Version of 10-18-2001 BASIC COMMUTATORS AS RELATORS

David A. Jackson, Anthony M. Gaglione, and Dennis Spellman

ABSTRACT. Charles Sims, [7] has asked whether or not the lower central subgroup ABSTRACT. Charles Sims, [7] has asked whether or not the lower central subgroup $\gamma_n(F)$ of a free group F coincides with the normal closure in F of the set of basic commutators of weight n. Here, we investigate variations ABSTRACT. Charles Sims, [*i*] has asked whether or not the lower central subgroup $\gamma_n(F)$ of a free group *F* coincides with the normal closure in *F* of the set of basic commutators of weight *n*. Here, we investigate va $\gamma_n(F)$ ot a tree group F concides with the normal closure in F ot the set of basic
commutators of weight n. Here, we investigate variations of this question where we
consider other varieties or other sets of commutators. consider other varieties or other sets of commutators. We also give brief new proofs
that this question has a positive answer in the previously known cases for weight n
at most 4 and for the case where $n = 5$ and F is fre that this question has a positive answer in the previously known cases for weight n that this question has a positive answer in the previously known cases for weight *n* at most 4 and for the case where $n = 5$ and *F* is free with rank 2. We show that for $n \geq 4$, $\gamma_n(F)$ is the normal closure in *F* of at most 4 and for the case where $n = 5$ and F is free with rank 2. We show that
for $n \geq 4$, $\gamma_n(F)$ is the normal closure in F of the set of basic commutators having
weights n through $2n - 4$, inclusive. This is a c for $n \geq 4$, $\gamma_n(F)$ is the normal closure in F of the set of basic commutators having
weights n through $2n-4$, inclusive. This is a common generalization of Sims' result
for weight 4 and a result of Martin Ward, but bo weights n thro
for weight 4 as
in the proof.

1: INTRODUCTION AND NOTATION

For a natural number $r \geq 2$, $F = F_r$ is the free group on the ordered alphabet $X = \{x_1, x_2, \ldots, x_r\}.$ number $r \geq 2$, $F = F_r$ is the free group on the ordered alphabet
 $f(x, x_r)$.

for $x^{-1}yx$ and [y, x] for the commutator $y^{-1}x^{-1}yx$. If A and B

w group G then AG is the pormal closure of A in G and [A B] is

 x_{f} $X = \{x_1, x_2, \ldots, x_r\}.$
We write y^x for $x^{-1}yx$ and $[y, x]$ for the commutator $y^{-1}x^{-1}yx$. If A and B are subsets of any group G, then A^G is the normal closure of A in G and $[A, B]$ is the subgroup of G that is genera We write y^x for $x^{-1}yx$ and $[y, x]$ for the commutator $y^{-1}x^{-1}yx$. If A and B are subsets of any group G, then A^G is the normal closure of A in G and $[A, B]$ is the subgroup of G that is generated by the set of eleme the subgroup of *G* that is generated by the set of elements $[a, b]$ where $a \in A$ and $b \in B$. We write $[z, x, y]$ for the commutator $[[z, x], y]$. The weight of a commutator *c* is denoted by wt(*c*). We follow Marshall Hall's c is denoted by $wt(c)$. We follow Marshall Hall's definition for basic commutators.

Fix an order on the alphabet X. (1) The basic commutators of weight one
Fix an order on the alphabet X. (1) The basic commutators of weight one
the letters of X taken in this order. (2) Having defined and ordered the Fix an order on the alphabet X. (1) The basic commutators of weight one
are the letters of X taken in this order. (2) Having defined and ordered the
hasic commutators of weight less than n the basic commutators of weight are the letters of X taken in this order. (2) Having defined and ordered the basic commutators of weight n are all are the letters of X taken in this order. (2) Having defined and ordered the
basic commutators of weight less than n, the basic commutators of weight n are all
of the commutators $[c_i, c_j]$ which satisfy the conditions: (a) basic commutators of weight less than *n*, the basic commutators of weight *n* are all
of the commutators $[c_i, c_j]$ which satisfy the conditions: (a) c_i and c_j are basic
commutators with $n = \text{wt}(c_i) + \text{wt}(c_j)$. (b) In t commutators with $n = \text{wt}(c_i) + \text{wt}(c_j)$. (b) In the order that has been chosen
for basic commutators of weight less than $n, c_j < c_i$. (c) If $c_i = [c_s, c_t]$ where c_s of the commutators $[c_i, c_j]$ which satisfy the conditions: (a) c_i and c_j are basic commutators with $n = \text{wt}(c_i) + \text{wt}(c_j)$. (b) In the order that has been chosen for basic commutators of weight less than $n, c_j < c_i$. (c) for basic commutators of weight less than n, $c_j < c_i$. (c) If $c_i = [c_s, c_t]$ where c_s and c_t are basic commutators, then $c_t \leq c_j$ in the order that has been chosen for basic commutators of weight less than n. (3) The ba basic commutators of weight less than n. (3) The basic commutators of weight n follow all of the basic commutators of weight less than n in the order for the basic commutators of weight n may be ordered arbitrarily. commutators of weight less than $n+1$, but the basic commutators of weight n may

While we may order the commutators of weight n arbitrarily, the choices that we be ordered arbitrarily.
While we may order the commutators of weight n arbitrarily, the choices that we
make will have consequences for which commutators of higher weights are basic and
which are not. For example, depen While we may order the commutators of weight *n* arbitrarily, the choices that we make will have consequences for which commutators of higher weights are basic and which are not. For example, depending upon which order we which are not. For example, depending upon which order we use for the commuta-
tors $[x_2, x_1]$ and $[x_3, x_1]$ in weight 2, either $[[x_2, x_1], [x_3, x_1]]$ or else $[[x_3, x_1], [x_2, x_1]]$

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will be a basic commutator in weight 4. Following the terminology of Sims [7] we will be a basic commutator in weight 4. Following the terminology of Sims [7] we
will say that our choices determine a basic sequence of commutators. Whenever
we represent a basic sequence of commutators as a subscripted l will be a basic commutator in weight 4. Following the terminology of Sims [7] we
will say that our choices determine a basic sequence of commutators. Whenever
we represent a basic sequence of commutators as a subscripted l we represent a basic sequence of commutators as a subscripted list, the subscripts will reflect the order that we have chosen for the basic sequence and we will have $c_j < c_k$ if and only if $j < k$. I reflect the order that we have chosen for the basic sequence and we will have
 $\langle c_k$ if and only if $j \langle k$.

Write γ_n for the n^{th} term, $\gamma_n = \gamma_n(F) := [\gamma_{n-1}(F), F]$ of the lower central

is for F Let $C = \{x_i, x_i\}$,

 $c_j < c_k$ if and only if $j < k$.
Write γ_n for the n^{th} term, $\gamma_n = \gamma_n(F) := [\gamma_{n-1}(F), F]$ of the lower central
series for F. Let $C = \{x_1, x_2, \ldots, x_r, c_{r+1}, \ldots\}$ be any fixed basic sequence of
commutators that begins with th Write γ_n for the n^{th} term, $\gamma_n = \gamma_n(F) := [\gamma_{n-1}(F), F]$ of the lower central series for F. Let $C = \{x_1, x_2, \ldots, x_r, c_{r+1}, \ldots\}$ be any fixed basic sequence of commutators that begins with the ordered alphabet X. Througho series for F. Let $C = \{x_1, x_2, \ldots, x_r, c_{r+1}, \ldots\}$ be any fixed basic sequence of
commutators that begins with the ordered alphabet X. Throughout this paper, we
will let N_n denote the normal closure in F of the set \mathcal commutators that begins with the ordered alphabet X. Throughout this paper, we will let N_n denote the normal closure in F of the set \mathcal{R}_n of basic commutators of weight n from C. In [7], Charles Sims raised the que weight *n* from *C*. In [7], Charles Sims raised the question of whether or not $\gamma_n = N_n$ and answered this question positively for $n \le 4$ and for $n = 5, r = 2$. Our concern throughout this paper will be variations upon thi some useful tools for computing with commutators. We use these to give a brief throughout this paper will be variations upon this question. In section 2, we present
some useful tools for computing with commutators. We use these to give a brief
proofs of Sims' results that $\gamma_4(F) = N_4$ and that γ_5 In section 3, we prove that the analogues of Sims' question for the varieties $[\mathfrak{A}, \mathfrak{A}]$ and $[\mathfrak{N}_2, \mathfrak{A}]$ have positive answers. In section 4, the principal result is that, for proots of Sims' results that $\gamma_4(F) = N_4$ and that $\gamma_5(F) = N_5$ when F has rank 2.
In section 3, we prove that the analogues of Sims' question for the varieties [20, 20]
and [90₂, 20] have positive answers. In section 4 $2n-4$ R $\mathbb{E}[Y_2, \mathfrak{A}]$ have positive answers. In section 4, the principal result is that, for ≥ 4 , $\gamma_n(F)$ is the normal closure in F of $\bigcup_{j=n}^{2n-4} \mathcal{R}_j$.
For the given alphabet X, a **simple** commutator of weight n

 $n \geq 4$, $\gamma_n(F)$ is the normal closure in F of $\bigcup_{j=n}^{n} R_j$.
For the given alphabet X, a **simple** commutator of weight n is a commutator
 $[x_{i_1}, x_{i_2}, \dots, x_{i_n}]$ where each x_i is a letter of X. It is not difficut to se 4, $\gamma_n(F)$ is the normal close

For the given alphabet X, a s
 x_{i_2}, \dots, x_{i_n} where each x_i is

normal closure in F of the set
 ζ_i is ζ_i $\cdots \zeta_i$ then the con For the given alphabet X, a **simple** commutator of weight n is a commutator $[x_{i_1}, x_{i_2}, \dots, x_{i_n}]$ where each x_i is a letter of X. It is not difficut to see that $\gamma_n(F)$ is the normal closure in F of the set of simple c $[x_{i_1}, x_{i_2}, \dots, x_{i_n}]$ where each x_i is a letter of X. It is not difficut to see that $\gamma_n(F)$ is
the normal closure in F of the set of simple commutators of weight n. If $i_1 > i_2$, and
 $i_2 \le i_3 \le \dots \le i_n$, then the commu . It is not difficut to see that $\gamma_n(F)$ is

imutators of weight *n*. If $i_1 > i_2$, and
 x_{i_2}, \dots, x_{i_n} is a basic commutator

tors This is noted by Sims [7] and $i_2 \leq i_3 \leq \cdots \leq i_n$, then the commutator $[x_{i_1}, x_{i_2}, \cdots, x_{i_n}]$ is a basic commutator
for any choice of basic sequence of commutators. This is noted by Sims [7] and
can be proved by a simple induction. Following stand for any choice of basic sequence of commutators. This is noted by Sims [7] and
can be proved by a simple induction. Following standard notation, [d, c; b, a] is
an abbreviation for $[[d, c], [b, a]]$ and more generally $[C; a_1, a$ can be proved by a simple induction. Following standard notation, $[d, c; b, a]$ is
an abbreviation for $[[d, c], [b, a]]$ and more generally $[C; a_1, a_2, \ldots a_k]$ abbreviates
 $[C, [a_1, a_2, \ldots a_k]]$.
The following lemma, together with Gr

[C, $[a_1, a_2, \ldots, a_k]$].
The following lemma, together with Groves' Lemma and Proposition W in the next section, will also be used in our related paper, [5].

