yx$ and $[z, x, y]$ for the commutator $[[z, x], y]$. A simple commutator of weight 1 is a generator and a simple commutator of weight $n > 1$ is a commutator $[c, x]$ where c is a simple commutator of weight $n-1$ and x is a generator. We follow Marshall Hall's definition for basic commutators. See, for example, Hall, 1976, H. Neumann, 1967 or Jackson, Gaglione and Spellman, 2002.

If $[x_1, x_2, \ldots, x_j], [y_1, y_2, \ldots, y_k]$ and $[z_1, z_2, \ldots, z_\ell]$ are simple commutators with weights $j \geq 2, k \geq 2$ and $\ell \geq 1$, respectively, we abbreviate $[[x_1, \ldots x_j], [y_1, \ldots y_k]]$ as $[x_1, \ldots x_j \,;\, y_1, \ldots, y_k]$ and we abbreviate $[[x_1, \ldots x_j], [y_1, \ldots y_k], [z_1, \ldots, z_\ell]]$ using $[x_1, \ldots x_j \,;\, y_1, \ldots, y_k \,;\, z_1, \ldots z_\ell]$.

Write γ_n for the n^{th} term, $\gamma_n = \gamma_n(G) := [\gamma_{n-1}(G), G]$ of the lower central series of a group G. Let $\mathfrak{C}$ be some fixed basic sequence of commutators. Following Jackson, Gaglione and Spellman, 2002, for natural numbers n, s, we will write $N_{n:s}$ for the normal closure in F_r of the basic commutators from $\mathfrak{C}$ having weights n through $n+s-1$, inclusive. With this notation, one question from Sims, 1987 can be expressed as asking whether or not $\gamma_n F_r = N_{n:1}$. It is known that this question has an affirmative answer for weight n at most five. See Sims, 1987, Jackson, Gaglione and Spellman, 2002, and Jackson, Gaglione and Spellman, submitted. The question remains open in general.

The following theorem was proved in Jackson, Gaglione and Spellman, 2002 for $F = F_r$ the free group on $r \geq 2$ generators, using earlier work of Martin Ward. See Hurley and Ward, 1996, Ward, 1969 and Ward, 1972.

Theorem 4.7. *For* $n \geq 4$, $\gamma_n(F_r) = N_{n:n-3}$.

In this work, we specialize to the case $r = 2$. We take $F = F_2$ to be the free group on the ordered alphabet $\{a, b\}$ and assume that all basic sequences of commutators begin with $a < b$ as basic commutators of weight 1. It is not difficult to see that (because the alphabet and weights are small) $N_{6:2}$ and $N_{6:3}$ do not depend upon the choice of basic sequence $\mathfrak{C}$. Throughout, we will write N for $N_{6:2}$. By the $n = 6$ case of Theorem 4.7, we see that $\gamma_6 F = N_{6:3}$. We improve this.

Main Theorem. $\gamma_6 F = N_{6:2}$.

To prove the Main Theorem, it will suffice to prove that basic commutators from $\mathfrak{C}$ having weight 8 are already in $N = N_{6:2}$. This is clear for simple basic commutators of weight 8 and for basic commutators from $\mathfrak{C}$ having the form $[c, d]$ where c has weight 6 and d has weight 2. Similarly, it is immediately clear that the basic commutator $[b, a, a, a, a]$ is central mod N, hence both $[b, a, a, a, a\,;\, b, a, a]$ and $[b, a, a, a, a\,;\, b, a, b]$ are trivial mod N. For $c = [b, a, a, a, b]$, $c = [b, a, a, b, b]$ or $c = [b, a, b, b, b]$, the weight 8 basic commutator $[c, [b, a, b]]$ is trivial mod N because $[c, [b, a]]$ is a basic commutator of weight 7 and $[c, b]$ is a basic commutator of weight 6. There are seven other basic commutators $[c, d]$ of weight 8 for which c has weight 5 and d has weight 3. We will prove the Main Theorem by showing that these seven commutators and the three basic commutators $[c, d]$ where both c and d have weight 4 are trivial mod N.

