DECISION AND SEPARABILITY PROBLEMS FOR BAUMSLAG-SOLITAR SEMIGROUPS

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Abstract. We show that the semigroups $S_{k,\ell}$ having semigroup presentations $\langle a, b \ : \ ab^k = b^\ell a \rangle$ are residually finite and finitely separable. Generally, these semigroups have finite separating images which are finite groups and other finite separating images which are semigroups of order-increasing transformations on a finite partially ordered set. These semigroups thus have vastly different residual and separability properties than the Baumslag-Solitar groups which contain them.

0: Introduction and Notation

Let k and ℓ be nonnegative integers. Throughout, we will write $S_{k,\ell}$ for the semigroup having semigroup presentation $\langle a, b \ : \ ab^k = b^\ell a \rangle$. We regard these semigroups as being, in some respects, analogous to the groups $G_{k,\ell}$ having group presentations $Gp\langle a, b \ : \ ab^k = b^\ell a \rangle$. The groups $G_{k,\ell}$ are generally referred to as Baumslag-Solitar groups. Since the Baumslag-Solitar groups have one-relator presentations, it has long been known, by a theorem of Magnus, [15], that they have solvable word problems. The Baumslag-Solitar groups are HNN-extensions over the integers, so brief modern proofs that they have solvable word problems follow from Britton's Lemma. By a theorem of Baumslag and Solitar, [3], many of the Baumslag-Solitar groups are non-hopfian and hence are not residually finite [16]. For a more recent account of Baumslag-Solitar groups, see also [2].

When $k \geq 1$ and $\ell \geq 1$, it is known by a theorem of Adjan, [1, Section II, Theorem 3], that the natural homomorphism from $S_{k,\ell}$ to $G_{k,\ell}$ is an embedding, and hence the presentations given for $S_{k,\ell}$ also have solvable word problems. Moreover, [1, Section II, Theorem 1], a left-cancellation law holds in $S_{k,\ell}$ provided that $\ell \geq 1$ and a right-cancellation law holds in $S_{k,\ell}$ provided that $k \geq 1$. In Section 1, we will give direct explicit solutions of the word problem for the semigroups $S_{k,\ell}$ (even in the fairly simple cases where k and/or ℓ are 0). Later, we will outline an alternate proof of the cancellation laws.

When A is a set, regarded as an alphabet of letters, we will use A^+ to denote the free semigroup of nonempty words on A. The free monoid, A^* on A is the set of all words on A, including the empty word. We will write 1 for the empty word.

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When the semigroup S is given by a semigroup presentation $\langle A : R\rangle$, write ρ for the congruence on A^+ that is generated by R. Then the elements of S are equivalence classes $[w]_\rho$ or $[w]$ of words w from A^+. We will write $s = s'$ or $[w] = [w']$ to indicate equality of elements of the semigroup S and use $w \equiv w'$ to indicate equality of words in the free semigroup A^+ or the free monoid A^*. For a word w from A^+, we will write $|w|$ for the length of w as a word on the letters of the alphabet A. If $x \in A$ is a letter, we will write $|w|_x$ for the number of occurrences of x in w.

Given a semigroup presentation $\langle A : R \rangle$, the membership problem for the presentation is to decide, given a word w on A and a finite set $\{w_1, w_2, \dots, w_n\}$ of words on A, whether or not w is equivalent, modulo the congruence on the free semigroup on A generated by R, to an element of the subsemigroup generated by the image of $\{w_1, w_2, \dots, w_n\}$. In Section 2, we give a brief proof that the membership problem is always solvable for the semigroups $S_{k,\ell}$.

A semigroup S is **residually finite** if for any pair s, s' of distinct elements of S, there is a finite semigroup Q and a semigroup homomorphism Φ of S onto Q such that $\Phi(s) \neq \Phi(s')$. Equivalently, S is residually finite if and only if it is a subdirect product of finite semigroups. (See the analgous proof for groups in [21] or the analogous exercise for monoids in [14].)

A semigroup S is **finitely separable**, see e.g. [8], [9], if for every subsemigroup M of S and every element $s \in S - M$, there is a finite semigroup Q and a semigroup homomorphism Φ of S onto Q such that $\Phi(s) \notin \Phi(M)$.

A semigroup S has **finitely separable subsets**, see [8], [9], if for every subset M of S and every element $s \in S - M$, there is a finite semigroup Q and a semigroup homomorphism Φ of S onto Q such that $\Phi(s) \notin \Phi(M)$. It is easily observed that a semigroup having finitely separable subsets is finitely separable and residually finite. We will denote by S^1 the monoid obtained from a semigroup S by adjoining a new unit 1 to S. In [9], Golubov has characterized semigroups with finitely separable subsets as follows: for elements x and y of the semigroup S, define $[y : x]$ to be $\{(u, v) \in S^1 \times S^1 : uxv = y\}$. Then S is a semigroup having finitely separable subsets if and only if for every fixed element $y \in S$ there are only a finite number of distinct sets $[y : x]$ as x ranges over all elements of S. In Section 2, will use this to show that the semigroups $S_{k,\ell}$ have finitely separable subsets. In Section 3, we will strengthen this result by showing that we can generally require that the finite images of $S_{k,\ell}$ have additional properties.

For any set X, the full transformation semigroup $\mathfrak{T}_X$ on X is the set of functions $X \to X$, with operation composition of functions. $\mathfrak{T}_X$ is a monoid. When X is a finite set with n elements then $\mathfrak{T}_X$ has order n^n. If $(\mathcal{P}, \preccurlyeq)$ is a partially ordered set, we will say that a transformation $f \in \mathfrak{T}_\mathcal{P}$ is **order-increasing** if $x \preccurlyeq xf$ for every $x \in \mathcal{P}$. Order-increasing transformations are also called extensive transformations. It is known that a finite semigroup S has a faithful representation by order-increasing transformations if and only if Green's relation $\mathcal{R}$ on S is the trivial relation. See Chapter 4 of Pin's book, [18], Varieties of Formal Languages, for a proof of the corresponding result for monoids. As usual, $x \preccurlyeq y$ means that either $x = y$ or else $x \prec y$, and we may describe the partial ordering on $\mathcal{P}$ either by the relation $\preccurlyeq$ on $\mathcal{P}$ or else by the relation $\prec$ on $\mathcal{P}$.

An element 0 in a semigroup S is a **zero element** if $0 \cdot s = s \cdot 0 = 0$ for every $s \in S$. Such an element is unique. If S is a semigroup with a zero element 0, then an element $s \in S$ is **nilpotent** if $s^n = 0$ for some natural number n. A semigroup S with a zero element is **nilpotent** if every element of S is nilpotent. It is not difficult to prove that every finite nilpotent semigroup is $\mathcal{R}$-trivial and hence has a faithful representation as a semigroup of order-increasing transformations on a finite partially ordered set.

We begin Section 3 by proving the following proposition (Proposition 3.1). If the semigroup S has a presentation $\langle A : R \rangle$ and $s \in S$ is represented by only a finite number of words on the alphabet A, then there is a homomorphism Φ of S onto a finite nilpotent semigroup such that $\Phi(s) \notin \Phi(S - s)$. As a first corollary of this proposition, provided that neither k nor ℓ is 0, the semigroups $S_{k,\ell}$ have finitely separable subsets where we can take the finite images to be finite nilpotent semigroups.