Basic Lemma. Let $r \geq 2$ and $n \geq 2$ be fixed. Suppose that $N_{n-1} = \gamma_{n-1}$. If for
every basic commutator c of weight $n-1$ and every $x \in X$ we have $[x] \in N$ **Example 1.** Every basic commutator c of weight $n-1$ and every $x \in X$, we have $[c, x] \in N_n$,
then $N - \infty$ then $N_n = \gamma_n$. every basic commutator c of weight $n-1$ and every $x \in X$, we have $[c, x] \in N_n$,
then $N_n = \gamma_n$.
Proof. Write \mathcal{R}_{n-1} for the set of basic commutators of weight $n-1$. It is clear that

Proof. Write \mathcal{R}_{n-1} for the set of basic commutators of weight $n-1$. It is clear that N_n is always a subgroup of γ_n , so we need to show that $\gamma_n \subseteq N_n$. By hypothesis,
 $\gamma_{n-1} = N_{n-1} S_0 \gamma_{n-1} = [\gamma_{n-1} F] = [N_{n-1$ *Proof.* Write \mathcal{R}_{n-1} for the set of basic commutators of weight $n-1$. It is clear that N_n is always a subgroup of γ_n , so we need to show that $\gamma_n \subseteq N_n$. By hypothesis, $\gamma_{n-1} = N_{n-1}$, so $\gamma_n = [\gamma_{n-1}, F] = [N_{n$ $\gamma_{n-1} = N_{n-1}$, so $\gamma_n = [\gamma_{n-1}, F] = [N_{n-1}, F] = [(\mathcal{R}_{n-1})^F, F] = [\mathcal{R}_{n-1}, F]$ which is
contained in N_n since $[c, x] \in N_n$ for every $c \in \mathcal{R}_{n-1}$ and every $x \in X$.
2: GROVES' LEMMA contained in N_n since $[c, x] \in N_n$ for every $c \in \mathcal{R}_{n-1}$ and every $x \in X$.

2: GROVES' LEMMA
Lemma 2.1. Suppose $a, b, and c$ are elements of any group G. Then

erman 2.1. Suppose a, b, and c are elements of any group G. Then
\n
$$
[c, b, a] = ([b, a, a]^{c^a}([b, a, c]^{-1})^a [b, a, a]^{-1})^{[c, b]} [b, a, c]^{[c, b]} [b, a, b]^{c^b}
$$
\n
$$
([b, a, c]^{-1})^b [b, a, b]^{-1} ([c, a, c]^{[c, b]} [c, a, b]^{c^b} ([c, a, c]^{-1})^b)^{[b, a]}.
$$

Proof. It will suffice to show that the equation is valid in the free group on $\{a, b, c\}$.
Write both sides of the equation as words in this free group and observe that the reduced forms are equal. See also the first par *Proof.* It will suffice to show that the equation is valid in the free group on $\{a, b, c\}$.
Write both sides of the equation as words in this free group and observe that the reduced forms are equal. See also the first p reduced forms are equal. See also the first parts of Groves' Lemma and Proposition W, below.

BAS
Lemma 2.2. $N_3 = \gamma_3$.

Lemma 2.2. $N_3 = \gamma_3$.
Proof. By the Basic Lemma and the easy observation that $N_2 = \gamma_2$, we need only check that $[x, x, x]$ is in N_2 when $i < i < k$. This follows from the equation in **Proof.** By the Basic Lemma and the easy observation that $N_2 = \gamma_2$, we need only check that $[x_k, x_j, x_i]$ is in N_3 when $i < j < k$. This follows from the equation in Lemma 2.1 check that $[x_k, x_j, x_i]$ is in N_3 when $i < j < k$. This follows from the equation in Lemma 2.1.

In [3] Havas and Richardson published a wonderfully short proof by J.R.J. Groves In [3] Havas and Richardson published a wonderfully short proof by J.R.J. Groves
that the commutator $[b, a, b, a]$ is in the normal closure of the set of basic commuta-
tors of weight 4 on the alphabet $\{a, b\}$. In the foll In [3] Havas and Richardson published a wonderfully short proof by J.R.J. Groves
that the commutator $[b, a, b, a]$ is in the normal closure of the set of basic commuta-
tors of weight 4 on the alphabet $\{a, b\}$. In the foll tors of weight 4 on the alphabet $\{a, b\}$. In the following lemma, we dissect Groves' proof for generalization and further use.

Groves' Lemma. Let A, B and C be elements in any group G .

(i) $[C, B, A] = [C, B]^{-1}[C, A]^{-1}[B, A, C]^{-1}([C, B][C, A][C, A, B])^{[B, A]}$

(i) $[C, B, A] = [C, B]^{-1}[C, A]^{-1}[B, A, C]^{-1}([C, B][C, A][C, A, B])^{[B, A]}$
(ii) If $[B, A, C], [C, A, B], [C, A; B, A]$ and $[C, B; C, A]$ are trivial in G, then B $A1 - [C, B; B, A]$ (ii) If $[B, A, C], [C, A, B], [C, A; B, A]$ and $[C, B; C, A]$ are trivial in G, then $[C, B, A] = [C, B; B, A]$.

(i) $I_1 = [C, B; B, A]$.

(iii) If $[B, A, C]$, $[C, B, B]$, $[C, A, A]$, $[C, A, B]$ and $[C, A, C]$ are trivial in G, then B Al and $[C, B; B, A]$ are trivial in G. (iii) If $[B, A, C], [C, B, B], [C, A, A], [C, A, C], [C, B, A]$ and $[C, B, B, A]$ are trivial in G. [C, B, A] and [C, B; B, A] are trivial in G.
(iv) If $[C, B], [C, A, A]$ and $[C, A, B]$ are trivial in G, then so is $[B, A, C]$.

(v) If $[C, A; B, A], [C, B; B, A]$ and $[C, B; C, A]$ are trivial in G, then when any two of $[C, B, A], [B, A, C]$ and $[C, A, B]$ are trivial in G, the third is also.

(v) If $[C, B, A], [B, A, C]$ and $[C, A, B]$ are trivial in G, the third is also.

(vi) If $[C, A, C], [C, B, C], [C, A; B, A]$ and $[C, B; B, A]$ are trivial in G, then
 B $A] = [B \ A \ C]^{-1}[C \ B; C \ A][C \ A \ B][B, A]$ (vi) If $[C, A, C], [C, B, C], [C, A; B, A]$ and $[C, B; B, A]$ are trivial in G, then
 $[C, B, A] = [B, A, C]^{-1}[C, B; C, A][C, A, B]^{[B,A]}$.
 Proof. (i) Note that $BA = AB[B, A]$ and apply the commutator identity $[x, yz] = [x, z][x, y][x, y, z] - [x, z][x, y]^2$ to bot

[x, z, x] [x, x, y] [x, x, x, x][x, x, x]

Proof. (i) Note that $BA = AB[B, A]$ and apply the commutator identity $[x, yz] = [x, z][x, y][x, y, z] = [x, z][x, y]^z$ to both $[C, BA]$ and $[C, (AB)[B, A]]$. Equate these *Proof.* (i) Note that BA
[x, z][x, y][x, y, z] = [x, z][
and solve for [C, B, A].
(ii) With [$B \sim A \sim C$] and solve for $[C, B, A]$.

(ii) With $[B, A, C] = 1$, and $[C, A, B] = 1$, the equation in part (i) simpli-

and solve for $[C, B, A]$.

(ii) With $[B, A, C] = 1$, and $[C, A, B] = 1$, the equation in part (i) simplifies to $[C, B, A] = [C, B]^{-1}[C, A]^{-1}([C, B][C, A])^{[B, A]}$. With $[C, A, B, A] = 1$, and $[C, B, C, A] = 1$ we can write this as $[C, B, A] = [C, B]^{-1}[C$ (i) With $[B, A, C] = 1$, and $[C, A, B] = 1$, the equation in part (i) simples to $[C, B, A] = [C, B]^{-1}[C, A]^{-1}([C, B][C, A])^{[B, A]}$. With $[C, A; B, A] = 1$, and $[C, B; C, A] = 1$, we can write this as $[C, B, A] = [C, B]^{-1}[C, A]^{-1}[C, A][C, B]^{[B, A]}$
 $[C, B]^{ [B,A]$ thes to $[C, B, A] = [C, B]^{-1}[C, A]^{-1}([C, A), A]$
 $[C, B; C, A] = 1$, we can write this as $[$
 $=[C, B]^{-1}[C, B]^{[B, A]} = [C, B; B, A].$

(iii) Observe first that with $[C, A, A]$ $=[C, B]^{-1}[C, B]^{[B,A]} = [C, B; B, A].$
(iii) Observe first that with $[C, A, A], [C, A, B]$, and $[C, A, C]$ trivial in G, we do

have that $[C, A; B, A]$ and $[C, B; C, A]$ are trivial in G also, so we have $[C, B, A]$ = (iii) Observe first that with [C, A, A], [C, A, B], and [C, A, C] trivial in G, we do
have that [C, A; B, A] and [C, B; C, A] are trivial in G also, so we have [C, B, A] =
[C, B; B, A] by part (ii). Write this as [C, B, A have that [C, A; B, A] and [C, B; C, A] are trivial in G also, so we have [C, B, A] = [C, B; B, A] by part (ii). Write this as $[C, B, A] = [C, B, (B^{-1}A^{-1}B)A]$, and use $[x, yz] = [x, z][x, y]^z$ to rewrite this as $[C, B, A] = [C, B, A][C, B,$ $[C, B; B, A]$ by part (ii). Write this as $[C, B, A] = [C, B, (B^{-1}A^{-1}B)A]$, and use $[x, yz] = [x, z][x, y]^z$ to rewrite this as $[C, B, A] = [C, B, A][C, B, B^{-1}A^{-1}B]^A$. Multiply this last on the left by $[C, B, A]^{-1}$ and then conclude that $[C, B,$ $[x, yz] = [x, z][x, y]^z$ to rewrite this as $[C, B, A] = [C, B, A][C, B, B^{-1}A^{-1}B]^A$. Multiply this last on the left by $[C, B, A]^{-1}$ and then conclude that $[C, B, B^{-1}A^{-1}B]$ is trivial. It then follows, using the triviality of $[C, B, B]$ th is trivial. It then follows, using the triviality of $[C, B, B]$ that $[C, B, A^{-1}]$ is trivial and hence $[C, B, A]$ is trivial. Fivial. It then follows, using the triviality of $[C, B, B]$ that $[C, B, A^{-1}]$ is trivial
d hence $[C, B, A]$ is trivial.
(iv) Since $[C, B]$ and hence $[C, B, A]$ are trivial, from the first part of Groves'
mma we have $1 - [C, A]^{-1}[B, A$

and hence $[C, B, A]$ is trivial.

(iv) Since $[C, B]$ and hence [

Lemma we have $1 = [C, A]^{-1}[B]$

and $[C, A, B]$ are trivial in C we e [C, B, A] are trivial, from the first part of Groves'
[B, A, C]⁻¹([C, A][C, A, B])^[B,A]. Since both [C, A, A]
we obtain $1 - (\text{[B A C]^{-1})}^{[C,A]}$ and the triviality of (iv) Since $[C, B]$ and hence $[C, B, A]$ are trivial, from the first part of Groves'
Lemma we have $1 = [C, A]^{-1}[B, A, C]^{-1}([C, A][C, A, B])^{[B, A]}$. Since both $[C, A, A]$
and $[C, A, B]$ are trivial in G, we obtain $1 = ([B, A, C]^{-1})^{[C, A]}$ and $)^{\lfloor C, \cdot \rfloor}$ Lemma we have 1
and $[C, A, B]$ are ti
 $[B, A, C]$ follows.
 (v) From part (i) $(d [C, A, B]$ are trivial in G, we obtain $1 = ([B, A, C]^{-1})^{[C, A]}$ and the tri $A, C]$ follows.
(v) From part (i), with $[C, B, B, A]$ and $[C, A, B, A]$ trivial, we have

om part (i), with
$$
[C, B; B, A]
$$
 and $[C, A; B, A]$ trivial, we have

$$
[C, B, A] = [C, B]^{-1}[C, A]^{-1}[B, A, C]^{-1}[C, B][C, A][C, A, B]^{[B, A]}
$$

 $[C, B, A] = [C, B]^{-1}[C, A]^{-1}[B, A, C]^{-1}[C, B][C, A][C, A, B]^{[B, A]}$
If $[B, A, C]$ and $[C, A, B]$ are trivial, then $[C, B, A] = [C, B; C, A]$. If $[C, B, A]$ and
 $[B, A, C]$ are trivial, then $1 - [C, B; C, A][C, A, B][B, A]$, if $[C, B, A]$ and $[C, A, B]$ are If $[B, A, C]$ and $[C, A, B]$ are trivial, then $[C, B, A] = [C, B; C, A]$. If $[C, B, A]$ and $[B, A, C]$ are trivial, then $1 = [C, B; C, A][C, A, B]^{[B, A]}$. If $[C, B, A]$ and $[C, A, B]$ are

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trivial, then $[B, A, C] = [C, B][C, A][C, B]^{-1}[C, A]^{-1} = [C, B; C, A]^{([C, B][C, A])^{-1}}$.
See also Proposition W. below. trivial, then $[B, A, C] = [C, B][C, A][C, B]^{-1}[C, A]^{-1} = [C, B; C, A]^{([C, B][C, A])^{-1}}$.
See also Proposition W, below.
(vi) Since [*C, B*] commutes with both [*B, 4*] and with *C,* it commutes with See also Proposition W, below.
(vi) Since $[C, B]$ commutes with both $[B, A]$ and with C, it commutes with

See also Proposition W, below.