The reader might find it a useful exercise at this point to write down a list of the 6 basic commutators of weight 5, the 9 basic commutators of weight 6, the 18 basic commutators of weight 7 and the 30 basic commutators of weight 8. Because we can order basic commutators within weight 3 and within weight 4 arbitrarily, this list is ambiguous in weights 6 and 8. For example, in weight 8, we will have either $[b, a, a, b\,;\, b, a, a, a]$ or else $[b, a, a, a\,;\, b, a, a, b]$ basic, depending upon our choice of order in weight 4. Observe that these choices will not change $N = N_{6:2}$ or $N_{6:3}$, since we will have either c or c^{-1} in the normal subgroup in the four relevant instances of ambiguity. Rather than appealing to this discussion below, we will assume that $[b, a, b\,;\, b, a, a]$ is basic of weight 6 and that $[b, a, a, b\,;\, b, a, a, a], [b, a, b, b\,;\, b, a, a, a]$ and $[b, a, b, b\,;\, b, a, a, b]$ are basic of weight 8.

The two useful identities in the following theorem are among the simplest of a vast collection of similar identities.

Theorem H. *Suppose that A, B and C are elements of any group G. Then*

$$(1) \qquad [C, B, A] = [C, A, C]^{[C,B]}[C, A, B]^{C^B}([C, A, C]^{-1})^B[C, A, B]^{-1} \left([B, A, B]^{C^B}\left([B, A, C]^{-1}\right)^B[B, A, B]^{-1}\right)^{[C,A]} [C, A, A][C, A, B]^A\left([C, A, A]^{-1}\right)^{B^A}$$

$$(2) \qquad [C, B, A] = \left([C, A, A]^{[B,A]^{-1}B^{-1}}[C, A, B]^{B^{-1}A}[C, A, A]^{-1}\left([C, A, B]^{-1}\right)^{B^{-1}}\right)^{[C,B]}\left([C, B, B]^{-1}\right)^{B^{-1}} [C, A, C]^{C^{-1}B^{-1}C}[C, A, B]^{B^{-1}C}[C, A, C]^{-1}\left([B, A, C]^{-1}\right)^{C^{-1}A^{-1}B^{-1}CA}[C, B, B]^{B^{-1}A}$$

Proof. For either equation, it will suffice to show that the equation is valid in the free group on $\{A, B, C\}$. Write both sides of the equation as words in this free group and observe that the reduced forms are equal. □

2. A few calculations

Lemma 1. $[b,a,a,a,b,a] \equiv 1 \mod N$.

Proof. Use part (2) of Theorem H with $C = [b,a,a,a], B = b$ and $A = a$. Then $[C,A,A],[C,A,B],[C,B,B]$ and $[B,A,C]^{-1}$ are basic commutators of weight 6, hence trivial mod N. Since $[C,A] = [b,a,a,a,a]$ is central mod N, $[C,A,C]$ is also trivial mod N, hence so is $[C,B,A]$. □

One easy consequence of Lemma 1 is that $[b,a,a,a,b]$ is central mod N. As a corollary to this, the weight 8 basic commutator $[b,a,a,a,b\,;\,b,a,a]$ is trivial mod N. The next lemma collects this and some earlier easy observations in a form suitable for reference later in this exposition.