A word on the alphabet A is an R-word, with respect to the semigroup presentation $\langle A : R \rangle$, if it is one of the words in a pair from R. A piece, relative to the presentation, is a word which occurs as a subword of R-words in at least two distinct ways. For $n \geq 1$, the presentation satisfies the small overlap hypothesis $C(n)$ if no R-word can be written as a product of fewer than n pieces. In his thesis, [19], John Remmers proved that if a semigroup S has a finite presentation $\langle A : R \rangle$, which satisfies the small overlap hypotheses, $C(3)$, then the word problem for the presentation is solvable. Actually, he stated and proved a much stronger result which has the following easy consequence: for any element $s \in S$, there are only a finite number of words on the alphabet A which represent s in this presentation. For a published version of this work, see Theorems 5.2.14 and 5.2.15 in Peter Higgins' book [10]. Using Remmers' geometric method, Cummings and Goldstein, [6], proved that the word problem is also solvable for finite semigroup presentations which satisfy the small overlap hypothesis $C(2)$ and also a certain semigroup hypothesis $T(4)$. As the reviewer of their paper points out, they actually prove a conclusion that is strong enough to obtain the same consequence that is noted above for Remmers' result. As a second corollary of Proposition 3.1, every semigroup which has a finite presentation satisfying either $C(3)$ or else both $C(2)$ and $T(4)$ has finitely separable subsets and we may take the finite images to be nilpotent semigroups.

In Section 4, we prove that, for k and ℓ both at least 1, $S_{k,\ell}$ is residually a finite group: by this we mean that given any two distinct elements $s, s' \in S_{k,\ell}$ there is a finite group G and a semigroup homomorphism Φ from $S_{k,\ell}$ onto G such that $\Phi(s) \neq \Phi(s')$.

1: Normal Forms for $S_{k,\ell}$

Example 1.1. Let u be the element of $S_{2,3}$ which is represented by $b^5ab^5a^2b^4ab^4$. If we use the relation $ab^2 = b^3a$ to push b^3's to the right, we shorten the length of the word at every step and end with $(b^2a \cdot ba \cdot ba \cdot a) \cdot b^8$. If we use $ab^2 = b^3a$ to push b^2's to the left, we lengthen the word at every step and end with $b^{44} \cdot (a \cdot ab \cdot a \cdot a)$.

The next two lemmas include definitions for explicit normal forms for elements of $S_{k,\ell}$. We will need these normal forms in later sections of the paper. Readers familiar with either the van der Waerden trick or with string-rewriting techniques will regard the proofs of these lemmas as obvious.

Lemma 1.2: Right-hand normal forms for $S_{k,\ell}$**.** *Suppose that* $\ell \geq 1$*.*

(1) Every element s *of* $S_{k,\ell}$ *can be written as* $[wb^n]$ *where* $n \geq 0$ *and* w *is a word on the set* $W_\ell = \{a, ba, b^2a, \ldots, b^{\ell-1}a\}$*. (We may have that* $n = 0$ *or that* w *is the empty word, but not both of these for any element* s *of the semigroup* $S_{k,\ell}$*.)*

(2) Let $\mathcal{NR} = \{\, wb^n : n \geq 0, w$ *is a word on* W_ℓ *and* wb^n *is nonempty* $\}$*. Then distinct elements of* $\mathcal{NR}$ *represent distinct elements of* $S_{k,\ell}$*. We will call the element of* $\mathcal{NR}$ *which represents* $s \in S_{k,\ell}$*, the right-hand normal form for* s*.*

(3) Suppose that s *is an element of* $S_{k,\ell}$*, that* u *is the right-hand normal form for* s *and that* v *is any other word on the alphabet* $\{a, b\}$ *which also represents* s *in* $S_{k,\ell}$*. Then* $|u| < |v|$ *if* $k < \ell$*,* $|u| = |v|$ *if* $k = \ell$*, and* $|u| > |v|$ *if* $k > \ell$*.*

Proof. (1) Observe first that $[b^{q\ell}a] = [ab^{qk}]$ in $S_{k,\ell}$ for any natural number q. Let u be a word on $\{a, b\}$ which represents s in $S_{k,\ell}$. We prove by induction on $|u|_a$ that $[u]$ is equal in $S_{k,\ell}$ to $[v]$ where $v \equiv wb^n$ is a word in the desired form. If $|u|_a = 0$, then $u \equiv b^n$ for some $n > 0$ and we are done. If $|u|_a > 0$, then we may write $u \equiv u'ab^{n_2}$ for some u' and n_2, and we are again done if u' is the empty word. If u' is not empty, by the induction hypothesis, we have that $[u']$ is equal in $S_{k,\ell}$ to $[w'b^{n_1}]$ where $n_1 \geq 0$ and w' is a word on $\{a, ba, \ldots, b^{\ell-1}a\}$. Divide n_1 by ℓ to write $n_1 = q\ell + j_1$ where $0 \leq j_1 < \ell$. Then $[u] = [w'b^{n_1}ab^{n_2}] = [w'b^{j_1}b^{q\ell}ab^{n_2}] = [(w' \cdot b^{j_1}a) \cdot b^{qk+n_2}]$.

(2) We adapt a well-known argument of van der Waerden [22] to show that distinct elements of $\mathcal{NR}$ represent distinct elements of $S_{k,\ell}$. Write $\mathcal{NR}^*$ for $\mathcal{NR} \cup \{1\}$ where 1 is the empty word, and let $\mathfrak{T}_{\mathcal{NR}^*}$ be the full transformation semigroup on $\mathcal{NR}^*$. We want to define transformations α and β on $\mathcal{NR}^*$. First, $1\alpha = a$ and $1\beta = b$. If wb^n is a nontrivial element of $\mathcal{NR}^*$, write $n = q\ell + j$, where $0 \leq j < \ell$ and let $wb^n\alpha = (w \cdot b^j a)b^{qk}$. Define β at wb^n by $wb^n\beta = wb^{n+1}$. Let $\mathfrak{S}$ be the subsemigroup of $\mathfrak{T}_{\mathcal{NR}^*}$ that is generated by α and β. It is routine to verify that $\alpha\beta^k = \beta^\ell\alpha$ as elements of $\mathfrak{S}$, so we have a homomorphism $\Phi : S_{k,\ell} \to \mathfrak{S}$ defined by $\Phi([a]) = \alpha$ and $\Phi([b]) = \beta$. An easy argument by induction verifies that $1\Phi([wb^n]) \equiv wb^n$. From this, we may conclude that Φ is an isomorphism and that every element of $S_{k,\ell}$ has a unique representative in $\mathcal{NR}$.

(3) This follows easily using the same induction on $|u|_a$ as in part (1) of this proof.

Lemma 1.3: Left-hand normal forms for $S_{k,\ell}$**.** *Suppose that* $k \geq 1$*.*

(1) Every element s *of* $S_{k,\ell}$ *can be written as* $[b^n w]$ *where* $n \geq 0$ *and* w *is a word on the set* $W_k = \{a, ab, ab^2, \ldots, ab^{k-1}\}$*. (We may have that* $n = 0$ *or that* w *is the empty word, but not both of these for any element* s *of the semigroup* $S_{k,\ell}$*.)*

(2) Let $\mathcal{NL} = \{\, b^n w : n \geq 0, w$ *is a word on* W_k *and* $b^n w$ *is nonempty* $\}$*. Then distinct elements of* $\mathcal{NL}$ *represent distinct elements of* $S_{k,\ell}$*. We will call the element of* $\mathcal{NL}$ *which represents* $s \in S$*, the left-hand normal form for* s*.*

(3) Suppose that s *is an element of* $S_{k,\ell}$*, that* u *is the left-hand normal form for* s *and that* v *is any other word on the alphabet* $\{a, b\}$ *which also represents* s *in* $S_{k,\ell}$*. Then* $|u| > |v|$ *if* $k < \ell$*,* $|u| = |v|$ *if* $k = \ell$*, and* $|u| < |v|$ *if* $k > \ell$*.*

Proof. This result and its proof are dual to Lemma 1.2.

