# UNCOUNTABLY MANY BASIC SEQUENCES OF COMMUTATORS

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Let $F = F_X$ be a non-Abelian free group with a finite ordered generating set $X$. In general, we will use $x_i$, $1 \leq i \leq |X|$ for the elements of $X$, but it will be notationally convenient to write $a$ for $x_1$ and $b$ for $x_2$. We require only that $|X| \geq 2$, so we might have that $X = \{a, b\}$.

We write $y^x$ for $x^{-1}yx$ and $[y, x]$ for the commutator $y^{-1}x^{-1}yx$. We write $[z, x, y]$ for the commutator $[[z, x], y]$. The weight of a commutator $c$ is denoted by $\mathrm{wt}(c)$. We follow Marshall Hall's definition for basic commutators. See, for example, [1] or [2].

(1) The basic commutators of weight one are the letters of $X$ taken in the given order on $X$. (2) Having defined and ordered the basic commutators of weight less than $n$, the basic commutators of weight $n$ are all of the commutators $[c_i, c_j]$ which satisfy the conditions: (*i*) $c_i$ and $c_j$ are basic commutators with $n = \mathrm{wt}(c_i)+\mathrm{wt}(c_j)$. (*ii*) In the order that has been chosen for basic commutators of weight less than $n$, $c_j < c_i$. (*iii*) If $c_i = [c_s, c_t]$ where $c_s$ and $c_t$ are basic commutators, then $c_t \leq c_j$ in the order that has been chosen for basic commutators of weight less than $n$. (3) The basic commutators of weight $n$ follow all of the basic commutators of weight less than $n$ in the order for the basic commutators of weight less than $n + 1$, but the basic commutators of weight $n$ may be ordered arbitrarily.

While we may order the commutators of weight $n$ arbitrarily, the choices that we make will have consequences for which commutators of higher weights are basic and which are not. For example, depending upon which order we use for the commutators $[x_2, x_1]$ and $[x_3, x_1]$ in weight 2, either $[[x_2, x_1], [x_3, x_1]]$ or else $[[x_3, x_1], [x_2, x_1]]$ will be a basic commutator in weight 4.

Our main objective in this brief note is to show that, essentially because we can arbitrarily order the basic commutators of weight $n > 2$, there are uncountably many different basic sequences of commutators which begin with the finite ordered alphabet $X$.

**Proposition.** *If $X$ is any finite, ordered alphabet with $|X| \geq 2$, then for the group $F_X$, there are uncountably many different basic sequences of commutators which begin with $X$.*

*Proof.* Engel commutators will be useful. The following inductive definition is standard: $[y, {}_1x] = [y, x]$ and $[y, {}_nx] = [[y, {}_{n-1}x], x]$ for $n > 1$. By a very easy induction on $n$, for $n \geq 2$, $[b, {}_{n-1}a]$ is a basic commutator of weight $n$ in every basic sequence of commutators which begins with $X$. From this, it follows easily that for $n \geq 3$, the commutator $[b, {}_{n-2}a, b]$ is also a basic commutator of weight $n$ in every basic sequence of commutators which begins with $X$.

If $\mathcal{C}$ is any basic sequence of commutators and $n \geq 1$, let $B_n(\mathcal{C})$ denote the ordered set of basic commutators of weight $n$ from $\mathcal{C}$ and let $\alpha(\mathcal{C}, n)$ denote the first element in $B_n(\mathcal{C})$.

Suppose that $\mathcal{C}$ and $\mathcal{C}'$ are both basic sequences of commutators which begin with $X$. We observe that $\mathcal{C} \neq \mathcal{C}'$ if the order on $B_n(\mathcal{C})$ is different from the order on $B_n(\mathcal{C}')$ even if these two sets have the same underlying unordered set. If $x, y \in B_n(\mathcal{C}) \cap B_n(\mathcal{C}')$ with $x < y$ in $\mathcal{C}$ but $y < x$ in $\mathcal{C}'$, then $[y, x] \in B_{2n}(\mathcal{C})$, but $[x, y] \in B_{2n}(\mathcal{C}')$ and similarly $[y, x, x] \in B_{3n}(\mathcal{C})$, but $[x, y, y] \in B_{3n}(\mathcal{C}')$. In particular, if for any $n$, $\alpha(\mathcal{C}, n) \neq \alpha(\mathcal{C}', n)$, then $\mathcal{C} \neq \mathcal{C}'$.

If we have any countable set of basic sequences of commutators, we may index these basic sequences as $\mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5, \ldots$. Define a basic sequence, $\mathcal{C}_*$, as follows. Choose any convenient order for the basic commutators $[x_j, x_i], i < j$ of weight two. Having defined and ordered the basic commutators from $\mathcal{C}_*$ of weight less than $n$, choose

$$\alpha(\mathcal{C}_*, n) = \begin{cases} [b, {}_{n-1}a] & \text{if } \alpha(\mathcal{C}_n, n) \neq [b, {}_{n-1}a] \\ [b, {}_{n-2}a, b] & \text{if } \alpha(\mathcal{C}_n, n) = [b, {}_{n-1}a] \end{cases}$$

Order the other commutators of weight $n$ in $\mathcal{C}_*$ arbitrarily. Then $\mathcal{C}_* \neq \mathcal{C}_n$ for any $n$. □

Having uncountably many basic sequences of commutators leads one to consider the limitations of using algorithms for generating basic sequences of commutators since "most" basic sequences of commutators cannot be generated by an algorithm. Define a basic sequence, $\mathcal{A}$, to be **algorithmic** if there is an algorithm which on input of any natural number $n$ will generate the ordered list of basic commutators of weight $n$ from $\mathcal{A}$. The sense of the following easy proposition is that for very many things that a mathematician might want to prove about a basic sequence of commutators, it can be assumed that the sequence is algorithmic.

**Proposition.** *Let $n$ be any natural number and let $\mathcal{C}$ be any basic sequence of commutators in $F$ which begins with the ordered alphabet $X$. Then there is an algorithmic basic sequence of commutators $\mathcal{A}$ (which begins with $X$) such that for every $m \leq n$, the set of basic commutators of weight $m$ from $\mathcal{C}$ is the same (as an ordered list) as the set of basic commutators of weight $m$ from $\mathcal{A}$.*

*Proof.* Most of the definition for $\mathcal{A}$ is forced by requiring that $\mathcal{A}$ is a basic sequence of commutators beginning with $X$, but to complete the definition, we need an algorithm which gives the order for basic commutators of some fixed weight $m$ from $\mathcal{A}$. One way to order $\mathcal{A}$ is the following. Let $B_m(\mathcal{A}) = B_m(\mathcal{C})$, as ordered lists, if $m \leq n$. For $m > n$, order the elements of $B_m(\mathcal{A})$, inductively, by $[c_1, c_2] < [d_1, d_2]$ if $c_1 < d_1$ or if $c_1 = d_1$ and $c_2 < d_2$. □

## References

[1] Marshall Hall, Jr. *The Theory of Groups*, second edition, Chelsea Publishing Company, New York, 1976

[2] Hanna Neumann *Varieties of Groups* Ergebnisse der Mathematik und ihrer Grenzgebiete, **37**, Springer-Verlag, New York, Berlin 1967

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