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BASIC COMMUTATORS AS RELATORS

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ABSTRACT. Charles Sims, [7] has asked whether or not the lower central subgroup $\gamma_n(F)$ of a free group F coincides with the normal closure in F of the set of basic commutators of weight n. Here, we investigate variations of this question where we consider other varieties or other sets of commutators. We also give brief new proofs that this question has a positive answer in the previously known cases for weight n at most 4 and for the case where $n = 5$ and F is free with rank 2. We show that for $n \geq 4$, $\gamma_n(F)$ is the normal closure in F of the set of basic commutators having weights n through $2n-4$, inclusive. This is a common generalization of Sims' result for weight 4 and a result of Martin Ward, but both of these earlier results are used in the proof.

1: INTRODUCTION AND NOTATION

For a natural number $r \geq 2$, $F = F_r$ is the free group on the ordered alphabet $X = \{x_1, x_2, \dots, x_r\}$.

We write y^x for $x^{-1}yx$ and $[y,x]$ for the commutator $y^{-1}x^{-1}yx$. If A and B are subsets of any group G, then A^G is the normal closure of A in G and $[A,B]$ is the subgroup of G that is generated by the set of elements $[a,b]$ where $a \in A$ and $b \in B$. We write $[z,x,y]$ for the commutator $[[z,x],y]$. The weight of a commutator c is denoted by $\mathrm{wt}(c)$. We follow Marshall Hall's definition for basic commutators. See, for example, [2] or [6].

Fix an order on the alphabet X. (1) The basic commutators of weight one are the letters of X taken in this order. (2) Having defined and ordered the basic commutators of weight less than n, the basic commutators of weight n are all of the commutators $[c_i, c_j]$ which satisfy the conditions: (a) c_i and c_j are basic commutators with $n = \mathrm{wt}(c_i) + \mathrm{wt}(c_j)$. (b) In the order that has been chosen for basic commutators of weight less than n, $c_j < c_i$. (c) If $c_i = [c_s, c_t]$ where c_s and c_t are basic commutators, then $c_t \leq c_j$ in the order that has been chosen for basic commutators of weight less than n. (3) The basic commutators of weight n follow all of the basic commutators of weight less than n in the order for the basic commutators of weight less than $n+1$, but the basic commutators of weight n may be ordered arbitrarily.

While we may order the commutators of weight n arbitrarily, the choices that we make will have consequences for which commutators of higher weights are basic and which are not. For example, depending upon which order we use for the commutators $[x_2, x_1]$ and $[x_3, x_1]$ in weight 2, either $[[x_2, x_1], [x_3, x_1]]$ or else $[[x_3, x_1], [x_2, x_1]]$

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will be a basic commutator in weight 4. Following the terminology of Sims [7] we will say that our choices determine a basic sequence of commutators. Whenever we represent a basic sequence of commutators as a subscripted list, the subscripts will reflect the order that we have chosen for the basic sequence and we will have $c_j < c_k$ if and only if $j < k$.

Write γ_n for the n^{th} term, $\gamma_n = \gamma_n(F) := [\gamma_{n-1}(F), F]$ of the lower central series for F. Let $\mathcal{C} = \{x_1, x_2, \ldots, x_r, c_{r+1}, \ldots\}$ be any fixed basic sequence of commutators that begins with the ordered alphabet X. Throughout this paper, we will let N_n denote the normal closure in F of the set $\mathcal{R}_n$ of basic commutators of weight n from $\mathcal{C}$. In [7], Charles Sims raised the question of whether or not $\gamma_n = N_n$ and answered this question positively for $n \leq 4$ and for $n = 5, r = 2$. Our concern throughout this paper will be variations upon this question. In section 2, we present some useful tools for computing with commutators. We use these to give a brief proofs of Sims' results that $\gamma_4(F) = N_4$ and that $\gamma_5(F) = N_5$ when F has rank 2. In section 3, we prove that the analogues of Sims' question for the varieties $[\mathfrak{A}, \mathfrak{A}]$ and $[\mathfrak{N}_2, \mathfrak{A}]$ have positive answers. In section 4, the principal result is that, for $n \geq 4$, $\gamma_n(F)$ is the normal closure in F of $\cup_{j=n}^{2n-4} \mathcal{R}_j$.

For the given alphabet X, a **simple** commutator of weight n is a commutator $[x_{i_1}, x_{i_2}, \cdots, x_{i_n}]$ where each x_i is a letter of X. It is not difficut to see that $\gamma_n(F)$ is the normal closure in F of the set of simple commutators of weight n. If $i_1 > i_2$, and $i_2 \leq i_3 \leq \cdots \leq i_n$, then the commutator $[x_{i_1}, x_{i_2}, \cdots, x_{i_n}]$ is a basic commutator for any choice of basic sequence of commutators. This is noted by Sims [7] and can be proved by a simple induction. Following standard notation, $[d, c; b, a]$ is an abbreviation for $[[d, c], [b, a]]$ and more generally $[C; a_1, a_2, \ldots a_k]$ abbreviates $[C, [a_1, a_2, \ldots a_k]]$.

The following lemma, together with Groves' Lemma and Proposition W in the next section, will also be used in our related paper, [5].

Basic Lemma. *Let $r \geq 2$ and $n \geq 2$ be fixed. Suppose that $N_{n-1} = \gamma_{n-1}$. If for every basic commutator c of weight $n-1$ and every $x \in X$, we have $[c, x] \in N_n$, then $N_n = \gamma_n$.*

Proof. Write $\mathcal{R}_{n-1}$ for the set of basic commutators of weight $n-1$. It is clear that N_n is always a subgroup of γ_n, so we need to show that $\gamma_n \subseteq N_n$. By hypothesis, $\gamma_{n-1} = N_{n-1}$, so $\gamma_n = [\gamma_{n-1}, F] = [N_{n-1}, F] = [(\mathcal{R}_{n-1})^F, F] = [\mathcal{R}_{n-1}, F]$ which is contained in N_n since $[c, x] \in N_n$ for every $c \in \mathcal{R}_{n-1}$ and every $x \in X$.