We can solve the word problem for $\langle a, b : ab^k = b^\ell a \rangle$ using Lemma 1.2 when $\ell > 0$ or by using Lemma 1.3 when $k > 0$. If $k = \ell = 0$, then $S_{0,0} = \{a, b\}^+$.

We can slightly strengthen Lemma 1.2 (and dually for Lemma 1.3) to show that if $k < \ell$ and v is an arbitrary word on $\{a, b\}$ which represents $s \in S$, then there is a sequence of elementary transitions from v to the right-hand normal form for s which shortens the length of the representative for s at every transition. If two words, v_1 and v_2 represent the same element of S, then by reducing both to their common right-hand normal form, we find a derivation for their equality which has length at most $|v_1| + |v_2|$. In particular, we see that the Dehn function is in this case at most linear. When $k = \ell$, it is not hard to see that the Dehn functions are bounded by quadratic functions.

2: $S_{k,\ell}$ Has Finitely Separable Subsets

Lemma 2.1. *Suppose that $\ell > 0$ and that u_1 and u_2 are words on the alphabet $\{a, b\}$ with $[u_1] = [u_2]$ in $S_{k,\ell}$. If $u_1 \equiv b^{n_1} a v_1$ and $u_2 \equiv b^{n_2} a v_2$ with $0 \le n_1 < \ell$ and $0 \le n_2 < \ell$, then $n_1 = n_2$ and $[v_1] = [v_2]$ as elements of $S_{k,\ell}$.*

Proof. Let w_1 be the right-hand normal form for v_1 and w_2 be the right-hand normal form for v_2. By the restrictions on n_1 and n_2, $b^{n_1} a w_1$ is the right-hand normal form for u_1 and $b^{n_2} a w_2$ is the right-hand normal form for u_2. Since $[u_1] = [u_2]$ in $S_{k,\ell}$, these normal forms are the same and we have that $n_1 = n_2$ and that $w_1 \equiv w_2$. By the definition of the normal forms, we have that $[v_1] = [w_1]$ and $[v_2] = [w_2]$ in $S_{k,\ell}$.

Lemma 2.2. *Let u and v be words on the alphabet $\{a, b\}$ such that $[u] = [v]$ in $S_{k,\ell}$. Then $|u|_a = |v|_a$.*

$$\textit{If}\quad 0 < k \le \ell,\ \textit{then}\quad |u|_b\,(k/\ell)^{|u|_a} \le |v|_b \le |u|_b\,(\ell/k)^{|u|_a}.$$

$$\textit{If}\quad 0 < \ell \le k,\ \textit{then}\quad |u|_b\,(\ell/k)^{|u|_a} \le |v|_b \le |u|_b\,(k/\ell)^{|u|_a}.$$

Proof. If $w \to w'$ is any elementary transition, replacing an occurrence of ab^k by $b^\ell a$ or vice versa, then $|w|_a = |w'|_a$. It follows that $|u|_a = |v|_a$.

By symmetry, it will suffice, for the rest, to prove that $|u|_b (k/\ell)^{|u|_a} \le |v|_b$ when $0 < k \le \ell$. We prove this by induction on $|u|_a$. If $|u|_a = |v|_a = 0$, then $u \equiv v \equiv b^n$, for some $n > 0$ and we are done, with equality in this case. Assume then that $|u|_a > 0$ and write $u \equiv b^{n_1} a u'$ and $v \equiv b^{n_2} a v'$. For $i = 1, 2$ divide n_i by ℓ to find $n_i = \ell q_i + j_i$ where $0 \le j_i < \ell$. Then $[u] = [b^{j_1} a b^{k q_1} u']$ in $S_{k,\ell}$ and $[v] = [b^{j_2} a b^{k q_2} v']$ in $S_{k,\ell}$. By Lemma 2.1, we see that $j_1 = j_2$ and $[b^{k q_1} u'] = [b^{k q_2} v']$ in $S_{k,\ell}$. Each of this last pair of words has $|u|_a - 1$ occurrences of a, so we may apply the induction hypothesis to conclude

$$|b^{k q_1} u'|_b\,(k/\ell)^{|u|_a - 1} \le |b^{k q_2} v'|_b \quad \text{or} \quad (k q_1 + |u'|_b)\,(k/\ell)^{|u|_a - 1} \le k q_2 + |v'|_b.$$

Then $|u|_b (k/\ell)^{|u|_a}$

$$\begin{aligned}
&=(\ell q_1 + j_1 + |u'|_b)(k/\ell)^{|u|_a} = (k q_1 + \frac{k}{\ell} j_1 + \frac{k}{\ell}|u'|_b)(k/\ell)^{|u|_a - 1} \\
&\le (k q_1 + j_1 + |u'|_b)(k/\ell)^{|u|_a - 1} \le j_1 + (k q_1 + |u'|_b)(k/\ell)^{|u|_a - 1} \\
&\le j_1 + k q_2 + |v'|_b \le j_1 + \ell q_2 + |v'|_b = |v|_b.
\end{aligned}$$

If $s \in S_{k,\ell}$ and u is any word on the alphabet $\{a, b\}$ which represents s, then we will define $|s|_a$ to be $|u|_a$. By the first conclusion of Lemma 2.2, this is unambiguous.

The next result is a known consequence of Theorem 2.5, below. The proof here gives a direct, explicit solution of the membership problem. For both Theorem 2.3 and Theorem 2.5, the most difficult cases are when $0 = k < \ell$ or $0 = \ell < k$. For these difficult cases of Theorem 2.3, the membership problem is known to be solvable in polynomial time by a result of R.V. Book, [4, Theorem 4.2], since the string-rewriting systems with rules $b^\ell a \to a$ or $ab^k \to a$ are convergent and monadic. The author thanks F. Otto for bringing Book's paper to his attention.

Theorem 2.3. *The membership problem is solvable for $\langle a, b : ab^k = b^\ell a\rangle$.*

Proof. Let v be a word on $\{a, b\}$ which represents some arbitrary element of $S_{k,\ell}$ and let $\mathcal{U} = \{u_1, u_2, \dots, u_N\}$ be a set of words representing the generators for some finitely generated subsemigroup of S. We need to exhibit an algorithm which will decide whether or not the element represented by v is in this subsemigroup.

If $k = \ell = 0$, then $S_{k,\ell}$ is the free semigroup on $\{a, b\}$. Let

$$W = \left\{ u_{i_1} u_{i_2} \dots u_{i_M} \ : \ u_i \in \mathcal{U} \ \text{ and } \sum_{m=1}^{M} |u_{i_m}| = |v| \right\}.$$

Then W is a finite set and we can examine whether or not any element of W is v.