(vi) Since $[C, B]$ commutes with both $[B, A]$ and with C, it commutes with $[B, A, C]$. Similarly, $[C, A]$ commutes with $[B, A, C]$. Since $[C, B]^{[B,A]} = [C, B]$ and $[C, A][B,A] = [C, A]$ the result then foll (vi) Since $[C, B]$ commutes with both $[B, A]$ and with C , it commute $[B, A, C]$. Similarly, $[C, A]$ commutes with $[B, A, C]$. Since $[C, B]^{[B,A]} = [C, C, A]^{[B,A]} = [C, A]$, the result then follows from the equation of part (i). $[C, A]^{[B,A]} = [C, A]$, the result then follows from the equation of part (i).
Parts (i), (ii), and (iii) and their proofs are visible in [3]. We find it convenient to

Parts (i), (ii), and (iii) and their proofs are visible in [3]. We find it convenient to also place the other parts under the common umbrella of "Groves' Lemma." Both Groves' Lemma and Proposition W below were originally Parts (i), (ii), and (iii) and their proofs are visible in [3]. We find it convenient to also place the other parts under the common umbrella of "Groves' Lemma." Both Groves' Lemma and Proposition W, below, were originall Groves' Lemma and Proposition W, below, were originally formulated as tools to use in the proof that Sims' question has a positive answer for weight 5 and rank 3. Since these results do simplify some proofs in this paper, use in the proof that Sims' question has a positive answer for weight 5 and rank 3. here also. We make repeated use of all cases of Groves' Lemma in [5]. Equal to simplify some proofs in this paper, we have include the also. We make repeated use of all cases of Groves' Lemma in [5]. For any group G and elements x, y , and z in G , define $W(x, y, z)$ by

p G and elements x, y, and z in G, define
$$
W(x, y, z)
$$

$$
W(x, y, z) = [z, y]^{-1} [z, x]^{-1} [y, x]^{-1} [z, y][z, x][y, x].
$$

 $W(x, y, z) = [z, y]^{-1}[z, x]^{-1}[y, x]^{-1}[z, y][z, x][y, x].$
 $W(x, y, z)$ can be written in a number of different moderately lengthy forms. We will find it convenient to be able to make brief reference to this element without $W(x, y, z)$ can be written in a number of different moderately lengthy forms. We will find it convenient to be able to make brief reference to this element without immediately committing to any one of the forms will find it convenient to be able to make brief reference to this element without immediately committing to any one of the forms.

Proposition W. If G is any group and x, y, and z are elements of G, then

\n
$$
\begin{aligned}\n &\text{(i)} \quad [z, y, x] = \left([y, x, z]^{-1} \right)^{[z, x][z, y]} W(x, y, z)[z, x, y]^{[y, x]} \\
 &\text{(ii)} \quad W(x, y, z) = [z, y; z, x][z, y; y, x]^{[z, x]} [z, x; y, x]\n \end{aligned}
$$
\n

(*ii*)
$$
W(x, y, z) = [z, y; z, x][z, y; y, x]^{[z,x]}[z, x; y, x]
$$

(*iii*) $W(x, y, z) = [z, x; y, x]^{[z,y]}[z, y; y, x][z, y; z, x]$

(*ii*)
$$
W(x, y, z) = [z, y; z, x][z, y; y, x]^{[z,x]}[z, x; y, x]
$$

(*iii*) $W(x, y, z) = [z, x; y, x]^{[z,y]}[z, y; y, x][z, y; z, x]^{[y,x]}$

Proof. These may be verified by substitution and reduction in the free group on *Proof.* These may be verified by substitution and reduction in the free group on $\{x, y, z\}$. The subgroup of G generated by x, y and z is an image of this free group.
The first part of the proposition can also be reg *Proof.* These may be verified by substitution and reduction in the free group on $\{x, y, z\}$. The subgroup of G generated by x, y and z is an image of this free group.
The first part of the proposition can also be reg The first part of the proposition can also be regarded as a restatement of the first part of Groves' Lemma.

Like Lemma 2.2 above and Theorem 2.6 below, the following result is known Like Lemma 2.2 above and Theorem 2.6 below, the following result is known
by Sims, [7]. The proofs in [7] rely on a computer implementation of a string
rewriting algorithm. Our proofs here are reasonably short and direct. Like Lemma 2.2 above and Theorem 2.6 below, the following result is known
by Sims, [7]. The proofs in [7] rely on a computer implementation of a string
rewriting algorithm. Our proofs here are reasonably short and direct. by Sims, [7]. The proofs in [7] rely on a computer impler
rewriting algorithm. Our proofs here are reasonably short an
anticipates that used in [5] to prove that $N_5 = \gamma_5$ for all r.
Theorem 2.5. $N_4 = \gamma_4$ for arbitra

Theorem 2.5. $N_4 = \gamma_4$ *for arbitrary* $r \ge 2$.
Proof. By the Basic Lemma and Lemma 2.2, it will suffice to show that $[c, x_{\ell}] \in N_4$ (or equivalently $[c, x_{\ell}] = 1$ modulo N_1) whenever c is a basic commutator of w *Proof.* By the Basic Lemma and Lemma 2.2, it will suffice to show that $[c, x_{\ell}] \in N_4$ (or equivalently $[c, x_{\ell}] \equiv 1$ modulo N_4) whenever c is a basic commutator of weight 3 and $x_{\ell} \in X$. Then $c = [x, x, x]$ where $i \leq$ *Proof.* By the Basic Lemma and Lemma 2.2, it will suffice to show that $[c, x_{\ell}] \in N_4$ (or equivalently $[c, x_{\ell}] \equiv 1 \mod 10$ N_4) whenever c is a basic commutator of weight 3 and $x_{\ell} \in X$. Then $c = [x_j, x_i, x_k]$ where $i < j$ (or equivalently $[c, x_\ell] \equiv 1$ modulo N_4) whenever c is a basic commutator of weight 3 and $x_\ell \in X$. Then $c = [x_j, x_i, x_k]$ where $i < j$ and $i \le k$. If $\ell \ge k$, then $[c, x_\ell]$ is a basic commutator of weight 4, so we may ass 3 and $x_{\ell} \in X$. Then $c = [x_j, x_i, x_k]$ where $i < j$ and $i \le k$. If $\ell \ge k$, then $[c, x_{\ell}]$ is a basic commutator of weight 4, so we may assume that we have $\ell < k$ as well as $i < j$ and $i \le k$. We use part (iii) of Groves' Le as $i < j$ and $i \le k$. We use part (iii) of Groves' Lemma with $C = [x_j, x_i], B = x_k$,
and $A = x_\ell$. Then $[B, A] = [x_k, x_\ell]$ which is basic of weight 2, so $[B, A, C] =$ $i \leq k$. We use part (iii) of Groves' Lemma with $C = [x_j, x_i], B = x_k$,

Then $[B, A] = [x_k, x_\ell]$ which is basic of weight 2, so $[B, A, C] =$

] which, when nontrivial in F, is either a basic commutator of weight

inverse of such depe and $A = x_{\ell}$. Then $[B, A] = [x_{k}, x_{\ell}]$ which is basic of weight 2, so $[B, A, C] = [x_{k}, x_{\ell}; x_{j}, x_{i}]$ which, when nontrivial in F, is either a basic commutator of weight 4, or else the inverse of such, depending upon the order $[x_k, x_\ell; x_j, x_i]$ which, when nontrivial in F , is either a basic commutator of weight 4, or else the inverse of such, depending upon the order we have chosen in C for the basic commutators of weight 2. Also $[C, B, B] = [x_j,$

BASIC COMMUTATORS AS RELATORS $\frac{5}{5}$
commutator of weight 4, so it will suffice to show that $[C, A]$ commutes with A, B
and C modulo N . We will see that the following four cases naturally occurcommutator of weight 4, so it will suffice to show that $[C, A]$ commutes with and C modulo N_4 . We will see that the following four cases naturally occur: and C modulo N_4 . We will see that the following four cases naturally occur:

$$
(1) \quad \ell = i \qquad (2) \quad \ell > i \text{ and } j \ge \ell \qquad (3) \quad \ell > i \text{ and } j < \ell \qquad (4) \quad \ell < i
$$

(1) $\ell = i$ (2) $\ell > i$ and $j \ge \ell$ (3) $\ell > i$ and $j < \ell$ (4) $\ell < i$

Case 1: $\ell = i$ In this case, we have $A = x_{\ell} = x_i$, so that $[C, A] = [x_j, x_i, x_i]$. <u>Case 1:</u> $\ell = i$ In this case, we have $A = x_{\ell} = x_i$, so that $[C, A] = [x_j, x_i, x_i]$.
Then $[x_j, x_i, x_i, x_i]$, $[x_j, x_i, x_i, x_j]$ and $[x_j, x_i, x_i, x_k]$ are all basic commutators of weight A so $[C, A]$ commutes modulo N , with $A = x_i, B = x_i$ Case 1: $\ell = i$ In this case, we have $A = x_{\ell} = x_i$, so that $[C, A] = [x_j, x_i, x_i, x_i]$
Then $[x_j, x_i, x_i, x_i], [x_j, x_i, x_i, x_j]$ and $[x_j, x_i, x_i, x_k]$ are all basic commutators of weight 4, so $[C, A]$ commutes modulo N_4 with $A = x_i, B = x_k$ a weight 4, so $[C, A]$ commutes modulo N_4 with $A = x_i, B = x_k$ and $C = [x_i, x_i]$.

 $B = x_k$ and $C = [x_j, x_i]$.
 x_{ℓ} is a basic commutator
 $A \begin{bmatrix} A & A \end{bmatrix} = [x_1, x_2, x_3]$ are of weight 3 and then $[C, A, B] = [x_j, x_i, x_\ell, x_k]$ and $[C, A, A] = [x_j, x_i, x_\ell, x_\ell]$ are $[C, A] = [x_j, x_i, x_\ell]$ is a basic commutator
 $[x_\ell, x_k]$ and $[C, A, A] = [x_j, x_i, x_\ell, x_\ell]$ are

ns to show that $[C, A, C]$ is trivial modulo Case 2: $\ell > i$ and $j \ge \ell$ With $\ell > i$, $[C, A] = [x_j, x_i, x_\ell]$ is a basic commutator
of weight 3 and then $[C, A, B] = [x_j, x_i, x_\ell, x_k]$ and $[C, A, A] = [x_j, x_i, x_\ell, x_\ell]$ are
basic commutators of weight 4. It remains to show that $[C, A, C]$ of weight 3 and then $[C, A, B] = [x_j, x_i, x_\ell, x_k]$ and $[C, A, A] = [x_j, x_i, x_\ell, x_\ell]$ are
basic commutators of weight 4. It remains to show that $[C, A, C]$ is trivial modulo
 N_4 and for this it will suffice to show that $[x_j, x_i, x_\ell, x_i$ N_4 and for this it will suffice to show that $[x_j, x_i, x_\ell, x_i]$ and $[x_j, x_i, x_\ell, x_j]$ are trivial modulo N_4 . The former is trivial by case 1. Since we assume here that N_4 and for this it will suffice to show that $[x_j, x_i, x_\ell, x_i]$ and $[x_j, x_i, x_\ell, x_i]$
trivial modulo N_4 . The former is trivial by case 1. Since we assume h
 $j \ge \ell$, $[x_j, x_i, x_\ell, x_j]$ is a basic commutator of weight 4 and w vial modulo N_4 . The former is trivial by case 1. Since we assume her
 $\sum \ell$, $[x_j, x_i, x_\ell, x_j]$ is a basic commutator of weight 4 and we are done.