2: Groves' Lemma

Lemma 2.1. *Suppose $a, b,$ and c are elements of any group G. Then*

$$[c,b,a] = \left([b,a,a]^{c^a}([b,a,c]^{-1})^a[b,a,a]^{-1}\right)^{[c,b]}[b,a,c]^{[c,b]}[b,a,b]^{c^b}$$
$$([b,a,c]^{-1})^b[b,a,b]^{-1}\left([c,a,c]^{[c,b]}[c,a,b]^{c^b}([c,a,c]^{-1})^b\right)^{[b,a]}.$$

Proof. It will suffice to show that the equation is valid in the free group on $\{a, b, c\}$. Write both sides of the equation as words in this free group and observe that the reduced forms are equal. See also the first parts of Groves' Lemma and Proposition W, below.

Lemma 2.2. $N_3 = \gamma_3$.

Proof. By the Basic Lemma and the easy observation that $N_2 = \gamma_2$, we need only check that $[x_k, x_j, x_i]$ is in N_3 when $i < j < k$. This follows from the equation in Lemma 2.1.

In [3] Havas and Richardson published a wonderfully short proof by J.R.J. Groves that the commutator $[b, a, b, a]$ is in the normal closure of the set of basic commutators of weight 4 on the alphabet $\{a, b\}$. In the following lemma, we dissect Groves' proof for generalization and further use.

Groves' Lemma. *Let A, B and C be elements in any group G.*

(i) $[C, B, A] = [C, B]^{-1}[C, A]^{-1}[B, A, C]^{-1}([C, B][C, A][C, A, B])^{[B,A]}$

(ii) If $[B, A, C], [C, A, B], [C, A; B, A]$ and $[C, B; C, A]$ are trivial in G, then $[C, B, A] = [C, B; B, A]$.

(iii) If $[B, A, C], [C, B, B], [C, A, A], [C, A, B]$ and $[C, A, C]$ are trivial in G, then $[C, B, A]$ and $[C, B; B, A]$ are trivial in G.

(iv) If $[C, B], [C, A, A]$ and $[C, A, B]$ are trivial in G, then so is $[B, A, C]$.

(v) If $[C, A; B, A], [C, B; B, A]$ and $[C, B; C, A]$ are trivial in G, then when any two of $[C, B, A], [B, A, C]$ and $[C, A, B]$ are trivial in G, the third is also.

(vi) If $[C, A, C], [C, B, C], [C, A; B, A]$ and $[C, B; B, A]$ are trivial in G, then $[C, B, A] = [B, A, C]^{-1}[C, B; C, A][C, A, B]^{[B,A]}$.

Proof. (i) Note that $BA = AB[B, A]$ and apply the commutator identity $[x, yz] = [x, z][x, y][x, y, z] = [x, z][x, y]^z$ to both $[C, BA]$ and $[C, (AB)[B, A]]$. Equate these and solve for $[C, B, A]$.

(ii) With $[B, A, C] = 1$, and $[C, A, B] = 1$, the equation in part (i) simplifies to $[C, B, A] = [C, B]^{-1}[C, A]^{-1}([C, B][C, A])^{[B,A]}$. With $[C, A; B, A] = 1$, and $[C, B; C, A] = 1$, we can write this as $[C, B, A] = [C, B]^{-1}[C, A]^{-1}[C, A][C, B]^{[B,A]}$ $= [C, B]^{-1}[C, B]^{[B,A]} = [C, B; B, A]$.

(iii) Observe first that with $[C, A, A], [C, A, B]$, and $[C, A, C]$ trivial in G, we do have that $[C, A; B, A]$ and $[C, B; C, A]$ are trivial in G also, so we have $[C, B, A] = [C, B; B, A]$ by part (ii). Write this as $[C, B, A] = [C, B, (B^{-1}A^{-1}B)A]$, and use $[x, yz] = [x, z][x, y]^z$ to rewrite this as $[C, B, A] = [C, B, A][C, B, B^{-1}A^{-1}B]^A$. Multiply this last on the left by $[C, B, A]^{-1}$ and then conclude that $[C, B, B^{-1}A^{-1}B]$ is trivial. It then follows, using the triviality of $[C, B, B]$ that $[C, B, A^{-1}]$ is trivial and hence $[C, B, A]$ is trivial.

(iv) Since $[C, B]$ and hence $[C, B, A]$ are trivial, from the first part of Groves' Lemma we have $1 = [C, A]^{-1}[B, A, C]^{-1}([C, A][C, A, B])^{[B,A]}$. Since both $[C, A, A]$ and $[C, A, B]$ are trivial in G, we obtain $1 = ([B, A, C]^{-1})^{[C,A]}$ and the triviality of $[B, A, C]$ follows.

(v) From part (i), with $[C, B; B, A]$ and $[C, A; B, A]$ trivial, we have

$$[C, B, A] = [C, B]^{-1}[C, A]^{-1}[B, A, C]^{-1}[C, B][C, A][C, A, B]^{[B,A]}$$

If $[B, A, C]$ and $[C, A, B]$ are trivial, then $[C, B, A] = [C, B; C, A]$. If $[C, B, A]$ and $[B, A, C]$ are trivial, then $1 = [C, B; C, A][C, A, B]^{[B,A]}$. If $[C, B, A]$ and $[C, A, B]$ are trivial, then $[B,A,C] = [C,B][C,A][C,B]^{-1}[C,A]^{-1} = [C,B;C,A]^{([C,B][C,A])^{-1}}$. See also Proposition W, below.