If $0 = k < \ell$, write $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$, where $|u_i|_a = 0$ if $u_i \in \mathcal{U}_1$ and $|u_i|_a > 0$ if $u_i \in \mathcal{U}_2$. If $\mathcal{U}_1$ is empty, let

$$W = \left\{ u_{i_1} u_{i_2} \dots u_{i_M} \ : \ u_i \in \mathcal{U} \ \text{ and } \sum_{m=1}^{M} |u_{i_m}|_a = |v|_a \right\}.$$

If v is equal in $S_{k,\ell}$ to some product of elements of $\mathcal{U}$, then this product must occur in W. Since W is finite, we can calculate the right-hand normal form for each element of W and check whether or not one of these is also the right-hand normal form for v. If $\mathcal{U}_1$ is nonempty, rearrange the order of $\mathcal{U}$, if necessary so that $\mathcal{U}_1 = \{u_1, u_2, \dots, u_P\} = \{b^{n_1}, \dots, b^{n_P}\}$ and $\mathcal{U}_2 = \{u_{P+1}, \dots, u_N\}$. Let

$$\begin{aligned} \mathcal{B} &= \left\{ u_1^{j_1} u_2^{j_2} \dots u_P^{j_P} \ : \ 0 \le j_i < \ell \ \text{ for } \ 1 \le i \le P \right\} \\ &= \left\{ b^{\sum_1^P j_i n_i} \ : \ 0 \le j_i < \ell \ \text{ for } \ 1 \le i \le P \right\}. \end{aligned}$$

Since $[b^\ell a] = [a]$, we observe that if z is any word in the subsemigroup of $\{a, b\}^+$ generated by the words $\mathcal{U}_1$ and u is any nonempty word on $\mathcal{U}_2$, then there is a word z' in $\mathcal{B}$ such that $[zu] = [z'u]$ in $S_{k,\ell}$. Write v in its right-hand normal form as $w\dot{b}^n$ where $n \ge 0$ and w is a word on $\{a, ba, \dots, b^{\ell-1}a\}$.

$$\text{Let } W = \left\{ u'_{i_1} u_{i_1} u'_{i_2} u_{i_2} \dots u'_{i_M} u_{i_M} b^{n_{j_1}} b^{n_{j_2}} \dots b^{n_{j_R}} \ : \right.$$
$$\left. u'_i \in \mathcal{B},\ u_i \in \mathcal{U}_2,\ \sum_{m=1}^{M} |u_{i_m}|_a = |v|_a \ \text{ and } \sum_{r=1}^{R} n_{j_r} \le n \right\}.$$

If v is equal in $S_{k,\ell}$ to some product of elements of $\mathcal{U}$, then such a product must occur in W. Since $\mathcal{B}$ is finite, W is also finite and we can again compare the right-hand normal form for v to the right-hand normal forms for elements of W.

The case where $0 = \ell < k$ is similar to the previous case.

Assume next that $0 < k \leq \ell$. In this case, define W by

$$W = \Bigg\{ u_{i_1} u_{i_2} \dots u_{i_M} \ : \ u_i \in \mathcal{U}, \\ \sum_{m=1}^{M} |u_{i_m}|_a = |v|_a \ \text{ and } \sum_{m=1}^{M} |u_{i_m}|_b \leq |v|_b (\ell/k)^{|v|_a} \Bigg\}.$$

Then we know by Lemma 2.2 that if v is equal in $S_{k,\ell}$ to some product u of elements of $\mathcal{U}$, this product must occur in W. Since W is finite, we can calculate the right-hand normal form for each element of W and check whether or not one of these is also the right-hand normal form for v.

A similar argument works if $0 < \ell \leq k$.

Recall that for elements y and x of a semigroup S, the set $[y : x]$ is defined by $[y : x] = \{(u, v) \in S^1 \times S^1 \ : \ uxv = y\}$.

Theorem 2.4, Golubov, [9]**.** *In order that the semigroup S have finitely separable subsets, it is necessary and sufficient that for every element $y \in S$, there are only a finite number of distinct sets $[y : x]$ as x ranges over all elements of S.*

Theorem 2.5. *The semigroups $S_{k,\ell}$ have finitely separable subsets.*

Proof. By the theorem of Golubov Theorem just cited, it will suffice to show that for each $s \in S_{k,\ell}$, there are at most a finite number of distinct sets $[s : x]$.

If $k = \ell = 0$, then $S_{0,0}$ is the free semigroup on $\{a, b\}$ and $[s : x]$ is nonempty only when x is a subword of s. If $k > 0$ and $\ell > 0$, then we see by Lemma 2.2 that there are only a finite number of words on $\{a, b\}$ which represent s. If $[s : x]$ is nonempty, then x must be represented by one of the subwords of this finite set of words. We see again that there can be only a finite number of elements $x \in S_{k,\ell}$ for which $[s : x]$ is nonempty. For the remainder of the proof, we may assume by symmetry that $0 = k < \ell$.

Suppose first that $s = [b^n]$ for some $n \geq 1$. Then $[s : x]$ is nonempty only when $x = [b^{n'}]$ with $1 \leq n' \leq n$.

For any element $z \in S_{0,\ell}$ with $|z|_a > 0$, we may write the right-hand normal form of z as $b^{j_p} a \dots b^{j_2} a b^{j_1} a b^n$ where $n \geq 0$ and $0 \leq j_i < \ell$ for $1 \leq i \leq p$. In $S_{0,\ell}$, multiplication takes the form $[b^{i_q} a b^{i_{q-1}} a \dots b^{i_1} a b^{i_0}] \cdot [b^{i'_p} a b^{i'_{p-1}} a \dots b^{i'_1} a b^{i'_0}] = [b^{i_q} a b^{i_{q-1}} a \dots b^{i_1} a b^{r} a b^{i'_{p-1}} a \dots b^{i'_1} a b^{i'_0}]$ where i_0 and i'_0 are arbitrary whole numbers, but we may assume that other exponents on b are at most $\ell - 1$ and that the exponent r is obtained as the remainder when $i_0 + i'_p$ is divided by ℓ. Assume then that s has right-hand normal form $b^{j_m} a \dots b^{j_2} a b^{j_1} a b^n$ where $m \geq 1$ and $0 \leq j_i < \ell$ for $1 \leq i \leq m$. If $s = uxv$, then we must have $m = |u|_a + |x|_a + |v|_a$. We observe that $[s : x]$ can be nonempty only when $|x|_a \leq |s|_a = m$. If $[s : x]$ is nonempty and

$|x|_a = m'$, we write the right-hand normal form for x as $b^{j'_{m'}}ab^{j'_{m'-1}}a \ldots ab^{j'_1}ab^{n'}$ if $m' > 0$ or as $b^{n'}$ if $m' = 0$. If $[s : x]$ contains an element $(u, 1)$ or an element (u, b^t) then we must have $n' \le n$. Since $m' \le m$ and $0 \le j'_i < \ell$ for $1 \le i \le m'$, there can be only a finite number of different elements x such that $[s : x]$ contains an element $(u, 1)$ or an element (u, b^t). If every element $(u, v) \in [s : x]$ has $|v|_a \ge 1$, divide n' by ℓ to write $n' = q\ell + n''$ where $0 \le q$ and $0 \le n'' < \ell$. For $i \ge 0$ define x_i to be $b^{j'_{m'}}ab^{j'_{m'-1}}a \ldots ab^{j'_1}ab^{n''+i\ell}$ (or $b^{n''+i\ell}$ if $m' = 0$). Then $[s : x] = [s : x_i] = [s : x_0]$ for $i \ge 0$. Since there are only finitely many possible choices for x_0, there can be only finitely many different nonempty sets $[s : x_0]$ for which every element $(u, v) \in [s : x_0]$ has $|v|_a \ge 1$.

3: Semigroups Having Finitely Separable Subsets

Proposition 3.1. *Suppose that $\langle A : R \rangle$ is a semigroup presentation for the semigroup S, that ρ is the congruence on A^+ generated by R and write $[w]$ for the ρ-congruence class of the word $w \in A^+$. For $s \in S$, let $C_s = \{ w' \in A^+ : [w'] = s\}$. If C_s is a finite set, then there is a finite nilpotent semigroup Q and a homomorphism Φ from S onto Q such that $\Phi(s') \ne \Phi(s)$ whenever $s' \in S$ and $s' \ne s$.*

Proof. An element $s_d \in S^1$ is a left divisor of s if $s_d s'_d = s$ for some $s'_d \in S^1$. We are not requiring left divisors to be proper or to be nontrivial, so 1 and s are left divisors of s. Let $\mathcal{A}_s$ be the set of left divisors of s and let Ω be the set of elements of S which are not left divisors of s. Let $X_s = \mathcal{A}_s \cup \{\Omega\}$, so that the set Ω is an element of the set X_s. If $s = s_1 s_2$, then C_{s_1} and C_{s_2} must also be finite: if $w_1 \in C_{s_1}$ and $w_2 \in C_{s_2}$, then $w_1 w_2 \in C_s$. If $s_d s'_d = s$, then s_d must be represented in S by one or more of the finitely many initial subwords of one or more of the finitely many words in C_s. Thus X_s is finite, and the transformation semigroup $\mathfrak{T}_{X_s}$ is finite also. We observe that Ω is nonempty.

