AXIOMS FOR THICKNESS OF FEATHERS

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Abstract. We approach thickness of feathers from an axiomatic point of view. We show that our axioms are independent and we use our axioms to establish general properties for arbitrary definitions of thickness which satisfy these axioms.

1. Axioms for Thickness

Informally, a feather is a semigroup derivation diagram with the labels on the edges removed. See [2] for a discussion of semigroup derivation diagrams and see [1] for a more formal definition of feathers.

If $M$ is a feather, we will use the notation $\hat{M}$ for the reflection of $M$ across some line in the plane and the notation $\leftarrow M$ for the feather obtained by reversing the direction of all of the edges in $M$. A medial vertex for a two-sided region $D$ is a vertex of $D$ which is neither the initial vertex nor the terminal vertex for $D$. A region $D$ of $M$ is a strongly interior region of $M$ if no edge of $D$ and no medial vertex of $D$ occurs on the boundary of $M$. Let $M$ be a feather with bottom side $\alpha_M$, top side $\omega_M$, initial vertex $v_\iota$ and terminal vertex $v_\tau$. A meridian in $M$ is a positive path in $M$ from $v_\iota$ to $v_\tau$. For example, $\alpha_M$ and $\omega_M$ are meridians.

We can define a partial order on the meridians in $M$ by $\mu \leq \nu$ if $\mu^{-1}\nu$ is the counterclockwise boundary of a feathery submap $N$ of $M$. We call such a feathery submap a layer. Here, we allow that $\mu = \nu$ and that $N$ is a feather without regions. A nontrivial layering of $M$ with $k$ layers is a sequence of meridians

$$\mathcal{L} = \{\mu_0 = \alpha_M, \mu_1, \mu_2, \ldots, \mu_k = \omega_M\}$$

such that for $1 \leq j \leq k$, the walk $\mu_{j-1}\mu_j^{-1}$ is the counterclockwise boundary of a feathery submap $N_j$ of $M$ where each $N_j$ has at least one region. The submaps $N_1, N_2, \ldots, N_k$ are the layers of the layering. A vertex $v$ in the feather $M$ is a cut vertex in $M$ if $M - v$ is disconnected. Every cut vertex in $M$ must be on both the top and bottom side of $M$. A feather is nonseparable if it has no cut vertices. A block of a feather is a maximal nonseparable submap. A block of a feather is itself a feather. A block is nontrivial if it contains at least one region. We call a block without any regions a trivial block. For any feather $M$, we may write $M = M_1M_2\ldots M_t$ where the $M_j$ are all of the blocks of $M$ and their common vertices are all of the cut vertices of $M$. A region $D$ of the map $M$ is appended on the top side of $M$ if $\omega_D$ is a segment of $\omega_M$ and is appended on the bottom side of $M$ if $\alpha_D$ is a segment of $\alpha_M$. A region $D$ is an appended region of $M$ if it is appended on either side of $M$.

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For an edge $e$ in a directed map, we will write $\iota(e)$ for the initial vertex of $e$ and $\tau(e)$ for the terminal vertex of $e$. A vertex $v$ in a directed map is **superfluous** if $\text{indegree}(v) = \text{outdegree}(v) = 1$.

If $M$ is any directed map having at least one edge, let $e$ be some edge in $M$ and replace $e$ by edges $e_1$ and $e_2$ and a vertex $v$ where $\iota(e_1) = \iota(e)$, $\tau(e_2) = \tau(e)$ and $\tau(e_1) = v = \iota(e_2)$. Write, momentarily, $M_{\tau(e_1,e_2)}$ for the resulting directed map and write $v_{\tau(e_1,e_2)}$ for the new vertex $v$. This is an awkward notation to typeset and to read. We will abbreviate $M_{\tau(e_1,e_2)}$ as simply $M_\tau$ and write just $v_\tau$ for the introduced vertex. We observe that $v_\tau$ is, by construction, a superfluous vertex. We will call $M_\tau$ the map obtained from $M$ by inserting a superfluous vertex on $e$.