(vi) Since $[C,B]$ commutes with both $[B,A]$ and with C, it commutes with $[B,A,C]$. Similarly, $[C,A]$ commutes with $[B,A,C]$. Since $[C,B]^{[B,A]} = [C,B]$ and $[C,A]^{[B,A]} = [C,A]$, the result then follows from the equation of part (i).

Parts (i), (ii), and (iii) and their proofs are visible in [3]. We find it convenient to also place the other parts under the common umbrella of "Groves' Lemma." Both Groves' Lemma and Proposition W, below, were originally formulated as tools to use in the proof that Sims' question has a positive answer for weight 5 and rank 3. Since these results do simplify some proofs in this paper, we have included them here also. We make repeated use of all cases of Groves' Lemma in [5].

For any group G and elements x, y, and z in G, define $W(x,y,z)$ by

$$W(x,y,z) = [z,y]^{-1}[z,x]^{-1}[y,x]^{-1}[z,y][z,x][y,x].$$

$W(x,y,z)$ can be written in a number of different moderately lengthy forms. We will find it convenient to be able to make brief reference to this element without immediately committing to any one of the forms.

Proposition W. *If G is any group and x, y, and z are elements of G, then*

$$(i)\quad [z,y,x] = \left([y,x,z]^{-1}\right)^{[z,x][z,y]} W(x,y,z)[z,x,y]^{[y,x]}$$
$$(ii)\quad W(x,y,z) = [z,y;z,x][z,y;y,x]^{[z,x]}[z,x;y,x]$$
$$(iii)\quad W(x,y,z) = [z,x;y,x]^{[z,y]}[z,y;y,x][z,y;z,x]^{[y,x]}$$

Proof. These may be verified by substitution and reduction in the free group on $\{x,y,z\}$. The subgroup of G generated by x, y and z is an image of this free group. The first part of the proposition can also be regarded as a restatement of the first part of Groves' Lemma.

Like Lemma 2.2 above and Theorem 2.6 below, the following result is known by Sims, [7]. The proofs in [7] rely on a computer implementation of a string rewriting algorithm. Our proofs here are reasonably short and direct. The method anticipates that used in [5] to prove that $N_5 = \gamma_5$ for all r.

Theorem 2.5. *$N_4 = \gamma_4$ for arbitrary $r \geq 2$.*

Proof. By the Basic Lemma and Lemma 2.2, it will suffice to show that $[c, x_\ell] \in N_4$ (or equivalently $[c, x_\ell] \equiv 1$ modulo N_4) whenever c is a basic commutator of weight 3 and $x_\ell \in X$. Then $c = [x_j, x_i, x_k]$ where $i < j$ and $i \leq k$. If $\ell \geq k$, then $[c, x_\ell]$ is a basic commutator of weight 4, so we may assume that we have $\ell < k$ as well as $i < j$ and $i \leq k$. We use part (iii) of Groves' Lemma with $C = [x_j, x_i], B = x_k$, and $A = x_\ell$. Then $[B,A] = [x_k, x_\ell]$ which is basic of weight 2, so $[B,A,C] = [x_k, x_\ell; x_j, x_i]$ which, when nontrivial in F, is either a basic commutator of weight 4, or else the inverse of such, depending upon the order we have chosen in $\mathcal{C}$ for the basic commutators of weight 2. Also $[C,B,B] = [x_j, x_i, x_k, x_k]$ is always a basic commutator of weight 4, so it will suffice to show that $[C,A]$ commutes with A, B and C modulo N_4. We will see that the following four cases naturally occur:

$$(1)\quad \ell = i \qquad (2)\quad \ell > i \text{ and } j \geq \ell \qquad (3)\quad \ell > i \text{ and } j < \ell \qquad (4)\quad \ell < i$$

Case 1: $\ell = i$ In this case, we have $A = x_\ell = x_i$, so that $[C,A] = [x_j, x_i, x_i]$. Then $[x_j, x_i, x_i, x_i], [x_j, x_i, x_i, x_j]$ and $[x_j, x_i, x_i, x_k]$ are all basic commutators of weight 4, so $[C,A]$ commutes modulo N_4 with $A = x_i, B = x_k$ and $C = [x_j, x_i]$.

Case 2: $\ell > i$ and $j \geq \ell$ With $\ell > i$, $[C,A] = [x_j, x_i, x_\ell]$ is a basic commutator of weight 3 and then $[C,A,B] = [x_j, x_i, x_\ell, x_k]$ and $[C,A,A] = [x_j, x_i, x_\ell, x_\ell]$ are basic commutators of weight 4. It remains to show that $[C,A,C]$ is trivial modulo N_4 and for this it will suffice to show that $[x_j, x_i, x_\ell, x_i]$ and $[x_j, x_i, x_\ell, x_j]$ are trivial modulo N_4. The former is trivial by case 1. Since we assume here that $j \geq \ell$, $[x_j, x_i, x_\ell, x_j]$ is a basic commutator of weight 4 and we are done.

Case 3: $\ell > i$ and $j < \ell$ Exactly as in case 2, to show that $[x_j, x_i, x_k, x_\ell]$ is trivial modulo N_4, it will suffice to show that $[C, A, x_j] = [x_j, x_i, x_\ell, x_j]$ is trivial modulo N_4. This is immediate from case 2, applied to $[x_j, x_i, x_{\hat{k}}, x_{\hat{\ell}}]$ where $\hat{k} = \ell$ and $\hat{\ell} = j$.