Lemma 2. *The following weight 8 basic commutators are trivial mod N:*

$$[b,a,a,a,a\,;\,b,a,a],[b,a,a,a,a\,;\,b,a,b],[b,a,a,a,b\,;\,b,a,b],$$
$$[b,a,a,b,b\,;\,b,a,b],[b,a,b,b,b\,;\,b,a,b] \text{ and } [b,a,a,a,b\,;\,b,a,a].$$

Proof. The triviality of the first five of these commutators was discussed in the paragraph following the statement of the Main Theorem and the triviality of the last commutator is an easy consequence of Lemma 1. □

Lemma 3. *If C is either of $[b,a,a]$ or $[b,a,b]$, then $[C,b]$ commutes mod N with $[b,a,a,a]$.*

Proof. Use part (1) of Theorem H with $A = [b,a,a,a], B = b$ and C either of $[b,a,b]$ or $[b,a,a]$, observing that $[C,B]$ is basic of weight 4. Then $[A,C] = [C,A]^{-1}$ is basic of weight 7, so $[C,A,A],[C,A,B]$ and $[C,A,C]$ are trivial mod N. Since $[A,B] = [b,a,a,a,b]$ is central mod N by Lemma 1, $[B,A] = [A,B]^{-1}$ is central and the commutators $[B,A,B]$ and $[B,A,C]$ are also trivial mod N. □

Lemma 4. $[b,a,b,b\,;\,b,a,a,b] \equiv 1 \mod N$.

Proof. Use part (1) of Theorem H with $C = [b,a,b], B = b$ and $A = [b,a,a,b]$. Then $[A,C] = [C,A]^{-1}$ is basic of weight 7, so $[C,A,A],[C,A,B]$ and $[C,A,C]$ are trivial mod N. Observe also that $[B,A] = [A,B]^{-1} = [b,a,a,b,b]^{-1}$ which commutes mod N with $B = b$, hence $[B,A,B]$ is trivial mod N. To see that $[B,A,C]$ is also trivial, we observe that $[B,A]^{-1} = [b,a,a,b,b]$ commutes with $C = [b,a,b]$ by Lemma 2. □

Lemma 5. $[b,a,a,b,b\,;\,b,a,a] \equiv 1 \mod N$.

Proof. Use part (2) of Theorem H with $C = [b,a,a,b], B = b$ and $A = [b,a,a]$. Then $[C,A]$ is a basic commutator of weight 7, so $[C,A,A],[C,A,B]$ and $[C,A,C]$ are trivial mod N. The commutator $[C,B,B]$ is a basic commutator of weight 6 and $[B,A]^{-1} = [b,a,a,b] = C$, so $[B,A,C]$ is also trivial. □

Lemma 6. $[b,a,b,b,b\,;\,b,a,a] \equiv 1 \mod N$.

Proof. Use part (2) of Theorem H with $C = [b,a,b,b], B = b$ and $A = [b,a,a]$. The proof is then the same as the proof of Lemma 5 except that we here use Lemma 4 to see that $[B,A,C]$ is trivial mod N. □

Lemma 7. *The following commutators are trivial mod N:*

$$[b,a,a\,;\,b,a\,;\,b,a,a], \qquad [b,a,a\,;\,b,a\,;\,b,a,b], \qquad [b,a,b\,;\,b,a\,;\,b,a,b],$$
$$[b,a,a\,;\,b,a\,;\,a], \qquad [b,a,a\,;\,b,a\,;\,b], \qquad \textit{and} \qquad [b,a,b\,;\,b,a\,;\,b].$$

Proof. The triviality of the weight 8 basic commutators in the top displayed line follows from the triviality of the weight 6 commutators below them. For example, we know that $[b,a,a\,;\,b,a]$ commutes with $[b,a]$ since $[b,a,a\,;\,b,a\,;\,b,a]$ is a basic commutator of weight 7, so if $[b,a,a\,;\,b,a]$ commutes with a as well as with $[b,a]$, then it commutes with $[b,a,a]$.