For a letter $a_i \in A$, define a transformation $\alpha_i \in \mathfrak{T}_{X_s}$ by $\Omega\alpha_i = \Omega$ and for $s_d \in \mathcal{A}_s$, $s_d\alpha_i = s_d[a_i]$ if $s_d[a_i]$ is also a left divisor of s and $s_d\alpha_i = \Omega$, otherwise.

Let Q be the subsemigroup of $\mathfrak{T}_{X_s}$ that is generated by $\{ \alpha_i : a_i \in A \}$. Define a semigroup homomorphism $\phi : A^+ \to Q$ by $\phi(a_i) = \alpha_i$. If $v_1 = v_2$ is an arbitrary relation from the set R of relations for S, write ν_1 and ν_2 for $\phi(v_1)$ and $\phi(v_2)$, respectively. We want to show that $\nu_1 = \nu_2$ so that ϕ induces a well-defined homomorphism $\Phi : S \to Q$. Observe that $\Omega\nu_1 = \Omega = \Omega\nu_2$. If s_p is a left divisor of s, then, by construction, $s_p\nu_i$ is either $s_p[v_i]$ or else Ω depending upon whether or not $s_p[v_i]$ is also a left divisor of s. Since $[v_1] = [v_2]$, we have that both $s_p\nu_1$ and $s_p\nu_2$ have value $s_p[v_1] = s_p[v_2]$ when this is a left divisor of s and both have value Ω otherwise.

If w is any word on A whose length is greater than every word in C_s, then $\Phi([w])$ is the zero element in Q. If $\prod \alpha_{i_j}$ is any element of Q, then for some sufficiently large N, $(\prod a_{i_j})^N$ cannot be one of the finitely many words in C_s and hence $(\prod \alpha_{i_j})^N$ must be the zero element in Q.

Finally, observe that $1\Phi(s) = s$ and that if s' is some element of S that is distinct from s, then either $1\Phi(s') = \Omega$, in the commonly occurring case where s' is not a left divisor of s, or else s' is one of the left divisors of s and $1\Phi(s') = s'$.

The author thanks S. Margolis for formulating and and giving a brief proof of the following result which should be regarded as a converse to Proposition 3.1. Suppose that A is a finite alphabet and that $\langle A : R\rangle$ is a semigroup presentation for the semigroup S. If S is residually a finite nilpotent semigroup, then every nonzero element of S has a finite congruence class for the congruence generated by R.

We can define a very natural partial order on the set $\mathcal{A}_s$ of left divisors of s by $s_1 \preceq s_2$ if and only if s_1 is a left divisor of s_2. It follows easily from the finiteness of C_s that $\preceq$ is antisymmetric. We extend this partial order to X_s by taking Ω to be the greatest element and observe that all of the transformations α_i are order-increasing.

Let Γ be the right Cayley graph of S^1 with respect to the generating set A. Then X_s can be obtained from the set of vertices in Γ by collapsing the set Ω to a single point. The partial order in the previous paragraph then follows from the natural order on Γ.

The proof of Proposition 3.1 works equally well if one replaces the set $\mathcal{A}_s$ of left divisors of s by the larger set of all divisors of s. Then Ω becomes the set of elements of S which are not divisors of s. This Ω is a two-sided ideal in S and the Rees quotient S/Ω is a finite nilpotent semigroup which separates the image of s from the images of all other elements of S. One might reasonably argue that this is a better way to prove the proposition, but this change will ruin the connection with the right Cayley graph discussed in the previous paragraph.

If the set R of relations in Proposition 3.1 is finite and we know some word w in the finite set C_s, then it is not difficult to see that we can effectively calculate all of C_s and then use this to construct X_s and Q. Since we first fix s and hypothesize only that the one set C_s is finite, we might construct this finite separating image Q for s even when the presentation $\langle A : R\rangle$ has an unsolvable word problem.

When the set R of relations in Proposition 3.1 is the empty set and S is the free semigroup on A, then each set C_s has exactly one element and the order $\prec$ on X_s is actually a linear order. As one consequence of Proposition 3.1, we obtain new proofs of well-known separability properties of free semigroups. Below, we state additional consequences.

Corollary 3.2. *Suppose neither k nor ℓ is 0. If $s \in S_{k,\ell}$, then there is a homomorphism Φ from $S_{k,\ell}$ onto a nilpotent semigroup such that $\Phi(s) \neq \Phi(s')$ for any $s' \in S_{k,\ell}$ that is distinct from s.*

Proof. If s is an element of $S_{k,\ell}$, let u be any word on the alphabet $\{a, b\}$ which represents s. By Lemma 2.2 we have a bound on length $|u'|_a + |u'|_b$ for all words u' on $\{a, b\}$ which represent s. Therefore, the set C_s of Proposition 3.1 is finite.

In the introduction, we stated the definition for the small overlap hypotheses, $C(n)$. Following [6] and [11], we state a definition for the semigroup hypothesis, $T(4)$. Assume for the semigroup presentation, $\langle A : R\rangle$, that the set R is transitively closed: that is, if $w_1 = w_2$ and $w_2 = w_3$ are both relations in R, then R also includes the relation $w_1 = w_3$. We will use $\mathcal{P}$ as a symbol for the presentation $\langle A : R\rangle$, here. Let $\bar{A}$ be a set disjoint from A but in one-to-one correspondence with A. The star (or coinitial) graph, $\mathcal{P}^{st}$, for the presentation $\mathcal{P}$, is a graph having $A \cup \bar{A}$ as its vertex set. The edge set for $\mathcal{P}^{st}$ is the union of three sets: (1) m-edges: If the segment $a_i a_j$ occurs in any R-word, we include an edge between the vertices a_i and $\bar{a}_j$ of $\mathcal{P}^{st}$. (2) i-edges: If $w_1 = w_2$ is a relation in R, a_i is the initial letter of

w_1, and a_j is the initial letter of w_2, then we include an edge in $\mathcal{P}^{st}$ between the vertices $\bar{a}_i$ and $\bar{a}_j$. We allow here the possibility that a_i and a_j are the same letter of A and the resulting edge is a loop. (3) t-edges: If $w_1 = w_2$ is a relation in R, a_i is the final letter of w_1, and a_j is the final letter of w_2, then we include an edge in $\mathcal{P}^{st}$ between the vertices a_i and a_j. Again, this edge will be a loop if the two words end in the same letter. Then the presentation $\mathcal{P}$ has the property $T(4)$ provided that every closed walk of length 3 in $\mathcal{P}^{st}$ either has all of its vertices in A or else has all of its vertices in $\bar{A}$. More generally, $\mathcal{P}$ has property $T(k+1)$ provided there are no closed walks in $\mathcal{P}^{st}$ of length k' for $3 \le k' \le k$ having the form $e\gamma_i e'\gamma_t$ where e and e' are m-edges, γ_i, if nonempty, is composed of i-edges, and γ_t, if nonempty, is composed of t-edges.