Case 4: $\ell < i$ We again have that $[C,A] = [x_j, x_i, x_\ell]$, but with $\ell < i$, this is no longer a basic commutator. We have in this case that $\ell < i < j$ and $\ell < i \leq k$. We need to use eventually that $[x_i, x_\ell, x_j]$ and $[x_j, x_\ell, x_i]$ are basic commutators of weight 3 and that both of these commute with x_i, x_j, x_k and x_ℓ. The commutators $[x_j, x_\ell, x_i, x_i], [x_j, x_\ell, x_i, x_j], [x_j, x_\ell, x_i, x_k]$ and $[x_i, x_\ell, x_j, x_j]$ are basic commutators of weight 4. With a change of notation, $[x_i, x_\ell, x_j, x_\ell]$ and $[x_j, x_\ell, x_i, x_\ell]$ are trivial modulo N_4 by case 1. Similarly, $[x_i, x_\ell, x_j, x_i]$ is trivial modulo N_4 by case 2, while $[x_i, x_\ell, x_j, x_k]$ is basic if $k \geq j$ and is trivial by case 2 if $k < j$.

By either the second or the third part of Proposition W, we see that $W(x_\ell, x_i, x_j)$ is trivial modulo N_4. Using this, and the first part of Proposition W, we have that $[x_j, x_i, x_\ell]$ is equivalent to $([x_i, x_\ell, x_j]^{-1})^{[x_j,x_\ell][x_j,x_i]}[x_j, x_\ell, x_i]^{[x_i,x_\ell]}$. Since we have just seen that $[x_i, x_\ell, x_j]$ and $[x_j, x_\ell, x_i]$ commute with x_i, x_j and x_ℓ, we have that $[x_j, x_i, x_\ell] \equiv [x_i, x_\ell, x_j]^{-1}[x_j, x_\ell, x_i]$ modulo N_4. Since these factors commute with x_k also, we are done.

Theorem 2.6. *If F is free with rank 2, then $N_5 = \gamma_5(F)$.*

Proof. Let F be free on the ordered alphabet $\{a, b\}$. Then the basic commutators of weight 4 are $[b,a,a,a], [b,a,a,b]$ and $[b,a,b,b]$. By Theorem 2.5 and the Basic Lemma, it will suffice to prove that $[b,a,a,b,a]$ and $[b,a,b,b,a]$ are trivial modulo N_5.

To show that $[b,a,a,b,a]$ and $[b,a,a,b;\, b,a]$ are trivial, we use part (iii) of Groves' Lemma with $C = [b,a,a]$, $B = b$ and $A = a$. Then $[B,A,C]^{-1}, [C,B,B], [C,A,A]$ and $[C,A,B]$ are basic commutators of weight 5. Since $[C,A] = [b,a,a,a]$, it commutes with a, b and $C = [b,a,a]$ modulo N_5.

To show that $[b,a,b,b,a]$ is trivial modulo N_5, we first show that $[b,a,b;\, b,a,a]$ is trivial modulo N_5 and that $[b,a,b,a]$ is equivalent modulo N_5 to $[b,a,a,b]$.

To show that $[b,a,b;\, b,a,a]$ is trivial modulo N_5, we use part (iv) of Groves' Lemma with $C = [b,a,a]$, $B = [b,a]$ and $A = b$. Then $[C,B]$ and $[C,A,A]$ are basic commutators of weight 5. Since $[C, A] = [b, a, a, b]$, it commutes with $B = [b, a]$ by the second paragraph of this proof.

To show that $[b, a, b, a]$ is equivalent to $[b, a, a, b]$ modulo N_5, we use part (vi) of Groves' Lemma with $C = [b, a]$, $B = b$ and $A = a$. Then all of the commutators $[C, A, C]$, $[C, B, C]$, $[C, A;\ B, A]$ and $[C, B;\ B, A]$ are basic commutators of weight 5, so we have $[b, a, b, a] \equiv [b, a;\ b, a]^{-1}[b, a, b;\ b, a, a][b, a, a, b]^{[b,a]}$. Since $[b, a, b;\ b, a, a]$ is trivial by the previous paragraph and $[b, a, a, b]$ commutes with $[b, a]$ by the second paragraph, we have $[b, a, b, a] \equiv [b, a, a, b]$.

To prove that $[b, a, b, b, a]$ is trivial modulo N_5, we use part (iii) of Groves' Lemma with $C = [b, a, b]$, $B = b$ and $A = a$. Then $[B, A, C]^{-1}$ and $[C, B, B]$ are basic commutators of weight 5. By the previous paragraph, we have that $[C, A]$ is equivalent to $[b, a, a, b]$, so $[C, A]$ commutes with $A = a$, $B = b$ and $C = [b, a, b]$.

3: Varieties with Commutator Laws

Following [6], write $\mathfrak{A}$ for the variety of abelian groups and $\mathfrak{N}_2$ for the variety of groups which are nilpotent with nilpotency class at most 2. In this section, we want to prove that Sims' question has a positive answer in the variety $[\mathfrak{A}, \mathfrak{A}]$ of metabelian groups and in the variety $[\mathfrak{N}_2, \mathfrak{A}]$.

Theorem 3.8. *$\gamma_n(F) \cdot K = N_n \cdot K$ if either $K = [\gamma_2(F), \gamma_2(F)]$ or if $K = [\gamma_3(F), \gamma_2(F)]$.*

Equivalently, for a group G that is metabelian or a group G from the variety $[\mathfrak{N}_2, \mathfrak{A}]$ (i.e., a group which satisfies the law $[[x_1, x_2, x_3], [x_4, x_5]]$), we can verify that G is nilpotent of class $n - 1$ by confirming that all of the basic commutators of weight n are trivial in G.