If we begin with a directed map $M$ having a superfluous vertex $v$, with $v = \tau(e_1)$ and $v = \iota(e_2)$, then we, can replace $e_1, e_2$ and $v$ in $M$ with a new edge $e$ where $\iota(e) = \iota(e_1)$ and $\tau(e) = \tau(e_2)$. We might write $M_{\iota(e_1,e_2)}$ for the resulting map and write $v_{\iota(e_1,e_2)}$ for the disappearing vertex $v$. Again, this is awkward. We write just $M_\iota$ for $M_{\iota(e_1,e_2)}$ and call $M_\iota$ the map obtained from $M$ by deleting the superfluous vertex $v = v_\iota = v_{\iota(e_1,e_2)}$.

Suppose that $M$ is a feather and that $D$ is an appended region on the bottom side of $M$. Write $\omega_M = \lambda_0 \omega_D \lambda_1$. We allow $\lambda_0$ and/or $\lambda_1$ to be empty. We define $M - D$ to be the feathery submap of $M$ which has boundary cycle $\lambda_0 \omega_D \lambda_1 \omega_M^{-1}$. Similarly, if $D$ is appended on the top side of $M$, but is not appended on the bottom side of $M$, write $\omega_M = \mu_0 \omega_D \mu_1$. Then $M - D$ is the feathery submap of $M$ which has boundary cycle $\alpha_M(\mu_0 \omega_D \mu_1)^{-1}$.

A **thickness function**, $\text{thck}()$ for feathers is a function from the set of feathers $\mathcal{F}$ to the nonnegative integers $\mathbb{Z}^+$ satisfying the following axioms for feathers $M$.

**Triviality:** If $|M| = 0$, then $\text{thck}(M) = 0$.

**Normalization:** If $|M| = 1$, then $\text{thck}(M) = 1$.

**Growth:** If $M$ contains a strongly interior region, then $\text{thck}(M) > 1$.

**Vertex:** If $e$ is some edge in the feather $M$ and $M_\tau$ is the feather obtained from $M$ by inserting a superfluous vertex on $e$, then $\text{thck}(M_\tau) = \text{thck}(M)$.

**Reflection:** $\text{thck}(\frak{M}) = \text{thck}(M)$.

**Reversal:** $\text{thck}(\frak{M}) = \text{thck}(M)$.

**Hereditary:** If $M$ is a subfeather of $\mathcal{M}$, then $\text{thck}(M) \leq \text{thck}(\mathcal{M})$.

**Joins:** If $M = M_1 M_2 \ldots M_t$ where the $M_j$ are all of the blocks of $M$, then $\text{thck}(M) = \max_{1 \leq j \leq t} \{ \text{thck}(M_j) \}$.

**Layers:** If $\mu$ is a meridian in $M$, $M_{\text{bottom}}$ is the layer of $M$ having boundary cycle $\alpha_M(\mu)^{-1}$ and $M_{\text{top}}$ is the layer of $M$ having boundary cycle $\mu(\omega_M)^{-1}$, then $\text{thck}(M) \leq \text{thck}(M_{\text{bottom}}) + \text{thck}(M_{\text{top}})$.

We have also considered a stronger version of the Growth axiom. A subfeather $N$ of the feather $M$ is a **strongly interior subfeather** of $M$ if no region $D$ of $N$ has any edge or medial vertex on the boundary of $M$. Equivalently, $N$ is a strongly interior subfeather of $M$ if every region $D$ of $N$ is a strongly interior region of $M$. See Figure 1 for an example. Observe that this definition does allow that we might have a strongly interior subfeather $N$ of $M$ where edges of $N$ from regionless blocks of $N$ do occur on the boundary of $M$. Then the Strong Growth axiom requires that $\text{thck}(M) > \text{thck}(N)$ if $N$ is a strongly interior subfeather of $M$. The (weak) Growth axiom is the special case of this when $N$ has only one region $D$. We will
say that a thickness function \( \text{thck}() \) is a **strong** thickness function if it satisfies the Strong Growth axiom.