<u>Case 3:</u> $\ell > i$ and $j < \ell$ Exactly as in case 2, to show that $[x_j, x_i, x_k$

<u>Case 3:</u> $\ell > i$ and $j < \ell$ Exactly as in case 2, to show that $[x_i, x_i, x_k, x_{\ell}]$ is $t_j \geq \ell$, $[x_j, x_i, x_\ell, x_j]$ is a basic commutator of weight 4 and we are done.

<u>Case 3:</u> $\ell > i$ and $j < \ell$ Exactly as in case 2, to show that $[x_j, x_i, x_k, x_\ell]$ is trivial modulo N_4 , it will suffice to show that $[C, A, x_j] =$ Case 3: $\ell > i$ and $j < \ell$ Exactly as in case 2, to show that $[x_j, x_i, x_k, x_{\ell}]$ is trivial modulo N_4 , it will suffice to show that $[C, A, x_j] = [x_j, x_i, x_{\ell}, x_j]$ is trivial modulo N_4 . This is immediate from case 2, applied and $\ell = j$. Case 4: $\ell < i$ We again have that $[C, A] = [x_j, x_i, x_k]$, but with $\ell < i$, this is longer a basic commutator. We have in this case that $\ell < i < i$ and $\ell < i < k$.

<u>Case 4:</u> $\ell < i$ We again have that $[C, A] = [x_j, x_i, x_\ell]$, but with $\ell < i$, this is no longer a basic commutator. We have in this case that $\ell < i < j$ and $\ell < i \leq k$. Case 4: $\ell < i$ We again have that $[C, A] = [x_j, x_i, x_\ell]$, but with $\ell < i$, this is
no longer a basic commutator. We have in this case that $\ell < i < j$ and $\ell < i \le k$.
We need to use eventually that $[x_i, x_\ell, x_j]$ and $[x_j, x_\ell, x_i]$ a no longer a basic commutator. We have in this case that $\ell < i < j$ and $\ell < i \leq k$.
We need to use eventually that $[x_i, x_\ell, x_j]$ and $[x_j, x_\ell, x_i]$ are basic commutators of
weight 3 and that both of these commute with x_i, x_j, x_k We need to use eventually that $[x_i, x_\ell, x_j]$ and $[x_j, x_\ell, x_i]$ are basic commutators of
weight 3 and that both of these commute with x_i, x_j, x_k and x_ℓ . The commutators
 $[x_j, x_\ell, x_i, x_i], [x_j, x_\ell, x_i, x_j], [x_j, x_\ell, x_i, x_k]$ and $[x_i,$ weight 3 and that both of these commute with x_i, x_j, x_k and x_ℓ . The commutators $[x_j, x_\ell, x_i, x_i], [x_j, x_\ell, x_i, x_j], [x_j, x_\ell, x_i, x_k]$ and $[x_i, x_\ell, x_j, x_j]$ are basic commutators of weight 4. With a change of notation, $[x_i, x_\ell, x_j, x$ $[x_j, x_\ell, x_i, x_i], [x_j, x_\ell, x_i, x_j], [x_j, x_\ell, x_i, x_k]$ and $[x_i, x_\ell, x_j, x_j]$ are basic commutators
of weight 4. With a change of notation, $[x_i, x_\ell, x_j, x_\ell]$ and $[x_j, x_\ell, x_i, x_\ell]$ are trivial
modulo N_4 by case 1. Similarly, $[x_i, x_\ell, x_j$ $[x_i, x_\ell, x_j, x_k]$ is basic if $k \geq j$ and is trivial by case 2 if $k < j$. modulo N_4 by case 1. Similarly, $[x_i, x_\ell, x_j, x_i]$ is trivial modulo N_4 by case 2, while of the second or the third part of Proposition W, we see that $W(x_\ell, x_i, x_i)$
By either the second or the third part of Proposition W, we see that $W(x_\ell, x_i, x_j)$
rivial modulo N. Heing this and the first part of Proposition

is trivial modulo N_4 . Using this, and the first part of Proposition W, we have that By either the second or the third part of Proposition W, we see that $W(x_\ell, x_i, x_j)$ is trivial modulo N_4 . Using this, and the first part of Proposition W, we have that $[x_j, x_i, x_\ell]$ is equivalent to $([x_i, x_\ell, x_j]^{-1})^{[x_j, x$ is trivial modulo N_4 . Using this, and the first part of Proposition W, we have that $[x_j, x_i, x_\ell]$ is equivalent to $([x_i, x_\ell, x_j]^{-1})^{[x_j, x_\ell][x_j, x_i]} [x_j, x_\ell, x_i]^{[x_i, x_\ell]}$. Since we have just seen that $[x_i, x_\ell, x_j]$ and $[x_j, x$ $[x_j, x_i, x_\ell]$ is equivalent to $([x_i, x_\ell, x_j]^{-1})^{[x_j, x_\ell][x_j, x_\ell]}[x_j, x_\ell, x_i]$. Since we have
just seen that $[x_i, x_\ell, x_j]$ and $[x_j, x_\ell, x_i]$ commute with x_i, x_j and x_ℓ , we have that
 $[x_j, x_i, x_\ell] \equiv [x_i, x_\ell, x_j]^{-1}[x_j, x_\ell, x_i]$ mod x_k also, we are done.
 Theorem 2.6. If F is free with rank 2, then $N_5 = \gamma_5(F)$.

Proof. Let F be free on the ordered alphabet $\{a, b\}$. Then the basic commutators **Proof.** Let F be free on the ordered alphabet $\{a, b\}$. Then the basic commutators of weight 4 are $[b, a, a, a]$, $[b, a, a, b]$ and $[b, a, b, b]$. By Theorem 2.5 and the Basic Lemma it will suffice to prove that $[b, a, b, a]$ and *Proof.* Let *F* be free on the ordered alphabet $\{a, b\}$. Then the basic commutators of weight 4 are $[b, a, a, a]$, $[b, a, a, b]$ and $[b, a, b, b]$. By Theorem 2.5 and the Basic Lemma, it will suffice to prove that $[b, a, a, b, a]$ Lemma, it will suffice to prove that $[b, a, a, b, a]$ and $[b, a, b, b, a]$ are trivial modulo N_5 .
To show that $[b, a, a, b, a]$ and $[b, a, a, b; b, a]$ are trivial, we use part (iii) of Groves'

To show that $[b, a, a, b, a]$ and $[b, a, a, b; b, a]$ are trivial, we use part (iii) of Groves'
Lemma with $C = [b, a, a]$, $B = b$ and $A = a$. Then $[B, A, C]^{-1}$, $[C, B, B]$, $[C, A, A]$
and $[C, A, B]$ are basic commutators of weight 5. Since $[C$ To show that $[b, a, a, b, a]$ and $[b, a, a, b; b, a]$ are trivial, we use part (iii) of Groves'
Lemma with $C = [b, a, a]$, $B = b$ and $A = a$. Then $[B, A, C]^{-1}$, $[C, B, B]$, $[C, A, A]$
and $[C, A, B]$ are basic commutators of weight 5. Since $[C$ and $[C, A, B]$ are basic commutators of weight 5. Since $[C, A] = [b, a, a, a]$, it commutes with a, b and $C = [b, a, a]$ modulo N_5 .
To show that $[b, a, b, b, a]$ is trivial modulo N_5 , we first show that $[b, a, b; b, a, a]$ is trivial m

mutes with a, b and $C = [b, a, a]$ modulo N_5 .
To show that $[b, a, b, b, a]$ is trivial modulo N_5 , we first show that $[b, a, b; b, a, a]$
is trivial modulo N_5 and that $[b, a, b; b, a, a]$ is equivalent modulo N_5 to $[b, a, a, b]$.

Lemma with $C = [b, a, b, b, a, a]$ is equivalent modulo N_5 to $[b, a, a, b]$.

To show that $[b, a, b; b, a, a]$ is trivial modulo N_5 , we use part (iv) of Groves'

Lemma with $C = [b, a, a]$, $B = [b, a]$ and $A = b$. Then $[C, B]$ and $[C, A, A$

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commutators of weight 5. Since $[C, A] = [b, a, a, b]$, it commutes with $B = [b, a]$ by
the second paragraph of this proof commutators of weight 5. Since $[C, A]$
the second paragraph of this proof.
To show that $[h, a, b, a]$ is equivalent the second paragraph of this proof.
To show that $[b, a, b, a]$ is equivalent to $[b, a, a, b]$ modulo N_5 , we use part (vi) of

the second paragraph of this proof.
To show that $[b, a, b, a]$ is equivalent to $[b, a, a, b]$ modulo N_5 , we use part (vi) of
Groves' Lemma with $C = [b, a]$, $B = b$ and $A = a$. Then all of the commutators
 $[C \, A \, C] \, [C \, B \, C] \, [C$ To show that $[b, a, b, a]$ is equivalent to $[b, a, a, b]$ modulo N_5 , we use part (vi) of
Groves' Lemma with $C = [b, a]$, $B = b$ and $A = a$. Then all of the commutators
 $[C, A, C]$, $[C, B, C]$, $[C, A; B, A]$ and $[C, B; B, A]$ are basic comm Groves' Lemma with $C = [b, a]$, $B = b$ and $A = a$. Then all of the commutators $[C, A, C]$, $[C, B, C]$, $[C, A; B, A]$ and $[C, B; B, A]$ are basic commutators of weight 5, so we have $[b, a, b, a] \equiv [b, a; b, a]^{-1}[b, a, b; b, a, a][b, a, a, b]^{[b,a]}$. Sin [C, A, C], [C, B, C], [C, A; B, A] and [C, B; B, A] are basic commutators of weight 5,
so we have $[b, a, b, a] \equiv [b, a; b, a]^{-1}[b, a, b; b, a, a][b, a, a, b]^{[b, a]}$. Since $[b, a, b; b, a, a]$ is
trivial by the previous paragraph and $[b, a, a, b]$ trivial by the previous paragraph and $[b, a, a, b]$ commutes with $[b, a]$ by the second paragraph, we have $[b, a, b, a] \equiv [b, a, a, b]$.
To prove that $[b, a, b, b, a]$ is trivial modulo N_5 , we use part (iii) of Groves' Lemma

paragraph, we have $[b, a, b, a] \equiv [b, a, a, b]$.