By Theorem 2.5, we may assume that $n \geq 5$. As noted in the introduction, $\gamma_n(F)$ is the normal closure in F of the set of simple commutators of weight n. Theorem 3.8 is then a consequence of the following theorem.

Theorem 3.7. *Suppose either that $n \geq 2$ and $K = [\gamma_2(F), \gamma_2(F)]$ or else that $n \geq 5$ and $K = [\gamma_3(F), \gamma_2(F)]$. If w is any simple commutator of weight n in F, then in F/K we will have one of the following three possibilities occur, where c_1 and c_2 are simple basic commutators of weight n.*

$$wK = c_1K \qquad or \qquad wK = c_1^{-1}K \qquad or \qquad wK = c_1^{-1}c_2K.$$

We will prove Theorem 3.7 after establishing a few easy facts about groups in the varieties $[\mathfrak{A}, \mathfrak{A}]$ and $[\mathfrak{N}_2, \mathfrak{A}]$. In terms of verifying nilpotence, Theorem 3.7 is substantially stronger than Theorem 3.8. To verify that a group G in one of these varieties is nilpotent of class $n - 1$ for some sufficiently large n, we need only show that the simple basic commutators of weight n are all trivial in G.

If G is any metabelian group, then it is well-known and easily verified that for any elements $c_1, c_2, c_3 \in G'$ and $x \in G$, we have $[c_1c_2, x] = [c_1, x][c_2, x]$, $[c_1^{-1}, x] = [c_1, x]^{-1}$ and $c_3^{c_2} = c_3$. If G is any group from the variety $[\mathfrak{N}_2, \mathfrak{A}]$, then it is similarly verified that for any elements $c_1, c_2 \in G', c_3 \in \gamma_3(G)$ and $x \in G$, we have $[c_1c_2, x] = [c_1, x][c_2, x]$, $[c_1^{-1}, x] = [c_1, x]^{-1}$ and $c_3^{c_2} = c_3$.

Recall that we have defined the element $W(a, b, c)$ in any group G by $W(a, b, c) = [c, b]^{-1}[c, a]^{-1}[b, a]^{-1}[c, b][c, a][b, a]$. It is then clear that $W(a, b, c)$ is trivial in any metabelian group G. If G is a group from the variety $[\mathfrak{N}_2, \mathfrak{A}]$, then from Proposition W, in section 2, we see that $W(a, b, c)$ is a product of three commuting factors $[c, b; c, a]$, $[c, b; b, a]$ and $[c, a; b, a]$.

Lemma 3.1. *If G is any metabelian group, then $[c, b, a] = [b, a, c]^{-1}[c, a, b]$ for any $a, b, c \in G$ and $[c, b, a] = [c, a, b]$ if $c \in G'$.*

Proof. Since $W(a, b, c)$ is trivial in G, the first conclusion follows from part (i) of Proposition W in section 2. Since $[b, a, c] = [[b, a], c]$ by definition, the second conclusion then follows from the first.

Lemma 3.2. *Suppose that G is a group in the variety $[\mathfrak{N}_2, \mathfrak{A}]$. If $a, b, c, d \in G$, then $[d, c; , b, a]$ is central in G. As a consequence, $W(a, b, c)$ is central in G.*

Proof. We do not require that a, b, c and d are distinct here, so we will have that $W(a, b, c)$ is central once we have that $[c, b; c, a]$, $[c, b; b, a]$ and $[c, a; b, a]$ are all central by the first conclusion of this lemma.

Let x be an arbitrary element of G. It will suffice to show that $[d, c; b, a; x]$ is trivial. We use part (v) of Groves' Lemma with $C = [d, c]$, $B = [b, a]$ and $A = x$ to prove that $[C, B, A] = [d, c; b, a; x]$ is trivial.

Lemma 3.3. *Suppose that G is a group in the variety $[\mathfrak{N}_2, \mathfrak{A}]$. Then for any $a, b, c \in G$, $[c, b, a] = [b, a, c]^{-1}[c, a, b]W(a, b, c)$. If $c \in G'$, then $[c, b, a] = [b, a, c]^{-1}[c, a, b]$. If $c \in \gamma_3(G)$, then $[c, b, a] = [c, a, b]$.*

Proof. The first conclusion again follows immediately from part (i) of Proposition W. From either part (ii) or part (iii) of Proposition W, we see that $W(a, b, c)$ is trivial for $G \in [\mathfrak{N}_2, \mathfrak{A}]$ if $c \in G'$. When $c \in \gamma_3(G)$, then $[b, a, c]$ is trivial.

Lemma 3.4. *Suppose that G is a group in the variety $[\mathfrak{N}_2, \mathfrak{A}]$. Then for any $a, b, c, d \in G$, $[d, c, b, a] = [d, c; b, a][d, c, a, b]$.*

Proof. Replace c by $[d, c]$ in the second conclusion of Lemma 3.3.

Lemma 3.5. *If G is any metabelian group, $x \in G', y_i \in G$ and $\pi \in S_t$ is any permutation, then $[x, y_1, y_2, \ldots, y_t] = [x, y_{\pi 1}, y_{\pi 2}, \ldots, y_{\pi t}]$.*

Proof. This is Lemma 34.51 from [6]. As observed in the proof there, this follows easily from Lemma 3.1.

Lemma 3.6. *Suppose that G is a group in the variety $[\mathfrak{N}_2, \mathfrak{A}]$ and that either $x \in \gamma_3(G)$ and $t \geq 1$ or else that $x \in G'$ and $t \geq 3$. If $y_i \in G$ for $1 \leq i \leq t$ and $\pi \in S_t$ is any permutation, then $[x, y_1, y_2, \ldots, y_t] = [x, y_{\pi 1}, y_{\pi 2}, \ldots, y_{\pi t}]$.*

Proof. When $x \in \gamma_3(G)$, then this lemma follows from the final conclusion of Lemma 3.3, using the fact that every permutation is a product of transpositions, in the same way that Lemma 3.5 follows from Lemma 3.1.