To prove the triviality of the commutators in the second displayed line, we use part (2) of Theorem H. For the three cases, we use, respectively, $C=[b,a,a], B=[b,a], A=a$, or $C=[b,a,a], B=[b,a], A=b$ or $C=[b,a,b], B=[b,a], A=b$. In all three cases, $[C,A,B]$ is a basic commutator of weight 6, $[C,A,C]$ and $[C,B,B]$ are basic commutators of weight 7 and $[B,A,C]$ is either freely trivial or else a basic commutator of weight 6. Hence, reducing the right hand side of part (2) of Theorem H, it will suffice to prove that $[C,A,A]^{[B,A]^{-1}B^{-1}}[C,A,A]^{-1}$ is trivial in each of the three cases. For this, we need only show that $[C,A,A]$ commutes with both B and $[B,A]$. Since $[C,A,A]$ is a simple basic commutator of weight 5 and B is a basic commutator of weight 2, we know that $[C,A,A]$ commutes with B. We will then be done if we show that $[C,A,A]$ commutes with $[B,A]$ in each of the three cases. In the first case, we need to see that $[C,A,A]=[b,a,a,a,a]$ commutes with $[B,A]=[b,a,a]$. In the second case, we need to see that $[C,A,A]=[b,a,a,b,b]$ commutes with $[B,A]=[b,a,b]$. In the third case, we need to see that $[C,A,A]=[b,a,b,b,b]$ commutes with $[B,A]=[b,a,b]$. In all three cases, this is known from Lemma 2. □

Lemma 8. $[b,a,b\,;\,b,a\,;\,b,a,a]\equiv 1 \mod N$.

Proof. Use part (2) of Theorem H with $C=[b,a,b], B=[b,a]$ and $A=[b,a,a]$. Then $[C,A]$ is a basic commutator of weight 6, hence $[C,A,A]$, $[C,A,B]$ and $[C,A,C]$ are trivial mod N. The commutator $[C,B,B]$ is a basic commutator of weight 7. To see that $[B,A,C]$ is also trivial mod N, observe that $[B,A]^{-1}=[A,B]=[b,a,a\,;\,b,a]$ commutes with $C=[b,a,b]$ by Lemma 7. □

Proof of Main Theorem. By Theorem 4.7, it will suffice to show that the basic commutators of weight 8 are in N. This is clear for simple basic commutators of weight 8 and for basic commutators $[x,y]$ where x has weight 6 and y has weight 2. The three basic commutators $[x,y]$ where x and y both have weight 4 are in N by Lemmas 3 and 4. The eight basic commutators $[x,y]$ where x is a simple commutator of weight 5 and y is a simple commutator of weight 3 are in N by Lemmas 2, 5 and 6. The four basic commutators $[x,y,z]$ where x and z have weight 3 while y has weight 2 are in N by Lemmas 7 and 8. □

References

[Hall, 1976] Marshall Hall, Jr. *The Theory of Groups*, second edition, Chelsea Publishing Company, New York 1976

[Havas and Richardson, 1983] G. Havas and J. S. Richardson *Groups of exponent five and class four* Commun. Alg. **11** (1983) 287–304

[Hurley and Ward, 1996] T. C. Hurley and M. A. Ward *Bases for commutator subgroups of a free group* Proc. Roy. Irish Acad. Sect. A **96** (1996) 43–65

[Jackson, Gaglione and Spellman, 2002] David A. Jackson, Anthony M. Gaglione and Dennis Spellman *Basic Commutators as Relators* J. Group Theory **5** (2002) 351–363

[Jackson, Gaglione and Spellman, submitted] David A. Jackson, Anthony M. Gaglione and Dennis Spellman *Weight Five Basic Commutators as Relators*

[H. Neumann, 1967] Hanna Neumann *Varieties of Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **37** Springer-Verlag New York, Berlin 1967

[Sims, 1987] Charles C. Sims *Verifying Nilpotence* J. Symbolic Computation **3** (1987) 231–247

[Ward, 1969] M.A. Ward *Basic Commutators* Phil. Trans. Roy. Soc. London, Ser.A **264** (1969) 343–412

[Ward, 1972] Martin Ward *Bases for Polynilpotent Groups* Proc. London Math. Soc., Ser (3) **24** (1972) 409–431

Department of Mathematics, Saint Louis University, St. Louis, M0 63103