To prove that $[b, a, b, b, a]$ is trivial modulo N_5 , we use part (iii) of Groves' Lemma

with $C = [b, a, b]$, $B = b$ and $A = a$. Then $[B, A, C]^{-1}$ and $[C, B, B]$ are basic com-

mutators To prove that $[b, a, b, b, a]$ is trivial modulo N_5 , we use part (iii) of Groves' Lemma
with $C = [b, a, b]$, $B = b$ and $A = a$. Then $[B, A, C]^{-1}$ and $[C, B, B]$ are basic com-
mutators of weight 5. By the previous paragraph, we ha mutators of weight 5. By the previous paragraph, we have that $[C, A]$ is equivalent to $[b, a, a, b]$, so $[C, A]$ commutes with $A = a$, $B = b$ and $C = [b, a, b]$.
3: VARIETIES WITH COMMUTATOR LAWS

3: VARIETIES WITH COMMUTATOR LAWS
Following [6], write $\mathfrak A$ for the variety of abelian groups and $\mathfrak N_2$ for the variety
groups which are pilpotent with pilpotency class at most 2. In this section, we Following [6], write $\mathfrak A$ for the variety of abelian groups and $\mathfrak N_2$ for the variety of groups which are nilpotent with nilpotency class at most 2. In this section, we want to prove that Sims' question has a posit of groups which are nilpotent with nilpotency class at most 2. In this section, we want to prove that Sims' question has a positive answer in the variety $[\mathfrak{A}, \mathfrak{A}]$ of metabelian groups and in the variety $[\mathfrak{N}_2,$ of groups which are nilpotent with nilpotency class at most 2. In this section, we
want to prove that Sims' question has a positive answer in the variety $[\mathfrak{A}, \mathfrak{A}]$ of
metabelian groups and in the variety $[\mathfrak{N}_2,$

 $[\gamma_3(F), \gamma_2(F)].$ (F), $\gamma_2(F)$

Equivalently, for a group G that is metabelian or a group G from the variety [$\mathfrak{N}_2, \mathfrak{A}$] (i.e., a group which satisfies the law $[[x_1, x_2, x_3], [x_4, x_5]]$), we can verify Equivalently, for a group G that is metabelian or a group G from the variety $[\mathfrak{N}_2, \mathfrak{A}]$ (i.e., a group which satisfies the law $[[x_1, x_2, x_3], [x_4, x_5]]$), we can verify that G is nilpotent of class $n - 1$ by confir [\mathfrak{N}_2 , \mathfrak{A}] (i.e., a group which satisfies the law $[[x_1, x_2, x_3], [x_4, x_5]]$), we can verify that G is nilpotent of class $n-1$ by confirming that all of the basic commutators of weight n are trivial in G.
By T

of weight *n* are trivial in *G*.
By Theorem 2.5, we may assume that $n \ge 5$. As noted in the introduction, $\gamma_n(F)$ is the normal closure in *F* of the set of simple commutators of weight *n*. Theorem 3.8 is then a consequence of the following theorem.

3.8 is then a consequence of the following theorem.
 Theorem 3.7. Suppose either that $n \geq 2$ and $K = [\gamma_2(F), \gamma_2(F)]$ or else that $n \geq 5$ and $K = [\gamma_3(F), \gamma_2(F)]$. If w is any simple commutator of weight n in F, then in F/K $n \geq 5$ and $K = [\gamma_3(F), \gamma_2(F)]$. If w is any simple commutator of weight n in F, then in F/K we will have one of the following three possibilities occur, where c_1 and c_2 are simple basic commutators of weight n.

> $wK = c_1K$ or $wK = c_1^{-1}K$ $1^{-1}K$ or $wK = c_1^{-1}c_2K$.

We will prove Theorem 3.7 after establishing a few easy facts about groups in We will prove Theorem 3.7 after establishing a few easy facts about groups in
the varieties $[\mathfrak{A}, \mathfrak{A}]$ and $[\mathfrak{N}_2, \mathfrak{A}]$. In terms of verifying nilpotence, Theorem 3.7 is
substantially stronger than Theorem 3.8 We will prove Theorem 3.7 after establishing a few easy facts about groups in
the varieties $[\mathfrak{A}, \mathfrak{A}]$ and $[\mathfrak{N}_2, \mathfrak{A}]$. In terms of verifying nilpotence, Theorem 3.7 is
substantially stronger than Theorem 3.8 substantially stronger than Theorem 3.8. To verify that a group G in one of these varieties is nilpotent of class $n - 1$ for some sufficiently large n , we need only show that the simple basic commutators of weight n varieties is nilpotent of class $n-1$ for some sufficiently large n, we need only show

If ^G is any metabelian group, then it is well-known and easily verified that for that the simple basic commutators of weight *n* are all trivial in *G*.

If *G* is any metabelian group, then it is well-known and easily verified that for

any elements $c_1, c_2, c_3 \in G'$ and $x \in G$, we have $[c_1c_2, x] = [c$ If G is any metabelian group, then it is well-known and easily verified that for
any elements $c_1, c_2, c_3 \in G'$ and $x \in G$, we have $[c_1c_2, x] = [c_1, x][c_2, x]$, $[c_1^{-1}, x] = [c_1, x]^{-1}$ and $c_3^{c_2} = c_3$. If G is any group fr any elements $c_1, c_2, c_3 \in G'$ and $x \in G$, we have $[c_1c_2, x] = [c_1, x][c_2, x]$, $[c_1^{-1}, x] = [c_1, x]^{-1}$ and $c_3^{c_2} = c_3$. If G is any group from the variety $[\mathfrak{N}_2, \mathfrak{A}]$, then it is similarly verified that for any ele $[c_1, x]^{-1}$ and $c_3^2 = c_3$. If G is any group
verified that for any elements $c_1, c_2 \in G$
 $[c_1, x][c_2, x], [c_1^{-1}, x] = [c_1, x]^{-1}$ and $c_3^{c_2}$.
Recall that we have defined the elements verified that for any elements $c_1, c_2 \in G'$, $c_3 \in \gamma_3(G)$ and $x \in G$, we have $[c_1c_2, x] = [c_1, x][c_2, x]$, $[c_1^{-1}, x] = [c_1, x]^{-1}$ and $c_3^{c_2} = c_3$.
Recall that we have defined the element $W(a, b, c)$ in any group G by $W(a,$

[c₁, x]|(c₂, x], [c₁⁻¹, x] = [c₁, x]⁻¹ and c₃⁻² = c₃.
Recall that we have defined the element $W(a, b, c)$ in any group G by $W(a, b, c)$ = [c, b]⁻¹[c, a]⁻¹[b, a]⁻¹[c, b][c, a][b, a]. It is then clear th

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metabelian group G. If G is a group from the variety $[\mathfrak{N}_2, \mathfrak{A}]$, then from Proposition
W in section 2, we see that $W(a, b, c)$ is a product of three commuting factors metabelian group G. If G is a group from the variety $[\mathfrak{N}_2, \mathfrak{A}]$, then from Proposition W, in section 2, we see that $W(a, b, c)$ is a product of three commuting factors $[c, b; c, a]$ $[c, b; b, a]$ and $[c, a; b, a]$ metabelian group *G*. If *G* is a group from the variety $[\mathfrak{N}_2, \mathfrak{A}]$, then from Proposition
W, in section 2, we see that $W(a, b, c)$ is a product of three commuting factors
[*c, b*; *c*, *a*], [*c, b*; *b*, *a*] and

Lemma 3.1. If G is any metabelian group
a, b, $c \in G$ and $[c, b, a] = [c, a, b]$ if $c \in G'$. $a, b, c \in G$ and $[c, b, a] = [c, a, b]$ if $c \in G'$.
Proof. Since $W(a, b, c)$ is trivial in G, the first conclusion follows from part (i)

Proof. Since $W(a, b, c)$ is trivial in G, the first conclusion follows from part (i) of Proposition W in section 2. Since $[b, a, c] = [[b, a], c]$ by definition, the second conclusion then follows from the first *Proof.* Since $W(a, b, c)$ is trivial in G , of Proposition W in section 2. Since [*i* conclusion then follows from the first.

conclusion then follows from the first.
 Lemma 3.2. Suppose that G is a group in the variety $[\mathfrak{N}_2, \mathfrak{A}]$. If $a, b, c, d \in G$,

then $[d, c; b, a]$ is central in G , as a consequence $W(a, b, c)$ is central in G **Lemma 3.2.** Suppose that G is a group in the variety $[\mathfrak{N}_2, \mathfrak{A}]$. If a, b, c, d is then $[d, c; b, a]$ is central in G. As a consequence, $W(a, b, c)$ is central in G.

then $[d, c; b, a]$ is central in G. As a consequence, $W(a, b, c)$ is central in G.
Proof. We do not require that a, b, c and d are distinct here, so we will have that $W(a, b, c)$ is central once we have that $[c, bc, a]$ is $k, b, a]$ *Proof.* We do not require that a, b, c and d are distinct here, so we will have that $W(a, b, c)$ is central once we have that $[c, b; c, a], [c, b; b, a]$ and $[c, a; b, a]$ are all central by the first conclusion of this lemma $W(a, b, c)$ is central once we have that $[c, b; c, a]$, $[c, b; b, a]$ and $[c, a; b, a]$ are all central by the first conclusion of this lemma. (a, b, c) is central once we have that $[c, b; c, a], [c, b; b, a]$ and $[c, a; b, a]$ are all central
the first conclusion of this lemma.
Let x be an arbitrary element of G. It will suffice to show that $[d, c; b, a; x]$ is
wish We use par

by the first conclusion of this lemma.
Let x be an arbitrary element of G. It will suffice to show that $[d, c; b, a; x]$ is
trivial. We use part (v) of Groves' Lemma with $C = [d, c]$, $B = [b, a]$ and $A = x$ to
prove that $[C, B, A] = [$ trivial. We use part (v) of Groves' Lemma with $C = [d, c], B = [b, a]$ and $A = x$ to prove that $[C, B, A] = [d, c; b, a; x]$ is trivial.

Lemma 3.3. Suppose that G is a group in the variety $[\mathfrak{N}_2, \mathfrak{A}]$. Then for any $a, b, c \in \mathfrak{A}$ **Lemma 3.3.** Suppose that G is a group in the variety $[\mathfrak{N}_2, \mathfrak{A}]$. Then for any $a, b, c \in G$, $[c, b, a] = [b, a, c]^{-1}[c, a, b]W(a, b, c)$. If $c \in G'$, then $[c, b, a] = [b, a, c]^{-1}[c, a, b]$.
If $c \in \infty(G)$, then $[c, b, a] = [c, a, b]$ If $c \in \gamma_3(G)$, then $[c, b, a] = [c, a, b]$. **3.3.** Suppose that G is a group $a] = [b, a, c]^{-1}[c, a, b]W(a, b, c)$
(G), then $[c, b, a] = [c, a, b]$.

Proof. The first conclusion again follows immediately from part (i) of Proposition *Proof.* The first conclusion again follows immediately from part (i) of Proposition W. From either part (ii) or part (iii) of Proposition W, we see that $W(a, b, c)$ is trivial for $C \in \mathbb{R}$, \mathfrak{A} if $c \in C'$. When c *Proof.* The first conclusion again follows immediately from part (i) of Pr
W. From either part (ii) or part (iii) of Proposition W, we see that W trivial for $G \in [\mathfrak{N}_2, \mathfrak{A}]$ if $c \in G'$. When $c \in \gamma_3(G)$, then $[b,$

trivial for $G \in [\mathfrak{N}_2, \mathfrak{A}]$ if $c \in G'$. When $c \in \gamma_3(G)$, then $[b, a, c]$ is trivial.
 Lemma 3.4. Suppose that G is a group in the variety $[\mathfrak{N}_2, \mathfrak{A}]$. Then for any $a, b, c, d \in G$ of $[a, b, a] = [d, c, b, a]$ **Lemma 3.4.** Suppose that G is a group in $a, b, c, d \in G$, $[d, c, b, a] = [d, c; b, a][d, c, a, b]$. $a, b, c, d \in G$, $[d, c, b, a] = [d, c; b, a][d, c, a, b]$.
Proof. Replace c by $[d, c]$ in the second conclusion of Lemma 3.3.

Lemma 3.5. If G is any metabelian group, $x \in G', y_i \in G$ and $\pi \in S_t$ is any permutation, then $[x, y_1, y_2, \ldots, y_t] = [x, y_{\pi 1}, y_{\pi 2}, \ldots, y_{\pi t}].$

permutation, then $[x, y_1, y_2, \dots, y_t] = [x, y_{\pi 1}, y_{\pi 2}, \dots, y_{\pi t}]$.
Proof. This is Lemma 34.51 from [6]. As observed in the proof there, this follows Proof. This is Lemma 34 easily from Lemma 3.1 .

easily from Lemma 3.1.