Suppose then that $x \in G'$ and $t \geq 3$. Note that it will suffice to prove the lemma in the case where x is a simple basic commutator $[d, c]$. We will assume that this is so, mostly so we can reuse the notation x. If $\pi 1 = 1$, then by the previous paragraph with $x = [d, c, y_1] \in \gamma_3(G)$ and π replaced by its restriction to $\{2, 3, \ldots, t\}$, we are done. If $\pi 1 = 2$, then $[d, c, y_1, y_2] = [d, c; y_1, y_2][d, c, y_2, y_1]$ by Lemma 3.4 and then $[d, c, y_1, y_2, y_3] = [d, c, y_2, y_1, y_3]$ since $[d, c; y_1, y_2]$ is central by Lemma 3.2. It then follows that $[d, c, y_1, y_2, y_3, \ldots, y_t] = [d, c, y_2, y_1, y_3, \ldots, y_t]$ and we obtain the conclusion by the result of the first paragraph with $x = [d, c, y_2]$.

If $\pi 1 > 2$, let τ be the transposition which interchanges 2 and $\pi 1$. Then we can use the conclusion of the first paragraph with $x = [d, c, y_1] \in \gamma_3(G)$ and the permutation τ to find $[d, c, y_1, y_2, y_3, \ldots, y_t] = [d, c, y_1, y_{\pi 1}, y_3, \ldots, y_2, \ldots, y_t]$. We are then done, since $[d, c, y_1, y_{\pi 1}, y_3, \ldots, y_t]$ is one of the cases we treated in the second paragraph.

Proof of Theorem 3.7. Much of the proof will be the same whether we are in the case where $K = [\gamma_2(F), \gamma_2(F)]$ or in the case where $K = [\gamma_3(F), \gamma_2(F)]$. Write $w = [x_{i_1}, x_{i_2}, \ldots, x_{i_n}]$. Since we are only concerned with nontrivial commutators, we may assume that either $i_1 < i_2$ or $i_2 < i_1$. (If $i_1 = i_2$ then w is trivial and we have $wK = c_1^{-1} c_1 K$.) We want to reduce immediately to the case where $i_2 < i_1$. If we do have $i_1 < i_2$, then $[x_{i_1}, x_{i_2}] = [x_{i_2}, x_{i_1}]^{-1}$ so $[x_{i_1}, x_{i_2}, x_{i_3}]K = [[x_{i_2}, x_{i_1}]^{-1}, x_{i_3}]K = [x_{i_2}, x_{i_1}, x_{i_3}]^{-1}K$ and eventually $[x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_n}]K = [x_{i_2}, x_{i_1}, x_{i_3}, \ldots, x_{i_n}]^{-1}K$. We will continue with the assumption that we have $i_1 > i_2$ and we will show that in this case we have either $wK = c_1 K$ or $w = c_1^{-1} c_2 K$. We next use either Lemma 3.5 or Lemma 3.6 to find that $[x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_n}]K = [x_{i_1}, x_{i_2}, x_{i_{\pi 3}}, x_{i_{\pi 4}}, \ldots, x_{i_{\pi n}}]K$ where π is chosen to be a permutation on $\{3, 4, \ldots, n\}$ such that $i_{\pi 3} \leq i_{\pi 4} \leq \cdots \leq i_{\pi n}$. We are then done if $i_2 \leq i_{\pi 3}$. If not, then we may assume, in order to better manage notation, that we have $i_1 > i_2 > i_3 \leq i_4 \leq i_5 \leq \cdots \leq i_n$.

In the case where $K = [\gamma_2(F), \gamma_2(F)]$, we next use Lemma 3.1 to express $[x_{i_1}, x_{i_2}, x_{i_3}]K$ as $[x_{i_2}, x_{i_3}, x_{i_1}]^{-1}[x_{i_1}, x_{i_3}, x_{i_2}]K$. Then we see $[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}]K = [[x_{i_2}, x_{i_3}, x_{i_1}]^{-1}[x_{i_1}, x_{i_3}, x_{i_2}], x_{i_4}]K = [x_{i_2}, x_{i_3}, x_{i_1}, x_{i_4}]^{-1}[x_{i_1}, x_{i_3}, x_{i_2}, x_{i_4}]K$. We can continue in this manner to find that $[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, \ldots, x_{i_n}]K$ is equal to the product $[x_{i_2}, x_{i_3}, x_{i_1}, x_{i_4}, \ldots, x_{i_n}]^{-1}[x_{i_1}, x_{i_3}, x_{i_2}, x_{i_4}, \ldots, x_{i_n}]K$. If it happens that $i_1 > i_4$ and/or that $i_2 > i_4$ we conclude by applying Lemma 3.5 again.

Suppose then that we are in the case where $K = [\gamma_3(F), \gamma_2(F)]$ and $n \geq 5$. In this case, we have $[x_{i_1}, x_{i_2}, x_{i_3}]K = [x_{i_2}, x_{i_3}, x_{i_1}]^{-1}[x_{i_1}, x_{i_3}, x_{i_2}]W(x_{i_3}, x_{i_2}, x_{i_1})K$ by Lemma 3.3. Since $W(x_{i_3}, x_{i_2}, x_{i_1})K$ is central by Lemma 3.2, we again find that $[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, \ldots, x_{i_n}] = [x_{i_2}, x_{i_3}, x_{i_1}, x_{i_4}, \ldots, x_{i_n}]^{-1}[x_{i_1}, x_{i_3}, x_{i_2}, x_{i_4}, \ldots, x_{i_n}]K$. If it happens that $i_1 > i_4$ and/or that $i_2 > i_4$ we conclude by applying Lemma 3.6 again.