Theorem 3.3, (Remmers, [19, 10]**).** *Suppose that the semigroup S has a presentation, $\langle A : R\rangle$, which satisfies the small overlap hypothesis, $C(3)$ and that δ is a bound on the lengths of the R-words. If $w_1, w_2 \in A^+$ are words that represent the same element of S, then $|w_2| \le \delta|w_1|$.*

Theorem 3.4, (Cummings and Goldstein, [6]**).** *Suppose that the semigroup S has a finite presentation, $\langle A : R\rangle$, which satisfies the small overlap hypotheses, $C(2)$ and $T(4)$ and that δ is a bound on the lengths of the R-words. If $w_1, w_2 \in A^+$ are words that represent the same element of S, then $|w_2| \le 2\delta|w_1|$.*

Corollary 3.5. *Suppose that the semigroup S has a finite presentation $\langle A : R\rangle$ which satisfies either the small overlap hypothesis $C(3)$ or else the hypotheses $C(2)$ and $T(4)$ and let s be some element of S. Then there is a homomorphism Φ from S onto a nilpotent semigroup such that $\Phi(s) \ne \Phi(s')$ for any $s' \in S$ that is distinct from s.*

Proof. If $s \in S$, let w be a word on A which represents s. When the presentation satisfies $C(3)$, we see by the theorem of Remmers that every word in the set C_s of Proposition 3.1 has length at most $\delta|w|$. When the presentation satisfies $C(2)$ and $T(4)$, we see by the theorem of Cummings and Goldstein that every word in C_s has length at most $2\delta|w|$. Since the alphabet A is finite, we see that C_s is finite in either case and the conclusion follows from Proposition 3.1.

4 $S_{k,\ell}$ Is Residually a Finite Group

Theorem 4.1. *If k and ℓ are positive integers whose greatest common divisor is 1, then the semigroup $S_{k,\ell}$ is residually a finite metabelian group.*

Proof. Let s and s' be distinct elements of $S_{k,\ell}$. If $|s|_a \ne |s'|_a$, let G be a finite cyclic group with generator c whose order is greater than $\max\{\,|s|_a, |s'|_a\,\}$. Define $\Phi : S_{k,\ell} \to G$ by $\Phi([a]) = c$ and $\Phi([b]) = 1_G$. Then it is easily seen that Φ is well-defined and that $\Phi(s) = c^{|s|_a} \ne c^{|s'|_a} = \Phi(s')$.

For the rest of the proof, we will assume that $|s|_a = |s'|_a = m$ for some $m \ge 0$. Let u and u' be the right-hand normal forms for s and s', respectively. Write these as $u \equiv b^{j_m}a \ldots b^{j_2}ab^{j_1}ab^{j_0}$ and $u' \equiv b^{j'_m}a \ldots b^{j'_2}ab^{j'_1}ab^{j'_0}$ where $0 \le j_i, j'_i < \ell$ for $1 \le i \le m$ and $j_0, j'_0 \ge 0$. For an arbitrary right-hand normal form $w \equiv b^{J_M}a \ldots b^{J_2}ab^{J_1}ab^{J_0}$ where $M = |w|_a, 0 \le J_i < \ell$ for $1 \le i \le M$ and $J_0 \ge 0$, define the polynomial P_w of degree at most M by $P_w(x) = \sum_{i=0}^{M} J_i x^i$.

We want to show that the rational numbers $P_u(k/\ell)$ and $P_{u'}(k/\ell)$ are not equal. We use induction on m and our claim is obvious when $m = 0$. When $m > 0$, define right-hand normal forms v and v' by $u \equiv b^{j_m}av$ and $u' \equiv b^{j'_m}av'$, respectively. By this construction, we have $P_u(x) = j_m x^m + P_v(x)$ and $P_{u'}(x) = j'_m x^m + P_{v'}(x)$. Suppose then, for the sake of obtaining a contradiction, that $P_u(k/\ell) = P_{u'}(k/\ell)$. Then $0 = \ell^m\big(P_u(k/\ell) - P_{u'}(k/\ell)\big) = (j_m - j'_m)k^m + \ell\big(\sum_{i=0}^{m-1}(j_i - j'_i)\ell^{m-i-1}k^i\big)$. Thus ℓ divides $(j_m - j'_m)k^m$. Since ℓ and k are relatively prime by hypothesis and $0 \le j_m, j'_m < \ell$, we must have $j_m = j'_m$. Then $P_v(k/\ell) = P_{v'}(k/\ell)$, so $v = v'$ by the induction hypothesis and then $u = u'$. This contradicts our assumption that s and s' are distinct elements.

Let p be any prime which is relatively prime to both k and ℓ. Write the nonzero rational number $P_u(k/\ell) - P_{u'}(k/\ell)$ in reduced form as $p^t \cdot \frac{r}{q}$, where r and q are integers with $r \neq 0, q > 0$ and p is neither a factor of r nor a factor of q. We note that q must be a factor of ℓ^m and that $t \ge 0$. Let n be any natural number that is greater than t. By our choice of n, we have that the integer $\ell^m\big(P_u(k/\ell) - P_{u'}(k/\ell)\big)$ is not equivalent to 0 modulo p^n. We write $\mathbb{Z}_{p^n}$ for the ring of integers modulo p^n. We generally abuse notation by writing just c for the equivalence class, $[c]$, in $\mathbb{Z}_{p^n}$ of the integer c.

Since p is relatively prime to ℓ and k, using a standard result from elementary number theory, see [17], there is an integer g with $g\ell \equiv 1 \mod p^n$ and an integer h with $hk \equiv 1 \mod p^n$. Let f be the unique natural number with $gk \equiv f \mod p^n$ and $0 < f < p^n$. It is then easy to see that $f\ell \equiv k \mod p^n$, so that f is in some sense k/ℓ in the ring of integers modulo p^n and that $f(\ell h) \equiv 1 \mod p^n$, so that f is an invertible element in this ring. Below, we will use the easy observation that $\ell^m f^i \equiv \ell^{m-i}k^i \mod p^n$ for $0 \le i \le m$.

Define 2×2 matrices α and β over the ring $\mathbb{Z}_{p^n}$ by $\alpha = \begin{bmatrix} f & 0 \\ 0 & 1 \end{bmatrix}$ and $\beta = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then α and β are invertible matrices and generate a subgroup G of $GL_2(\mathbb{Z}_{p^n})$. It is easily checked that $\beta^r = \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}$ for all r. Using our construction of f with $f\ell \equiv k \mod p^n$, we also see that $\beta^\ell\alpha = \begin{bmatrix} f & 0 \\ k & 1 \end{bmatrix} = \alpha\beta^k$. Thus, we have a well-defined homomorphism $\Phi : S_{k,\ell} \to G$ defined by $\Phi([a]) = \alpha$ and $\Phi([b]) = \beta$. A routine induction on m shows that $\Phi(s) = \Phi([u]) = \begin{bmatrix} f^m & 0 \\ P_u(f) & 1 \end{bmatrix}$ and that $\Phi(s') = \begin{bmatrix} f^m & 0 \\ P_{u'}(f) & 1 \end{bmatrix}$. If $P_u(f) \equiv P_{u'}(f) \mod p^n$, then, $0 \equiv \ell^m\big(P_u(f) - P_{u'}(f)\big)$ $= \ell^m \sum_{i=0}^m (j_i - j'_i)f^i \equiv \sum_{i=0}^m (j_i - j'_i)\ell^{m-i}k^i = \ell^m\big(P_u(k/\ell) - P_{u'}(k/\ell)\big)$ and this contradicts our choice of n.