Lemma 3.6. Suppose that G is a group in the variety $[\mathfrak{N}_2, \mathfrak{A}]$ and that either $x \in \gamma_3(G)$ and $t \geq 1$ or else that $x \in G'$ and $t \geq 3$. If $y_i \in G$ for $1 \leq i \leq t$ and **Lemma 3.6.** Suppose that G is a group in the variety $[\mathfrak{N}_2, \mathfrak{A}]$ and that either $x \in \gamma_3(G)$ and $t \geq 1$ or else that $x \in G'$ and $t \geq 3$. If $y_i \in G$ for $1 \leq i \leq t$ and $\pi \in S_t$ is any permutation, then $[x, y_1,$

 $\pi \in S_t$ is any permutation, then $[x, y_1, y_2, \dots, y_t] = [x, y_{\pi 1}, y_{\pi 2}, \dots, y_{\pi t}].$
Proof. When $x \in \gamma_3(G)$, then this lemma follows from the final conclusion of Lemma 3.3 using the fact that every permutation is a product o Proof. When $x \in \gamma_3(G)$, then this lemma follows from the final conclusion of
Lemma 3.3, using the fact that every permutation is a product of transpositions,
in the same way that Lemma 3.5 follows from Lemma 3.1 Lemma 3.3, using the fact that every permutation is a product of transpositions, in the same way that Lemma 3.5 follows from Lemma 3.1.

Suppose then that $x \in G'$ and $t \geq 3$. Note that it will suffice to prove the lemma in the same way that Lemma 3.5 follows from Lemma 3.1.
Suppose then that $x \in G'$ and $t \ge 3$. Note that it will suffice to prove the lemma
in the case where x is a simple basic commutator $[d, c]$. We will assume that this Suppose then that $x \in G'$ and $t \geq 3$. Note that it will suffice to prove the lemma
in the case where x is a simple basic commutator $[d, c]$. We will assume that this is
so, mostly so we can reuse the notation x. If $\pi 1$ in the case where x is a simple basic commutator $[d, c]$. We will assume that this is
so, mostly so we can reuse the notation x. If π 1 = 1, then by the previous paragraph
with $x = [d, c, y_1] \in \gamma_3(G)$ and π replaced by so, mostly so we can reuse the notation x. If π 1 = 1, then by the previous paragraph
with $x = [d, c, y_1] \in \gamma_3(G)$ and π replaced by its restriction to $\{2, 3, ..., t\}$, we are
done. If π 1 = 2, then $[d, c, y_1, y_2] = [d, c;$ done. If $\pi 1 = 2$, then $[d, c, y_1, y_2] = [d, c; y_1, y_2][d, c, y_2, y_1]$ by Lemma 3.4 and then $[d, c, y_1, y_2, y_3] = [d, c, y_2, y_1, y_3]$ since $[d, c; y_1, y_2]$ is central by Lemma 3.2. It

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then follows that $[d, c, y_1, y_2, y_3, \ldots, y_t] = [d, c, y_2, y_1, y_3, \ldots, y_t]$ and we obtain the
conclusion by the result of the first parameter with $x - [d, c, y_1]$ then follows that $[d, c, y_1, y_2, y_3, \dots, y_t] = [d, c, y_2, y_1, y_3, \dots, y_t]$ and conclusion by the result of the first paragraph with $x = [d, c, y_2]$.
If $\pi 1 > 2$ let τ be the transposition which interchanges 2 and conclusion by the result of the first paragraph with $x = [d, c, y_2]$.
If $\pi 1 > 2$, let τ be the transposition which interchanges 2 and $\pi 1$. Then we

conclusion by the result of the first paragraph with $x = [d, c, y_2]$.

If $\pi 1 > 2$, let τ be the transposition which interchanges 2 and $\pi 1$. Then we

can use the conclusion of the first paragraph with $x = [d, c, y_1] \in \$ If π 1 > 2, let τ be the transposition which interchanges 2 and π 1. Then we
can use the conclusion of the first paragraph with $x = [d, c, y_1] \in \gamma_3(G)$ and the
permutation τ to find $[d, c, y_1, y_2, y_3, \ldots, y_t] = [d, c, y$ permutation τ to find $[d, c, y_1, y_2, y_3, \dots, y_t] = [d, c, y_1, y_{\pi 1}, y_3, \dots, y_2, \dots, y_t]$. We are then done, since $[d, c, y_1, y_{\pi 1}, y_3, \dots, y_t]$ is one of the cases we treated in the second paragraph. are then done, since $[d, c, y_1, y_{\pi 1}, y_3, \ldots, y_t]$ is one of the cases we treated in the

Proof of Theorem 3.7. Much of the proof will be the same whether we are in the **Proof of Theorem 3.7.** Much of the proof will be the same whether we are in the case where $K = [\gamma_2(F), \gamma_2(F)]$ or in the case where $K = [\gamma_3(F), \gamma_2(F)]$. Write $w = [x, x]$. Since we are only concerned with pontrivial commutators *Proof of Theorem 3.7.* Much of the proof will be the same whether we are in the case where $K = [\gamma_2(F), \gamma_2(F)]$ or in the case where $K = [\gamma_3(F), \gamma_2(F)]$. Write $w = [x_{i_1}, x_{i_2}, \ldots, x_{i_n}]$. Since we are only concerned with nontri $w = [x_{i_1}, x_{i_2}, \dots, x_{i_n}]$. Since we are only concerned with nontrivial commutators, we may assume that either $i_1 < i_2$ or $i_2 < i_1$. (If $i_1 = i_2$ then w is trivial $w = [x_{i_1}, x_{i_2}, \ldots, x_{i_n}]$. Since we are only concerned with nontrivial commutators,
we may assume that either $i_1 < i_2$ or $i_2 < i_1$. (If $i_1 = i_2$ then w is trivial
and we have $wK = c_1^{-1}c_1K$.) We want to reduce immedi we may assume that either $i_1 < i_2$ or $i_2 < i_1$. (If $i_1 = i_2$ then w is trived and we have $wK = c_1^{-1}c_1K$.) We want to reduce immediately to the case when $i_2 < i_1$. If we do have $i_1 < i_2$, then $[x_{i_1}, x_{i_2}] = [x_{i_2}, x$ $]^{-1}$ so $[x_{i_1}, x_{i_2}, x_{i_3}]K =$ $[[x_{i_2}, x_{i_1}]^{-1}, x_{i_3}]K = [x_{i_2}, x_{i_1}, x_{i_3}]^{-1}K$ and eventually $[x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_n}]K =$ we have $wK = c_1^T c_1 K$.) We want to reduce immediately to the case where i_1 , i_1 . If we do have $i_1 < i_2$, then $[x_{i_1}, x_{i_2}] = [x_{i_2}, x_{i_1}]^{-1}$ so $[x_{i_1}, x_{i_2}, x_{i_3}]K$, $x_{i_1}]^{-1}$, $x_{i_3}K = [x_{i_2}, x_{i_1}, x_{i_3}]^{-1}K$ $[x_{i_2}, x]$ $\langle x_i, x_{i1}, x_{i2}, x_{i3} \rangle$ = $\langle x_{i1}, x_{i2}, x_{i3}, \ldots, x_{in} \rangle$ = $\langle x_{i1}, x_{i2}, x_{i3}, \ldots, x_{i,n} \rangle$ = $\langle x_{i1}, x_{i$ $\left[[x_{i_2}, x_{i_1}]^{-1}, x_{i_3}]K = [x_{i_2}, x_{i_1}, x_{i_3}]^{-1}K$ and eventually $[x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_n}]K$
 $\left[x_{i_2}, x_{i_1}, x_{i_3}, \ldots, x_{i_n}\right]^{-1}K$. We will continue with the assumption that we have i_1
 i_2 and we will show tha $1^{-1}c_2K$. $[x_{i_2}, x_{i_1}, x_{i_3}, \ldots, x_{i_n}]^{-1}K$. We will continue with the assumption that we have i_1
 i_2 and we will show that in this case we have either $wK = c_1K$ or $w = c_1^{-1}c_2$.

We next use either Lemma 3.5 or Lemma 3.6 to We next use either Lemma 3.5 or Lemma 3.6 to find that $[x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_n}]K =$ \hat{u}_2 and we will show that in this case we have either $wK = c_1K$ or $w = c_1^*c_2K$.
We next use either Lemma 3.5 or Lemma 3.6 to find that $[x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_n}]K =$
 $[x_{i_1}, x_{i_2}, x_{i_{\pi 3}}, x_{i_{\pi 4}}, \ldots, x_{i_{\pi n}}]K$ We next use either Lemma 3.5 or Lemma 3.6 to find that $[x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_n}]K = [x_{i_1}, x_{i_2}, x_{i_{\pi 3}}, x_{i_{\pi 4}}, \ldots, x_{i_{\pi n}}]K$ where π is chosen to be a permutation on $\{3, 4, \ldots, n\}$ such that $i_{\pi 3} \le i_{\pi 4} \$ in order to better manage notation, that we have $i_1 > i_2 > i_3 \leq i_4 \leq i_5 \leq$
 $\leq i_n$.

In the case where $K = [\gamma_2(F), \gamma_2(F)]$, we next use Lemma 3.1 to express
 $x_i, x_j | K$ as $[x_i, x_j, x_j]^{-1}[x_j, x_j, x_j]K$. Then we see $[x_i, x_j, x_j]K$

assume, in order to better manage notation, that we have $i_1 > i_2 > i_3 \leq i_4 \leq i_5 \leq$
 $\cdots \leq i_n$.

In the case where $K = [\gamma_2(F), \gamma_2(F)]$, we next use Lemma 3.1 to express
 $[x_{i_1}, x_{i_2}, x_{i_3}]K$ as $[x_{i_2}, x_{i_3}, x_{i_1}]^{-1}[x_{i_$ $[[x_{i_2}, x_{i_3}, x_{i_1}]^{-1}[x_{i_1}, x_{i_3}, x_{i_2}], x_{i_4}]K = [x_{i_2}, x_{i_3}, x_{i_1}, x_{i_4}]^{-1}[x_{i_1}, x_{i_2}],$ i the case where $K = [\gamma_2(F), \gamma_2(F)]$, we next use Lemma 3.1 to express $x_{i_2}, x_{i_3} | K$ as $[x_{i_2}, x_{i_3}, x_{i_1}]^{-1} [x_{i_1}, x_{i_3}, x_{i_2}] K$. Then we see $[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}] K =$
 $,x_{i_3}, x_{i_1}]^{-1} [x_{i_1}, x_{i_3}, x_{i_2}]$, $x_{i_4} | K$ $[x_{i_1}, x_{i_2}, x_{i_3}]K$ as $[x_{i_2}, x_{i_3}, x_{i_1}]^{-1}[x_{i_1}, x_{i_3}, x_{i_2}]K$. Then we see $[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}]K$ =
 $[[x_{i_2}, x_{i_3}, x_{i_1}]^{-1}[x_{i_1}, x_{i_3}, x_{i_2}], x_{i_4}]K$ = $[x_{i_2}, x_{i_3}, x_{i_1}, x_{i_4}]^{-1}[x_{i_1}, x_{i_3}, x_{i_2}, x_{i_4}]K$. W $[[x_{i_2}, x_{i_3}, x_{i_1}]^{-1}[x_{i_1}, x_{i_3}, x_{i_2}], x_{i_4}]K = [x_{i_2}, x_{i_3}, x_{i_1}, x_{i_4}]^{-1}[x_{i_1}, x_{i_3}, x_{i_2}, x_{i_4}]K$. We
can continue in this manner to find that $[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, \ldots, x_{i_n}]K$ is equal to the
product $[x_{i_2}, x_{i_$ $]^{-1}[x_{i_1}, x]$ can continue in this manner to find that $[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, \ldots, x_{i_n}]K$ is eq
product $[x_{i_2}, x_{i_3}, x_{i_1}, x_{i_4}, \ldots, x_{i_n}]^{-1}[x_{i_1}, x_{i_3}, x_{i_2}, x_{i_4}, \ldots, x_{i_n}]K$. If it hap
 $i_1 > i_4$ and/or that $i_2 > i_4$ we conclude b Suppose then that $i_2 > i_4$ we conclude by applying Lemma 3.5 agains Suppose then that we are in the case where $K = [\gamma_3(F), \gamma_2(F)]$
this case we have $[x_1, x_2, x_3]$ $K - [x_3, x_4]$. Then $X = [\gamma_3(F), \gamma_2(F)]$