4 Using higher commutators

For $s \geq 1$, we will use the notation $N_{n:s}$ for the normal closure in F of the basic commutators from $\mathcal{C}$ having weights n through $n + s - 1$, inclusive. That is,

$$N_{n:s} = \left(\bigcup_{j=n}^{n+s-1} \{ c \in \mathcal{C} \mid wt(c) = j \} \right)^F .$$

In this notation, the normal closure, N_n, of the set of basic commutators of weight n, becomes $N_{n:1}$. We want to show, consecutively, that $\gamma_n(F)$ is equal to $N_{n:n-1}, N_{n:n-2}$ and $N_{n:n-3}$. The first of these is contained in work of Martin Ward, see [8, 9, 4] and all of the cases rely heavily on the following theorem of Martin Ward. We will use only the part of this theorem that says that the given set of "basic invertators" generate $\gamma_n(F)$. We will not need the more difficult conclusion that this set is a free basis for $\gamma_n(F)$.

Theorem 4.1. *(Martin Ward, [4,9]) For $n \geq 2$, $\gamma_n(F)$ is freely generated by the set of all commutators of the form $[c, b_1^{\beta_1}, b_2^{\beta_2}, \ldots, b_q^{\beta_q}]$ where $q \geq 1$, c and the b_i are basic commutators having weights less than n, $c > b_1 \leq b_2 \leq \cdots \leq b_q$, $\beta_i = \pm 1$, $wt([c, b_1^{\beta_1}]) \geq n$, if $c = [c_1, c_2]$ then $c_2 \leq b_1$ and if $b_i = b_j$, then $\beta_i = \beta_j$.*

Proposition 4.2. *For $n \geq 2$, $\gamma_n(F) = N_{n:n-1}$.*

Proof. It is clear that $N_{n:n-1} \subseteq \gamma_n(F)$, and it will suffice to show that each of the free generators, $[c, b_1^{\beta_1}, b_2^{\beta_2}, \ldots, b_q^{\beta_q}]$ from the theorem of Martin Ward is in $N_{n:n-1}$. For this, we use induction on q. In the base case, $q = 1$, assume first that we have a free generator, $[c, b_1]$, where $\beta_1 = +1$. Then by the hypotheses for Theorem 4.1, we have that $[c, b]$ is a basic commutator, that $wt(c) < n$, $wt(b_1) < n$ and $wt([c, b_1]) \geq n$. We see that $[c, b_1]$ is a basic commutator with $n \leq wt([c, b_1]) \leq 2n - 2$, so $[c, b_1] \in N_{n:n-1}$. If we have instead the free generator $[c, b_1^{-1}]$, then $[c, b_1^{-1}] = ([c, b_1]^{-1})^{b_1^{-1}}$ which is also in $N_{n:n-1}$. Suppose then by induction, that $C = [c, b_1^{\beta_1}, b_2^{\beta_2}, \ldots, b_{q-1}^{\beta_{q-1}}]$ has been shown to be in $N_{n:n-1}$. Then $[C, b_q] = C^{-1}C^{b_q}$ and $[C, b_q^{-1}] = ([C, b_q]^{-1})^{b_q^{-1}}$ are in $N_{n:n-1}$ also.

Lemma 4.3. *Assume that $n \geq 3$ and that a and b are basic commutators of weight $n - 1$. Then $[b, a] \in N_{n:n-2}$.*

Proof. Since $[a, b] = [b, a]^{-1}$, this lemma is symmetric in a and b. Write $a = [a_2, a_1]$ and $b = [b_2, b_1]$ where a_1, a_2, b_1, and b_2 are basic commutators. By symmetry, we may assume that $a_1 \geq b_1$. Then $[b, a_1]$ and $[b, a_2]$ are both basic commutators and these have weight at least n and at most $2n - 3$, so b commutes with both a_1 and a_2 modulo $N_{n:n-2}$. Hence b commutes with $a = [a_2, a_1]$ modulo $N_{n:n-2}$.

Proposition 4.4. *For $n \geq 3$, $\gamma_n(F) = N_{n:n-2}$.*

Proof. By Proposition 4.2, it will suffice to prove that $N_{n:n-1} \subseteq N_{n:n-2}$. Suppose that $c = [b, a]$ is a basic commutator with $n \leq wt(c) \leq 2n - 2$. If $wt(c) \leq 2n - 3$ or if $wt(b) \geq n$, then it is immediate that $c \in N_{n:n-2}$. Since $wt(a) \leq wt(b)$, this leaves only the possibility that $wt(a) = wt(b) = n - 1$ and this case was covered by the previous lemma.

To prove that $\gamma_n(F) = N_{n:n-3}$ for $n \geq 4$, we will prove that $N_{n:n-2} \subseteq N_{n:n-3}$. The following two lemmas will be useful.

Lemma 4.5. *Assume that $n \geq 4$, that c is a simple basic commutator of weight $n-3$, that x, y, and z are letters with $x \leq y \leq z$ and that $[c, x]$ is a basic commutator of weight $n - 2$. Then $[c, x, z, y] \in N_{n:n-3}$.*

Proof. If $n = 4$, we already know that $\gamma_4(F) = N_4 = N_{4:1}$, so $[c, x, z, y] \in N_{4:1}$. For the rest of this proof, we assume that $n \geq 5$, so that c is a basic commutator with weight at least 2. Since the conclusion is clear when $y = z$, we will also assume that $y < z$.

