$GL_d(R)$ IS NOT RESIDUALLY p

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Proposition . Suppose that $p = 2k + 1$ is an odd prime, that α and β are elements having p-power order in a group G and that $\alpha\beta^k = \beta^{k+1}\alpha$ in G. Then $\beta = 1$.
Proof: Suppose that α has order p^s and that β has order p^t . If $t = 0$, w **oposition** . Suppose that $p = 2k + 1$ is an odd prime, that α and β are element
ving p-power order in a group G and that $\alpha \beta^k = \beta^{k+1} \alpha$ in G. Then $\beta = 1$.
Proof: Suppose that α has order p^s and that β

 t . Then $\beta = 1$.
If $t = 0$, we are Proof: Suppose that α has order p^s and that β has order.
done. We will prove the proposition by showing that $\beta^{p^{t-1}}$
Observe first that $\beta^{-k} \alpha \beta^k - \beta \alpha$ so that $\alpha \beta \alpha$ and α p^{x-1} Proof: Suppose that α has order p^s and that β has order p^t . If $t = 0$, we and the summer we have the proposition by showing that $\beta^{p^{t-1}} = 1$.
Observe first that $\beta^{-k}\alpha\beta^k = \beta\alpha$, so that $\alpha, \beta\alpha$ and $\alpha\$

 s . By done. We will prove the proposition
Observe first that $\beta^{-k}\alpha\beta^{k} = \beta\alpha$
induction on r, $\alpha\beta^{r}$ has order p^s for on by showing that $\beta^p = 1$.
 $\beta \alpha$, so that α , $\beta \alpha$ and $\alpha \beta$ all have for all natural numbers r, since

duction on r,
$$
\alpha\beta^r
$$
 has order p^s for all natural numbers r, since
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$$
1 = (\alpha\beta^r)^{p^s} = (\alpha\beta^k\beta^{r-k})^{p^s} = (\beta^{k+1}\alpha\beta^{r-k})^{p^s} = \beta^{k+1}(\alpha\beta^{r+1})^{p^s}\beta^{-(k+1)}
$$
\nIf $t = 1$, then β has order $p = 2k + 1$ and from $\alpha\beta^k = \beta^{k+1}\alpha$, we obtain

If $t = 1$, then β has order $p = 2k + 1$ a
 $\beta^k \alpha \beta^k = \alpha$. More generally, β has order p^t . These stop for an easy induction on *i* showing the t Then $\beta p^t - ($ 1 and from $\alpha \beta^k = \beta^{k+1} \alpha$, we obtain

Then $\beta^{p^t - (k+1)} \alpha \beta^k = \alpha$. This is the

that $\beta^{p^t - j(k+1)} \alpha \beta^k = \alpha$ for all natural If $t = 1$, then β has order $p = 2k + 1$ and from $\alpha\beta^w = \beta^{w+2}\alpha$, we obtain $\beta^k \alpha \beta^k = \alpha$. More generally, β has order p^t . Then $\beta^{p^t - (k+1)} \alpha \beta^k = \alpha$. This is the base step for an easy induction on j showing tha base step for an easy induction on j showing that $\beta^{p^t-j(k+1)}\alpha\beta^{jk}=\alpha$ for all natural numbers *j*. With $j = p^{t-1}$, w ly, β has order p^{ι} . Then $\beta^{p^{\iota} - (\kappa + 1)} \alpha \beta^{\iota}$
tion on j showing that $\beta^{p^t - j(k+1)} \alpha \beta^{j k}$,
, we have $p^t - j(k+1) = p^t - p^{t-1}(k)$ $\alpha_k \beta^k = \alpha$. This is the
 $\beta^k = \alpha$ for all natural
 $(k + 1) = k p^{t-1}$ and $\beta^{kp^{t-1}}\alpha\beta^{kp^{t-1}}=\alpha.$ mbers *j*. With $j = j$
 $\alpha^{b^{t-1}} \alpha \beta^{kp^{t-1}} = \alpha$.

Since *p* is odd, *p*^{*s*} is is odd. For typesetting convenience, write N for $\frac{1}{2}(p^s - 1)$ and
is odd. For typesetting convenience, write N for $\frac{1}{2}(p^s - 1)$ and

 s_i

$$
r \text{ for } kp^{t-1}. \text{ Then}
$$

\n
$$
1 = (\alpha \beta^r)^{p^s} = (\alpha \beta^r \alpha \beta^r)^N \alpha \beta^r = (\alpha \cdot \beta^{kp^{t-1}} \alpha \beta^{kp^{t-1}})^N \alpha \beta^r = (\alpha^2)^N \alpha \beta^r = \beta^r
$$

\nSince *k* is relatively prime to *p*, and $r = kp^{t-1}$, we have $\beta^{p^{t-1}} = 1$.

 p^{i-1}

Corollary 1. Suppose that $p = 2k + 1$ is an odd prime and let R be any ring in which k and $k + 1$ are invertible elements. Let $d \geq 2$ be a natural number. Then the general linear group $GL_d(R)$ is not residually a finite p-group.

proof: Let E_{ij} be the $d \times d$ matrix over R with 1 in row i, column j and zeroes
expecting Let E_{ij} be the $d \times d$ matrix over R with 1 in row i, column j and zeroes
expecting Let I be the $d \times d$ identity matrix and le Proof: Let E_{ij} be the $d \times d$ matrix over R with 1 in row *i*, column *j* and zeroes
elsewhere. Let I be the $d \times d$ identity matrix and let b be the elementary transvection
matrix $h = I + E_{ij}$. We want to show that whenev Proof: Let E_{ij} be the $d \times d$ matrix over R with 1 in row *i*, column *j* and zeroes
elsewhere. Let I be the $d \times d$ identity matrix and let b be the elementary transvection
matrix $b = I + E_{d1}$. We want to show that whene elsewhere. Let I be the $d \times d$ identity matrix and let b be the elementary transvection
matrix $b = I + E_{d1}$. We want to show that whenever ϕ is a homomorphism of
 $GL_d(R)$ onto a finite p-group G, then we have $\phi(b) = 1$. L matrix $b = I + E_{d1}$. We want to show that whenever ϕ is a homomorphism of $GL_d(R)$ onto a finite p-group G, then we have $\phi(b) = 1$. Let a be the elementary diagonal matrix with $a_{11} = k/(k+1)$, $a_{ii} = 1$ for $2 \le i \le d$, and $GL_d(R)$ onto a finite p-group G, then we have $\phi(b) = 1$. Let a be the elementary diagonal matrix with $a_{11} = k/(k+1)$, $a_{ii} = 1$ for $2 \le i \le d$, and $a_{ij} = 0$ if $i \ne j$.
Then it is easily checked that $b^r = I + rE_{d1}$ for ever diagonal matrix with $a_{11} = k/(k+1)$, $a_{ii} = 1$ for $2 \le i \le d$, and $a_{ij} = 0$ if $i \ne j$.
Then it is easily checked that $b^r = I + rE_{d1}$ for every natural number r and that $ab^k = b^{k+1}a$. Write α for $\phi(a)$ and β for $\$ and $\alpha\beta^k = \beta^{k+1}\alpha$, so $\phi(b) = \beta = 1$ by the proposition. sily checked that $b^r = I + rE_{d1}$ for every n
Write α for $\phi(a)$ and β for $\phi(b)$. Then α is
 $k+1\alpha$, so $\phi(b) = \beta = 1$ by the proposition.

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