Suppose then that we are in the case where $K = [\gamma_3(F), \gamma_2(F)]$ and $n \geq 5$. $i_1 > i_4$ and/or that $i_2 > i_4$ we conclude by applying Lemma 3.5 again.
Suppose then that we are in the case where $K = [\gamma_3(F), \gamma_2(F)]$ and $n \ge 5$.
In this case, we have $[x_{i_1}, x_{i_2}, x_{i_3}]K = [x_{i_2}, x_{i_3}, x_{i_1}]^{-1}[x_{i_1}, x_{i_$ Suppose then that we are in the case where $K = [\gamma_3(F), \gamma_2(F)]$ and $n \geq 5$.
In this case, we have $[x_{i_1}, x_{i_2}, x_{i_3}]K = [x_{i_2}, x_{i_3}, x_{i_1}]^{-1}[x_{i_1}, x_{i_3}, x_{i_2}]W(x_{i_3}, x_{i_2}, x_{i_1})K$
by Lemma 3.3. Since $W(x_{i_3}, x_{i_2}, x_{i_1})K$ $[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, \ldots, x_{i_n}] = [x_{i_2}, x_{i_3}, x_{i_1}, x_{i_4}, \ldots, x_{i_n}]^{-1} [x_{i_1}, x_{i_3}, x_{i_2}, x_{i_4}, \ldots, x_{i_n}] K.$ his case, we have $[x_{i_1}, x_{i_2}, x_{i_3}]K = [x_{i_2}, x_{i_3}, x_{i_1}]^{-1}[x_{i_1}, x_{i_3}, x_{i_2}]W(x_{i_3}, x_{i_2}, x_{i_1})K$

Lemma 3.3. Since $W(x_{i_3}, x_{i_2}, x_{i_1})K$ is central by Lemma 3.2, we again find that
 $x_{i_2}, x_{i_3}, x_{i_4}, \ldots, x_{i_n}] = [x_{i_$ If it happens that $i_1 > i_4$ and/or that $i_2 > i_4$ we conclude by applying Lemma 3.6 again.

4 Using higher commutators

For $s \ge 1$, we will use the notation $N_{n:s}$ for the normal closure in F of the basic
moutators from C having weights n through $n + s - 1$ inclusive. That is For $s \ge 1$, we will use the notation $N_{n:s}$ for the normal closure in F of the bacommutators from C having weights n through $n + s - 1$, inclusive. That is,

$$
N_{n:s} = \left(\bigcup_{j=n}^{n+s-1} \left\{c \in \mathcal{C} \mid wt(c) = j\right\}\right)^F.
$$

In this notation, the normal closure, N_n , of the set of basic commutators of In this notation, the normal closure, N_n , of the set of basic commutators of weight *n*, becomes $N_{n:1}$. We want to show, consecutively, that $\gamma_n(F)$ is equal to $N_{n:1}$. Also and $N_{n:1}$ of the first of these is co In this notation, the normal closure, N_n , of the set of basic commutators of weight *n*, becomes $N_{n:1}$. We want to show, consecutively, that $\gamma_n(F)$ is equal to $N_{n:2}$, $N_{n:2}$ and $N_{n:2}$. The first of these is $N_{n:n-1}$, $N_{n:n-2}$ and $N_{n:n-3}$. The first of these is contained in work of Martin Ward, see [8, 9, 4] and all of the cases rely heavily on the following theorem of Martin Ward. We will use only the part of this theore see $[8, 9, 4]$ and all of the cases rely heavily on the following theorem of Martin

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"basic invertators" generate $\gamma_n(F)$. We will not need the more difficult conclusion

that this set is a free basis for $\alpha(F)$

that this set is a free basis for $\gamma_n(F)$.
 Theorem 4.1. (Martin Ward, [4,9]) For $n \ge 2$, $\gamma_n(F)$ is freely generated by the **Theorem 4.1.** (Martin Ward, [4,9]) For $n \geq 2$, $\gamma_n(F)$ is freely generated by the set of all commutators of the form $[c, b_1^{\beta_1}, b_2^{\beta_2}, \ldots, b_q^{\beta_q}]$ where $q \geq 1$, c and the b_i are basic commutators baying weights **Theorem 4.1.** (Martin Ward, [4,9]) For $n \ge 2$, $\gamma_n(F)$ is freely generated by the set of all commutators of the form $[c, b_1^{\beta_1}, b_2^{\beta_2}, \ldots, b_q^{\beta_q}]$ where $q \ge 1$, c and the b_i are basic commutators having weights l set of all commutators of the form $[c, b_1^r, b_2^r, ..., b_q^q]$ where $q \ge 1$, c are basic commutators having weights less than $n, c > b_1 \le b_2 \le ... \le b_q$,
wt $([c, b_1^{\beta_1}]) \ge n$, if $c = [c_1, c_2]$ then $c_2 \le b_1$ and if $b_i = b_j$, then $wt([c, b_1^{\beta_1}]) \ge n$, if $c = [c_1, c_2]$ then $c_2 \le b_1$ and if $b_i = b_j$, then $\beta_i = \beta_j$.
 Proposition 4.2. For $n \ge 2$, $\gamma_n(F) = N_{n:n-1}$.

Proposition 4.2. For $n \ge 2$, $\gamma_n(F) = N_{n:n-1}$.
Proof. It is clear that $N_{n:n-1} \subseteq \gamma_n(F)$, and it will suffice to show that each of the free generators $\left[e^{-\beta_1} h^{\beta_2} \right]$ e^{β_3} from the theorem of Mortin Ward is *Proof.* It is clear that $N_{n:n-1} \subseteq \gamma_n(F)$, and it will suffice to show that each of the free generators, $[c, b_1^{a_1}, b_2^{a_2}, \ldots, b_q^{a_q}]$ from the theorem of Martin Ward is in $N_{n:n-1}$. For this, we use induction on q. In For this, we use induction on q. In the base case, $q = 1$, assume first that we have a free generator, $[c, b_1]$, where $\beta_1 = +1$. Then by the hypotheses for Theorem 4.1, we have that $[c, b]$ is a basic commutator, that $wt(c) < n$, $wt(b_1) < n$ and $wt([c, b_1]) \geq n$. We see that $[c, b_1]$ is a basic commutator with $n \leq wt([c, b_1]) \leq$ 4.1, we have that $[c, b]$ is a basic commutator, that $wt(c) < n$, $wt(b_1) < n$ and $wt([c, b_1]) \ge n$. We see that $[c, b_1]$ is a basic commutator with $n \le wt([c, b_1]) \le 2n - 2$, so $[c, b_1] \in N_{n:n-1}$. If we have instead the free generato $wt([c, b_1]) \geq n$. We see that $[c, b_1]$ is a basic commutator with $n \leq wt([c, b_1]) \leq 2n-2$, so $[c, b_1] \in N_{n:n-1}$. If we have instead the free generator $[c, b_1^{-1}]$, then $[c, b_1^{-1}] = ([c, b_1]^{-1})^{b_1^{-1}}$ which is also in $N_{n:n-1}$ $\begin{bmatrix} -1 \\ 1 \end{bmatrix} = ([c, b_1]^{-1})^{b_1^{-1}}$ w $[c, b_1^{-1}] = ([c, b_1]^{-1})^{b_1^{-1}}$ which is also in $N_{n:n-1}$. Suppose then by induction, that
 $C = [c, b_1^{\beta_1}, b_2^{\beta_2}, \dots, b_{q-1}^{\beta_{q-1}}]$ has been shown to be in $N_{n:n-1}$. Then $[C, b_q] = C^{-1}C^{b_q}$ $C = [c, b_1^{\beta_1}, b_2^{\beta_2}, \dots, b_{q-1}^{\beta_{q-1}}]$ has
and $[C, b_q^{-1}] = ([C, b_q]^{-1})^{b_q^{-1}}$ as has been shown to be in $\frac{1}{q}$ are in $N_{n:n-1}$ also.

Lemma 4.3. Assume that $n \geq 3$ and that a and b are basic commutators of weight $n-1$. Then [b, a] ∈ $N_{n:n-2}$.

n − 1. Then [b, a] ∈ $N_{n:n-2}$.
Proof. Since $[a, b] = [b, a]^{-1}$, this lemma is symmetric in a and b. Write $a = [a_2, a_1]$
and $b = [b_2, b_1]$ where a_1, a_2, b_2 and b_2 are basic commutators. By symmetry we and $b = [b_2, b_1]$ where a_1, a_2, b_1 , and b_2 are basic commutators. By symmetry, we *Proof.* Since $[a, b] = [b, a]^{-1}$, this lemma is symmetric in a and b. Write $a = [a_2, a_1]$
and $b = [b_2, b_1]$ where a_1, a_2, b_1 , and b_2 are basic commutators. By symmetry, we
may assume that $a_1 \geq b_1$. Then $[b, a_1]$ an and $b = [b_2, b_1]$ where a_1, a_2, b_1 , and b_2 are basic commutators. By symmetry, we may assume that $a_1 \ge b_1$. Then $[b, a_1]$ and $[b, a_2]$ are both basic commutators and these have weight at least n and at most $2n - 3$ these have weight at least *n* and at most $2n-3$, so *b* commutes with both a_1 and a_2 modulo $N_{n:n-2}$. Hence *b* commutes with $a = [a_2, a_1]$ modulo $N_{n:n-2}$.

Proposition 4.4. For $n \geq 3$, $\gamma_n(F) = N_{n:n-2}$.

Proposition 4.4. For $n \geq 3$, $\gamma_n(F) = N_{n:n-2}$.
Proof. By Proposition 4.2, it will suffice to prove that $N_{n:n-1} \subseteq N_{n:n-2}$. Suppose that $c = [b, a]$ is a basic commutator with $n \leq wt(c) \leq 2n-2$ If $wt(c) \leq 2n-3$ or *Proof.* By Proposition 4.2, it will suffice to prove that $N_{n:n-1} \subseteq N_{n:n-2}$. Suppose that $c = [b, a]$ is a basic commutator with $n \le wt(c) \le 2n-2$. If $wt(c) \le 2n-3$ or if $wt(b) > n$, then it is immediate that $c \in N$, since $wt(a)$ that $c = [b, a]$ is a basic commutator with $n \le wt(c) \le 2n-2$. If $wt(c) \le 2n-3$ or if $wt(b) \ge n$, then it is immediate that $c \in N_{n:n-2}$. Since $wt(a) \le wt(b)$, this leaves only the possibility that $wt(a) = wt(b) = n - 1$ and this case was covered by the previous lemma. only the possibility that $wt(a) = wt(b) = n - 1$ and this case was covered by the

previous lemma.
To prove that $\gamma_n(F) = N_{n:n-3}$ for $n \ge 4$, we will prove that $N_{n:n-2} \subseteq N_{n:n-3}$.
The following two lemmas will be useful.