Suppose first that $y = x$. We use part (iii) of Groves' Lemma with $C = [c, x], B = z$ and $A = x$. Then $[B, A, C]^{-1}, [C, B, B], [C, A, A]$ and $[C, A, B]$ are all basic commutators of weight n. Since $[C, A] = [c, x, x]$, we see that $[C, A, c]$ is a basic commutator of weight $2n - 4$ and that $[C, A, x]$ is a basic commutator of weight n. Thus, $[C, A]$ commutes with both c and x modulo $N_{n:n-3}$, so $[C, A]$ commutes with $C = [c, x]$ modulo $N_{n:n-3}$.

Now suppose that $y > x$. We use part (iii) of Groves' Lemma with $C = [c, x], B = z$ and $A = y$. Then $[B, A, C]^{-1}, [C, B, B], [C, A, A]$ and $[C, A, B]$ are all basic commutators of weight n. Since $[C, A] = [c, x, y]$ is a simple basic commutator and $wt(c) \geq 2$, $[C, A, c]$ is a basic commutator of weight $2n - 4$ and $[C, A]$ commutes with c modulo $N_{n:n-3}$. The previous paragraph applies to $[c, x, y, x]$, so $[C, A]$ also commutes with x and hence with $C = [c, x]$ modulo $N_{n:n-3}$.

Lemma 4.6. *Assume that $n \geq 4$, that c_1 and c_2 are simple basic commutators of weight $n - 3$, that x, y, and z are letters with $x < y \leq z$ and that $[c_1, x]$ and $[c_2, y]$ are basic commutators of weight $n - 2$. Then $[[c_2, y, z], [c_1, x]] \in N_{n:n-3}$.*

Proof. If $n = 4$, we already know that $[[c_2, y, z], [c_1, x]] \in \gamma_4(F) = N_4 = N_{4:1}$. For the rest of this proof, we assume that $n \geq 5$, so c_1 and c_2 are simple basic commutators with weight $n - 3 \geq 2$.

We use part (iv) of Groves' Lemma with $C = [c_1, x], B = [c_2, y]$ and $A = z$ to prove that $[B, A, C]$ is trivial modulo $N_{n:n-3}$. Since $[C, B]$ has weight $2n - 4$ and either $[C, B]$ or $[C, B]^{-1}$ is a basic commutator, we know that $[C, B]$ is trivial modulo $N_{n:n-3}$. We observe that $[C, A, A]$ is a simple basic commutator of weight n. To show that $[C, A, B]$ is also trivial modulo $N_{n:n-3}$, it will suffice to show that $[C, A, c_2]$ and $[C, A, y]$ are trivial modulo $N_{n:n-3}$. The latter follows from the previous lemma. Since $wt(c_2) \geq 2$, $[C, A, c_2]$ is a basic commutator of weight $2n-4$. Since $n \geq 5$, this weight is at least $n+1$ and $[C, A, c_2]$ is also trivial modulo $N_{n:n-3}$.

Theorem 4.7. *For $n \geq 4$, $\gamma_n(F) = N_{n:n-3}$.*

Proof. This is known when $n = 4$, so we may assume that $n \geq 5$.

By Proposition 4.4. it will suffice to show that $N_{n:n-2} \subseteq N_{n:n-3}$. Suppose that $c = [b, a]$ is a basic commutator with $n \leq wt(c) \leq 2n - 3$. If $wt(c) \leq 2n - 4$ or if $wt(b) \geq n$, then it is immediate that $c \in N_{n:n-3}$. Since $wt(b) \geq wt(a)$, it will suffice to prove that $[b, a] \in N_{n:n-3}$ when a and b are basic commutators with $wt(a) = n - 2$ and $wt(b) = n - 1$.

Next, we want to reduce to the case where a and b are simple basic commutators. Write $a = [a_2, a_1]$ and $b = [b_2, b_1]$ where a_1, a_2, b_1 and b_2 are basic commutators. Suppose first that $wt(a_1) \geq 2$ and that $a_1 \geq b_1$. Then both $[b, a_1]$ and $[b, a_2]$ are basic commutators and these have weight at least $n + 1$ and at most $2n - 5$. Hence, in this case, b commutes with a_1, a_2 and a modulo $N_{n:n-3}$. Similarly, suppose that $wt(b_1) \geq 2$ and that $b_1 \geq a_1$. Then both $[a, b_1]$ and $[a, b_2]$ are basic commutators and these have weight at least n and at most $2n-5$. Hence, in this case, a commutes with b_1, b_2 and b modulo $N_{n:n-3}$. If, in the case where $wt(a_1) \geq 2$, we have $b_1 > a_1$, then b_1 also has weight at least two: if, in the case where $wt(b_1) \geq 2$, we have $a_1 > b_1$, then a_1 also has weight at least two. Thus, for the rest of this proof, we may assume that a and b are both simple basic commutators.

Write a as $a = [c_1, x]$ where c_1 is a simple basic commutator with weight $n-3$ and x is a letter. Also write $b = [c_2, y, z]$ where y and z are letters with $y \leq z$, c_2 is a simple basic commutator with weight $n-3$ and $[c_2, y]$ is a simple basic commutator with weight $n-2$. If $x \geq z$, then $[b, x]$ is a simple basic commutator with weight n and $[b, c_1]$ is a basic commutator with weight $2n-4$. We see that b commutes with c_1, x and $a = [c_1, x]$ modulo $N_{n:n-3}$. If $y \leq x < z$, then $[b, x]$ is trivial modulo $N_{n:n-3}$ by Lemma 4.5 and $[b, c_1]$ is a basic commutator of weight with weight $2n-4$. We again see that b commutes with c_1, x and a modulo $N_{n:n-3}$. Finally, if $x < y \leq z$, then $[b, a] \in N_{n:n-3}$ by Lemma 4.6.

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