Routine calculations show that commutators in G have the form $\begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}$ for some element r in $\mathbb{Z}_{p^n}$ and then that any two such commutators commute with each other, so G is a metabelian group.

The original proof of Theorem 4.1 was much longer and the author sincerely thanks the anonymous referee for comments which have strengthened the conclusion and greatly shortened the proof. Those familiar with the p-adic integers, will understand that the proof above is a self-contained version of the following much shorter proof as outlined by the referee. Rather than using the normal forms for s and s' to choose a sufficiently large n, begin by embedding $S_{k,\ell}$ into the general linear group of 2×2 matrices over the rational numbers, replacing the f in the definition of α by k/ℓ. We still need to check that this is an embedding. Then, with p relatively prime to k and ℓ, the group generated by α and β embeds into the general linear group of 2×2 matrices over the p-adic integers, which is known, see [7], to be residually finite.

We want to show that $S_{k,\ell}$ is still residually a finite group if we remove the hypothesis that k and ℓ are relatively prime in the last theorem. If $d > 1$, let Π_d be the monoid having monoid presentation $Mon\langle a, b : b^d = 1\rangle$. We will show in Lemma 4.4 that Π_d is residually a finite group. There is a natural homomorphism from $S_{kd,\ell d}$ onto Π_d and also a natural homomorphism from $S_{kd,\ell d}$ onto $S_{k,\ell}$. These induce a natural homomorphism $\theta : S_{kd,\ell d} \to S_{k,\ell} \times \Pi_d$ where $\theta([u]) = ([u], [u])$ for any word u on $\{a, b\}$. We want to show that θ is one-to-one in some important cases. The following lemma will be useful.

Lemma 4.2. *If $\ell \geq 1$, then the semigroup $S_{k,\ell}$ is a left-cancellation semigroup. If $k \geq 1$, then the semigroup $S_{k,\ell}$ is a right-cancellation semigroup.*

Proof. Since the presentations for $S_{k,\ell}$ have no left cycles when $\ell \geq 1$ and have no right cycles when $k \geq 1$ this follows immediately from well-known results of Adjan [1]. For a geometric proof of these results of Adjan and related results, see the paper by Remmers [20], the account of Remmers' work given in Peter Higgins' book [10], and the paper by Kashintsev, [13] .

We outline a reasonably short direct proof of this lemma. To prove that $S_{k,\ell}$ is a left-cancellation semigroup, define for $[v] \in S_{k,\ell}$ a transformation $\lambda_{[v]} \in \mathfrak{T}_{S_{k,\ell}}$ by $\lambda_{[v]}([u]) = [vu]$. It is easily seen that $\lambda_{[a]}$ is one-to-one and it will suffice to show that $\lambda_{[b]}$ is also one-to-one. For this, we quickly see that $\lambda_{[b]}([u]) \neq \lambda_{[b]}([u'])$ if $|u|_a \neq |u'|_a$. If $|u|_a = |u'|_a = m$, but $u \neq u'$ then an induction on m shows that $\lambda_{[b]}([u]) \neq \lambda_{[b]}([u'])$.

We will write $\mathrm{GCD}(m, n)$ for the greatest common divisor of two integers, m and n. To distinguish between $[u]$ as an element of $S_{kd,\ell d}$, as an element of $S_{k,\ell}$, or as an element of Π_d, we will generally write s for $[u]$ as an element of $S_{kd,\ell d}$. Using $\mathrm{pr}_i((z_1, z_2, \ldots, z_n))$ for the i^{th} projection of an n-tuple $\mathbf{z}$, we can then write $\mathrm{pr}_1((\theta[u]))$ or $\mathrm{pr}_1(\theta(s))$ for occurrences of $[u]$ as an element of $S_{k,\ell}$ and write $\mathrm{pr}_2((\theta[u]))$ or $\mathrm{pr}_2(\theta(s))$ for occurrences of $[u]$ as an element of Π_d.

Lemma 4.3. *Define $\theta : S_{kd,\ell d} \to S_{k,\ell} \times \Pi_d$ by $\theta([u]) = ([u], [u])$ for any word u on the alphabet $\{a, b\}$. Then θ is a well-defined homomorphism. If either $\mathrm{GCD}(k, d) = 1$ or $\mathrm{GCD}(\ell, d) = 1$, then θ is one-to-one.*

Proof. By induction on n, it is easily seen that $[b^{\ell n} a] = [a b^{kn}]$ in $S_{k,\ell}$ for every natural number n. In particular, $[b^{\ell d} a] = [a b^{kd}]$ in $S_{k,\ell}$. Since $[b^{\ell d}] = [b^{kd}] = 1$ in Π_d, we have $\theta([b^{\ell d} a]) = ([b^{\ell d} a], [a]) = ([a b^{kd}], [a]) = \theta([a b^{kd}])$ and θ is well-defined.

We observe that if s is an element of any of the semigroups $S_{kd,\ell d}, S_{k,\ell}$ or Π_d and w and w' are both words on the alphabet $\{a, b\}$ which represent s as an element of that semigroup, then $|w|_a = |w'|_a$, so we may define $|s|_a$ to be this common value. When $s \in S_{kd,\ell d}$, then $|s|_a = |\,\mathrm{pr}_1(\theta(s))|_a = |\,\mathrm{pr}_2(\theta(s))|_a$.

Suppose that s and s' are distinct elements of $S_{kd,\ell d}$. If $|s|_a \neq |s'|_a$ in $S_{kd,\ell d}$, then $|\operatorname{pr}_1(\theta(s))|_a \neq |\operatorname{pr}_1(\theta(s))|_a$ and $|\operatorname{pr}_2(\theta(s))|_a \neq |\operatorname{pr}_2(\theta(s))|_a$ so $\theta(s) \neq \theta(s')$. We may thus assume that $|s|_a = |s'|_a = M$. If $M = 0$, then $s = [b^{j_0}]$ and $s' = [b^{j'_0}]$ with $j_0 \neq j'_0$. Then $\operatorname{pr}_1(\theta(s)) = [b^{j_0}] \neq [b^{j'_0}] = \operatorname{pr}_1(\theta(s'))$ in $S_{k,\ell}$, so we may continue with $|s|_a = |s'|_a = M > 0$. For this remaining case, we need to make use of the hypothesis that either $\mathrm{GCD}(k,d) = 1$ or that $\mathrm{GCD}(\ell,d) = 1$. By symmetry, we will assume that $\mathrm{GCD}(\ell,d) = 1$ and then write elements of $S_{kd,\ell d}$ and $S_{k,\ell}$ using their right-hand normal forms. We will assume that s and s' are distinct elements of $S_{kd,\ell d}$ with $|s|_a = |s'|_a = M > 0$ and also $\operatorname{pr}_2(\theta(s)) = \operatorname{pr}_2(\theta(s'))$ and we will prove that we must then have $\operatorname{pr}_1(\theta(s)) \neq \operatorname{pr}_1(\theta(s'))$.

