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$GL_d(R)$ IS NOT RESIDUALLY p

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Proposition . *Suppose that $p = 2k+1$ is an odd prime, that α and β are elements having p-power order in a group G and that $\alpha\beta^k = \beta^{k+1}\alpha$ in G. Then $\beta = 1$.*

Proof: Suppose that α has order p^s and that β has order p^t. If $t = 0$, we are done. We will prove the proposition by showing that $\beta^{p^{t-1}} = 1$.

Observe first that $\beta^{-k}\alpha\beta^k = \beta\alpha$, so that $\alpha, \beta\alpha$ and $\alpha\beta$ all have order p^s. By induction on r, $\alpha\beta^r$ has order p^s for all natural numbers r, since

$$1 = \left(\alpha\beta^r\right)^{p^s} = \left(\alpha\beta^k\beta^{r-k}\right)^{p^s} = \left(\beta^{k+1}\alpha\beta^{r-k}\right)^{p^s} = \beta^{k+1}\left(\alpha\beta^{r+1}\right)^{p^s}\beta^{-(k+1)}$$

If $t = 1$, then β has order $p = 2k+1$ and from $\alpha\beta^k = \beta^{k+1}\alpha$, we obtain $\beta^k\alpha\beta^k = \alpha$. More generally, β has order p^t. Then $\beta^{p^t-(k+1)}\alpha\beta^k = \alpha$. This is the base step for an easy induction on j showing that $\beta^{p^t-j(k+1)}\alpha\beta^{jk} = \alpha$ for all natural numbers j. With $j = p^{t-1}$, we have $p^t - j(k+1) = p^t - p^{t-1}(k+1) = kp^{t-1}$ and $\beta^{kp^{t-1}}\alpha\beta^{kp^{t-1}} = \alpha$.

Since p is odd, p^s is odd. For typesetting convenience, write N for $\frac{1}{2}(p^s-1)$ and r for kp^{t-1}. Then

$$1 = (\alpha\beta^r)^{p^s} = (\alpha\beta^r\alpha\beta^r)^N\alpha\beta^r = (\alpha\cdot\beta^{kp^{t-1}}\alpha\beta^{kp^{t-1}})^N\alpha\beta^r = (\alpha^2)^N\alpha\beta^r = \beta^r$$

Since k is relatively prime to p, and $r = kp^{t-1}$, we have $\beta^{p^{t-1}} = 1$.

Corollary 1. *Suppose that $p = 2k+1$ is an odd prime and let R be any ring in which k and $k+1$ are invertible elements. Let $d \geq 2$ be a natural number. Then the general linear group $GL_d(R)$ is not residually a finite p-group.*

Proof: Let E_{ij} be the $d\times d$ matrix over R with 1 in row i, column j and zeroes elsewhere. Let I be the $d\times d$ identity matrix and let b be the elementary transvection matrix $b = I + E_{d1}$. We want to show that whenever ϕ is a homomorphism of $GL_d(R)$ onto a finite p-group G, then we have $\phi(b) = 1$. Let a be the elementary diagonal matrix with $a_{11} = k/(k+1)$, $a_{ii} = 1$ for $2 \leq i \leq d$, and $a_{ij} = 0$ if $i \neq j$. Then it is easily checked that $b^r = I + rE_{d1}$ for every natural number r and that $ab^k = b^{k+1}a$. Write α for $\phi(a)$ and β for $\phi(b)$. Then α and β have p-power order and $\alpha\beta^k = \beta^{k+1}\alpha$, so $\phi(b) = \beta = 1$ by the proposition.