The following two lemmas will be useful.
 Lemma 4.5. Assume that $n \ge 4$, that c is a simple basic commutator of weight $n-3$, that x, y, and z are letters with $x \le y \le z$ and that $[c, x]$ is a basic commutator of weight $n-3$, that x, y, and z are letters with $x \leq y \leq z$ and that $[c, x]$ is a basic commutator

of weight $n-2$. Then $[c, x, z, y] \in N_{n:n-3}$.
Proof. If $n = 4$, we already know that $\gamma_4(F) = N_4 = N_{4:1}$, so $[c, x, z, y] \in N_{4:1}$. For the rest of this proof, we assume that $n \geq 5$, so that c is a basic commutator with *Proof.* If $n = 4$, we already know that $\gamma_4(F) = N_4 = N_{4:1}$, so $[c, x, z, y] \in N_{4:1}$. For the rest of this proof, we assume that $n \ge 5$, so that c is a basic commutator with weight at least 2. Since the conclusion is clea *Proof.* If $n = 4$, we already know that $\gamma_4(F) = N_4 = N_{4:1}$, so $[c, x, z, y] \in N_{4:1}$. For the rest of this proof, we assume that $n \ge 5$, so that c is a basic commutator with weight at least 2. Since the conclusion is clea the rest of this
weight at least
that $y < z$.

Suppose first that $y = x$. We use part (iii) of Groves' Lemma with $C = [c, x]$, $B =$ Suppose first that $y = x$. We use part (iii) of Groves' Lemma with $C = [c, x]$, $B = z$ and $A = x$. Then $[B, A, C]^{-1}$, $[C, B, B]$, $[C, A, A]$ and $[C, A, B]$ are all basic com-
mutators of weight n . Since $[C, A] = [c, x, x]$ we see that Suppose first that $y = x$. We use part (iii) of Groves' Lemma with $C = [c, x]$, $B = z$ and $A = x$. Then $[B, A, C]^{-1}$, $[C, B, B]$, $[C, A, A]$ and $[C, A, B]$ are all basic com-
mutators of weight n. Since $[C, A] = [c, x, x]$, we see that $[C$ mutators of weight *n*. Since [C, A] = [c, x, x], we see that [C, A, c] is a basic commutator of weight *n*. Thus, [C, A] commutes with both c and x modulo $N_{n:n-3}$, so [C, A] commutes with $C = [c, x]$ modulo $N_{n:n-3}$. Thus, [C, A] commutes with both c and x modulo $N_{n:n-3}$, so [C, A] commutes with $C = [c, x]$ modulo $N_{n:n-3}$.
Now suppose that $y > x$. We use part (iii) of Groves' Lemma with $C = [c, x]$, $B =$

 $C = [c, x]$ modulo $N_{n:n-3}$.
Now suppose that $y > x$. We use part (iii) of Groves' Lemma with $C = [c, x]$, $B = z$ and $A = y$. Then $[B, A, C]^{-1}$, $[C, B, B]$, $[C, A, A]$ and $[C, A, B]$ are all basic com-
mutators of weight n Since $[C, A]$ Now suppose that $y > x$. We use part (iii) of Groves' Lemma with $C = [c, x]$, $B = z$ and $A = y$. Then $[B, A, C]^{-1}$, $[C, B, B]$, $[C, A, A]$ and $[C, A, B]$ are all basic commutators of weight n. Since $[C, A] = [c, x, y]$ is a simple basic c mutators of weight *n*. Since $[C, A] = [c, x, y]$ is a simple basic commutator and $wt(c) \geq 2$, $[C, A, c]$ is a basic commutator of weight $2n - 4$ and $[C, A]$ commutes with c modulo $N_{n:n-3}$. The previous paragraph applies to $[c, x, y, x]$, so $[C, A]$ also commutes with x and hence with $C = [c, x]$ modulo $N_{n:n-3}$.

Lemma 4.6. Assume that $n \geq 4$, that c₁ and c₂ are simple basic commutators of **Lemma 4.6.** Assume that $n \geq 4$, that c_1 and c_2 are simple basic commutators of weight $n-3$, that x, y, and z are letters with $x < y \leq z$ and that $[c_1, x]$ and $[c_2, y]$ are basic commutators of weight $n-2$. Then **Lemma 4.6.** Assume that $n \geq 4$, that c_1 and c_2 are simple basic commuture weight $n - 3$, that x, y , and z are letters with $x < y \leq z$ and that $[c_1, x]$ are basic commutators of weight $n - 2$. Then $[[c_2, y, z], [c_$

are basic commutators of weight $n-2$. Then $[[c_2, y, z], [c_1, x]] \in N_{n:n-3}$.
Proof. If $n = 4$, we already know that $[[c_2, y, z], [c_1, x]] \in \gamma_4(F) = N_4 = N_{4:1}$.
For the rest of this proof, we assume that $n \ge 5$, so c_1 and c_2 For the rest of this proof, we assume that $n \geq 5$, so c_1 and c_2 are simple basic commutators with weight $n-3 \geq 2$.
We use part (iv) of Groves' Lemma with $C = [c_1, x]$, $B = [c_2, y]$ and $A = z$

to prove that $[n-3 \geq 2]$.

We use part (iv) of Groves' Lemma with $C = [c_1, x], B = [c_2, y]$ and $A = z$

to prove that $[B, A, C]$ is trivial modulo $N_{n:n-3}$. Since $[C, B]$ has weight $2n - 4$

and either $[C, B]$ or $[C, B]^{-1}$ is a b We use part (iv) of Groves' Lemma with $C = [c_1, x], B = [c_2, y]$ and $A = z$
to prove that $[B, A, C]$ is trivial modulo $N_{n:n-3}$. Since $[C, B]$ has weight $2n - 4$
and either $[C, B]$ or $[C, B]^{-1}$ is a basic commutator, we know that and either $[C, B]$ or $[C, B]^{-1}$ is a basic commutator, we know that $[C, B]$ is trivial modulo $N_{n:n-3}$. We observe that $[C, A, A]$ is a simple basic commutator of weight n. To show that $[C, A, B]$ is also trivial modulo $N_{n:n$ that $[C, A, c_2]$ and $[C, A, y]$ are trivial modulo $N_{n:n-3}$. The latter follows from the *n*. To show that [*C*, *A*, *B*] is also trivial modulo $N_{n:n-3}$, it will suffice to show that [*C*, *A*, *c*₂] and [*C*, *A*, *y*] are trivial modulo $N_{n:n-3}$. The latter follows from the previous lemma. Since $wt(c_$ that $[C, A, c_2]$ and $[C, A, y]$ are trivial modulo $N_{n:n-3}$. The latter follows from the previous lemma. Since $wt(c_2) \ge 2$, $[C, A, c_2]$ is a basic commutator of weight $2n-4$.
Since $n \ge 5$, this weight is at least $n+1$ and

Theorem 4.7. For $n \ge 4$, $\gamma_n(F) = N_{n:n-3}$.

Proof. This is known when $n = 4$, so we may assume that $n \geq 5$.
By Proposition 4.4. it will suffice to show that $N_{n:n-2} \subseteq N_{n:n-3}$. Suppose that $c = [b, a]$ is a basic commutator with $n \leq wt(c) \leq 2n - 3$. If $wt(c) \leq 2n -$ By Proposition 4.4. it will suffice to show that $N_{n:n-2} \subseteq N_{n:n-3}$. Suppose that $c = [b, a]$ is a basic commutator with $n \le wt(c) \le 2n-3$. If $wt(c) \le 2n-4$ or if $wt(b) \ge n$, then it is immediate that $c \in N_{n:n-3}$. Since $wt(b) \ge wt$ $s = [b, a]$ is a basic commutator with $n \le wt(c) \le 2n - 3$. If $wt(c) \le 2n - 4$ or if $wt(b) \ge n$, then it is immediate that $c \in N_{n:n-3}$. Since $wt(b) \ge wt(a)$, it will suffice to prove that $[b, a] \in N_{n:n-3}$ when a and b are basic commut suffice to prove that $[b, a] \in N_{n:n-3}$ when a and b are basic commutators with $wt(a) = n - 2$ and $wt(b) = n - 1$.
Next, we want to reduce to the case where a and b are simple basic commutators.

 $wt(a) = n - 2$ and $wt(b) = n - 1$.

Next, we want to reduce to the case where a and b are simple basic commutators.

Write $a = [a_2, a_1]$ and $b = [b_2, b_1]$ where a_1, a_2, b_1 and b_2 are basic commutators.

Suppose first that Next, we want to reduce to the case where a and b are simple basic commutators.
Write $a = [a_2, a_1]$ and $b = [b_2, b_1]$ where a_1, a_2, b_1 and b_2 are basic commutators.
Suppose first that $wt(a_1) \ge 2$ and that $a_1 \ge b_1$ Suppose first that $wt(a_1) \geq 2$ and that $a_1 \geq b_1$. Then both $[b, a_1]$ and $[b, a_2]$ are basic commutators and these have weight at least $n+1$ and at most $2n-5$. Hence, in this case, b commutes with a_1, a_2 and a modulo $N_{n:n-3}$. Similarly, suppose that basic commutators and these have weight at least $n+1$ and at most $2n-5$. Hence,
in this case, b commutes with a_1, a_2 and a modulo $N_{n:n-3}$. Similarly, suppose that
 $wt(b_1) \geq 2$ and that $b_1 \geq a_1$. Then both $[a, b_1$ in this case, *b* commutes with a_1, a_2 and *a* modulo $N_{n:n-3}$. Similarly, suppose that $wt(b_1) \geq 2$ and that $b_1 \geq a_1$. Then both $[a, b_1]$ and $[a, b_2]$ are basic commutators and these have weight at least *n* and $wt(b_1) \geq 2$ and that $b_1 \geq a_1$. Then both $[a, b_1]$ and $[a, b_2]$ are basic commutators
and these have weight at least *n* and at most $2n-5$. Hence, in this case, *a* commutes
with b_1, b_2 and *b* modulo $N_{n:n-3}$. I and these have weight at least *n* and at most $2n-5$. Hence, in this case, *a* commutes
with b_1, b_2 and *b* modulo $N_{n:n-3}$. If, in the case where $wt(a_1) \ge 2$, we have $b_1 > a_1$,
then b_1 also has weight at least t then b_1 also has weight at least two: if, in the case where $wt(b_1) \geq 2$, we have $a_1 > b_1$, then a_1 also has weight at least two. Thus, for the rest of this proof, we may assume that a and b are both simple basic c

Write a as $a = [c_1, x]$ where c_1 is a simple basic commutator with weight $n-3$ and x is a letter. Also write $b = [c_2, y, z]$ where y and z are letters with $y \le z$, c_2 is a simple basic commutator with weight $n - 3$ and $[c_2, y]$ is a simple basic and x is a letter. Also write $b = [c_2, y, z]$ where y and z are letters with $y \le z$, c_2 is a simple basic commutator with weight $n - 2$. If $x \ge z$, then $[b, x]$ is a simple basic commutator with weight $n - 2$. If $x \ge z$, c₂ is a simple basic commutator with weight $n-3$ and $[c_2, y]$ is a simple basic commutator with weight $n-2$. If $x \ge z$, then $[b, x]$ is a simple basic commutator with weight n and $[b, c_1]$ is a basic commutator with with weight *n* and [*b*, *c*₁] is a basic commutator with weight $2n - 4$. We see that *b* commutes with *c*₁, *x* and *a* = [*c*₁, *x*] modulo $N_{n:n-3}$. If $y \le x < z$, then [*b*, *x*] is with weight *n* and [*b*, *c*₁] is a basic commutator with weight $2n - 4$. We see that *b* commutes with c_1 , *x* and $a = [c_1, x]$ modulo $N_{n:n-3}$. If $y \le x < z$, then [*b*, *x*] is trivial modulo $N_{n:n-3}$ by Lemma 4.5 b commutes with c_1 , x and $a = [c_1, x]$ modulo $N_{n:n-3}$. If $y \le x < z$, then $[b, x]$ is trivial modulo $N_{n:n-3}$ by Lemma 4.5 and $[b, c_1]$ is a basic commutator of weight with weight $2n-4$. We again see that b commutes wi

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