Write $u \equiv b^{j_M} \cdots b^{j_2}ab^{j_1}ab^{j_0}$ for the right-hand normal form for s in $S_{kd,\ell d}$ and $u' \equiv b^{j'_M} \cdots b^{j'_2}ab^{j'_1}ab^{j'_0}$ for the right-hand normal form for s' in $S_{kd,\ell d}$ where $0 \leq j_i, j'_i < \ell d$ for $1 \leq i \leq M$. The assumption that we have $\operatorname{pr}_2(\theta(s)) = \operatorname{pr}_2(\theta(s'))$ is equivalent to the requirement that we have $j_i \equiv j'_i \mod d$ for $0 \leq i \leq M$. Let m be the largest i with $0 \leq i \leq M$ such that $j_i \neq j'_i$. If $m = 0$, then $j_0 \neq j'_0$, but $j_i = j'_i$ for $1 \leq i \leq M$. Let $v \equiv b^{j_M} \cdots b^{j_2}ab^{j_1}a$ so that $u \equiv vb^{j_0}$ and $u' \equiv vb^{j'_0}$. If we had $\operatorname{pr}_1(\theta(s)) = \operatorname{pr}_1(\theta(s'))$, then we would have $[v][b^{j_0}] = [u] = [u'] = [v][b^{j'_0}]$ in $S_{k,\ell}$. Since $S_{k,\ell}$ is a left-cancellation semigroup by Lemma 4.2, we would then have $[b^{j_0}] = [b^{j'_0}]$ in $S_{k,\ell}$ and $j_0 = j'_0$, a contradiction.

Assume then that $m \geq 1$. By symmetry, we may assume that $j_m > j'_m$. In this paragraph, define words v, u, and u' by $v \equiv b^{j_M}ab^{j_{M-1}} \ldots ab^{j_{m+1}}ab^{j'_m}$, $u \equiv b^{j_m - j'_m}ab^{j_{m-1}} \ldots ab^{j_1}ab^{j_0}$ and $u' \equiv ab^{j'_{m-1}}a \ldots ab^{j'_1}ab^{j'_0}$ so that $s = [v][u]$ and $s' = [v][u']$ either as elements of $S_{kd,\ell d}$ or as elements of $S_{k,\ell}$. By Lemma 4.2 again, it will suffice to show that $[u] \neq [u']$ as elements of $S_{k,\ell}$. (If $j'_M = 0$ and $j_M > 0$, then $m = M, s = [u], s' = [u']$ and v is vacuous, but generally, we want to reduce to this situation.) As noted above, we are assuming that $j_i \equiv j'_i \mod d$ for $0 \leq i \leq M$. In particular, $j_m \equiv j'_m \mod d$. To begin the reduction of $u \equiv b^{j_m - j'_m}ab^{j_{m-1}} \ldots ab^{j_1}ab^{j_0}$ to its right-hand normal form in $S_{k,\ell}$, we should divide $j_m - j'_m$ by ℓ to find a quotient q and a remainder r : $\quad j_m - j'_m = q\ell + r$ where $q \geq 0$ and $0 \leq r < \ell$. Then, in $S_{k,\ell}$, $\quad [u] = [b^r ab^{qk + j_{m-1}}ab^{j_{m-2}} \ldots b^{j_1}ab^{j_0}]$. We continue by reducing $b^{qk+j_{m-1}}ab^{j_{m-2}} \ldots b^{j_1}ab^{j_0}$ to its right-hand normal form in $S_{k,\ell}$ and similarly reducing u' to its right-hand normal form in $S_{k,\ell}$. If we show that $r > 0$, then we are done, since the right-hand normal forms for $[u]$ and $[u']$ in $S_{k,\ell}$ will have different exponents on the left-most b. Suppose, for the sake of obtaining a contradiction, that $r = 0$. Then $j_m - j'_m \equiv 0 \mod \ell$ and $j_m - j'_m \equiv 0 \mod d$. Since we assume here that $\mathrm{GCD}(\ell, d) = 1$, we then have $j_m - j'_m \equiv 0 \mod (\ell d)$. But $0 \leq j_m, j'_m < \ell d$, so this leads to $j_m = j'_m$, contradicting our choice of m.

Lemma 4.4. *Suppose $d > 1$ and that Π_d is the monoid having monoid presentation $Mon\langle a, b \ : \ b^d = 1\rangle$. If s and s' are distinct elements of Π_d, then there is a finite group G and a homomorphism Φ from Π_d onto G such that $\Phi(s) \neq \Phi(s')$.*

Proof. Π_d is the monoid free product of the infinite free cyclic monoid with generator a with the cyclic group having generator b of order d. Every element s in this free product Π_d can be uniquely represented by a word $b^{j_m}ab^{j_{m-1}}a \ldots b^{j_1}ab^{j_0}$ where $0 \leq j_i < d$ for $0 \leq i \leq m = |s|_a$. If s and s' are two elements of Π_d with $|s|_a = m$ and $|s'|_a = m'$, choose $n > \max\{m, m'\}$ and let $G_1 = Gp\langle a, b \ : \ a^n = 1,\, b^d = 1\rangle$.

Then, both as a monoid, and as a group, G_1 is the free product of the finite cyclic group of order n having generator a and the finite cyclic group of order d having generator b. There is a natural homomorphism from Π_d onto G_1 and the elements s and s' have distinct images σ and σ' under this homomorphism. Then $\sigma^{-1}\sigma'$ is a nontrivial element of G_1. Since G_1 is a free product of two finite groups, G_1 is a residually finite group. See [5, Section 1.3]. If the group G is a finite image of G_1 in which $\sigma^{-1}\sigma'$ is nontrivial, then the images of σ and σ' are distinct in G.

Theorem 4.5. *If $K > 0$ and $L > 0$, then the semigroup $S_{K,L}$ is residually a finite group.*

Proof. If $K = L = 1$, then $S_{K,L}$ is the free commutative semigroup on $\{a, b\}$, and every element of $S_{K,L}$ can be written uniquely in the form $a^m b^n$. Given two elements $a^{m_1}b^{n_1}$ and $a^{m_2}b^{n_2}$ choose $p > \max\{m_1, m_2\}$ and $q > \max\{n_1, n_2\}$. Let G be the direct product of two cyclic groups with orders p and q, respectively. Then it is easy to see that the function taking a to a generator of the first direct factor and b to a generator of the second direct factor is a homomorphism which separates $a^{m_1}b^{n_1}$ and $a^{m_2}b^{n_2}$.

If $1 \leq K < L$ and $\mathrm{GCD}(K, L) = 1$, then the result follows from Theorem 4.1 and the case where $1 \leq L < K$ and $\mathrm{GCD}(K, L) = 1$ is similar.

If $K = L > 1$, then $S_{K,K}$ embeds in the direct product $S_{1,1} \times \Pi_K$ by Lemma 4.3. If s and s' are distinct elements of $S_{K,K}$, then they have distinct images in either $S_{1,1}$ or else in Π_K. If they have distinct images in $S_{1,1}$, then we can separate these images in a finite group by the first paragraph of this proof. If they have distinct images in Π_K, then we are done by Lemma 4.4.

The case where $1 \leq L < K$ and $\mathrm{GCD}(K, L) = D > 1$ is similar to the case where $1 \leq K < L$ and $\mathrm{GCD}(K, L) = D > 1$ and we treat only the latter case. For this case, let $k = K/D$ and $\ell = L/D$. Every prime factor of D must be relatively prime to at least one of k or ℓ. Let d_1 be the product of all of the prime factors of D which are relatively prime to k and let d_2 be the product of the remaining prime factors of D. Then d_2 is relatively prime to ℓ and to ℓd_1. If $d_1 = 1$, then $D = d_2$ is relatively prime to ℓ and if $d_2 = 1$, then $D = d_1$ is relatively prime to k. In either case, we have that the homomorphism $\theta : S_{K,L} \to S_{k,\ell} \times \Pi_D$ is one-to-one by Lemma 4.3. If neither d_1 nor d_2 is 1, then $K = kd_1d_2$ and $L = \ell d_1 d_2$ where d_2 is relatively prime to ℓd_1 and d_1 is relatively prime to k. Making two applications of Lemma 4.3, we see that $S_{K,L}$ embeds in the product $(S_{k,\ell} \times \Pi_{d_1}) \times \Pi_{d_2}$ where each of the three direct factors is residually a finite group.

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