2. **Two Classic Results and Two More Recent Results**

We will make use of the following two results of J. H. Remmers. See [4] for a proof of Lemma 2.1 and see Lemma 1.10 in [3] for a proof of Lemma 2.2 or see Lemma 1.7.3 and Lemma 5.2.5 in [2].

**Lemma 2.1. (Remmers)** Suppose that \( \mathcal{M} \) is a feather. If \( v \) is a vertex of \( \mathcal{M} \), then there is a positive walk from the initial vertex of \( \mathcal{M} \) to the terminal vertex of \( \mathcal{M} \) which contains \( v \). If \( e \) is an edge of \( \mathcal{M} \), then there is a positive walk from the initial vertex of \( \mathcal{M} \) to the terminal vertex of \( \mathcal{M} \) which contains \( e \). \( \square \)

**Lemma 2.2. (Remmers)** If \( \mathcal{M} \) is a feather which has at least one region, then there are regions \( D_b \) and \( D_t \) of \( \mathcal{M} \) which are appended on the bottom and top sides of \( \mathcal{M} \), respectively. If \( \mathcal{M} \) has at least two regions, then we can choose regions \( D_b \) and \( D_t \) which are distinct. \( \square \)

We will also want the following two results from [1].

**Lemma 2.3.** Suppose that \( \mathcal{M} \) is a feather and that \( \phi_0, \phi_1 \) and \( \phi_2 \) are meridians of \( \mathcal{M} \) with \( \phi_0 \leq \phi_1 \leq \phi_2 \). If \( e \) is an edge of \( \mathcal{M} \) which occurs on both \( \phi_0 \) and \( \phi_2 \), then \( e \) also occurs on \( \phi_1 \). \( \square \)

**Lemma 2.4.** If \( \mathcal{M} \) is a feather and \( \mathcal{N} \) is a layer of \( \mathcal{M} \) which contains every appended region of \( \mathcal{M} \), then \( \mathcal{N} \) is all of \( \mathcal{M} \). \( \square \)

3. **Another Example of Thickness Function**

We presented several explicit and potentially useful examples of thickness functions in [1]. We give another example here.
Example 1. We define the function $\text{thck}_{(0,1,2)}(\cdot)$ by

$$\text{thck}_{(0,1,2)}(\mathcal{M}) = \begin{cases} 0 & \text{if } |\mathcal{M}| = 0 \\ 1 & \text{if } |\mathcal{M}| > 0 \text{ but no block of } \mathcal{M} \text{ has more than one region} \\ 2 & \text{otherwise.} \end{cases}$$

It is easily verified that $\text{thck}_{(0,1,2)}(\cdot)$ is a thickness function. Observe that $\text{thck}_{(0,1,2)}(\mathcal{M}) = \text{thck}_{(0,1,2)}(\mathcal{N}) = 2$ for the feathers $\mathcal{M}$ and $\mathcal{N}$ in Figure [1], hence $\text{thck}_{(0,1,2)}(\cdot)$ is not a strong thickness function.

4. Independence of the Axioms

Lemma 4.1. The Triviality axiom is independent of the other axioms.

Proof. Define a function $f : \mathbf{F} \rightarrow \mathbb{Z}^+$ by

$$f(\mathcal{M}) = \begin{cases} 1 & \text{if no block of } \mathcal{M} \text{ has more than one region} \\ 2 & \text{otherwise.} \end{cases}$$

Since $f(\mathcal{M}) = 1$ when $|\mathcal{M}| = 0$, we see that $f$ fails to satisfy the Triviality axiom. It is not hard to see that $f$ satisfies all of the other thickness axioms. □

Lemma 4.2. The Normalization axiom is independent of the other axioms.

Proof. Define a function $f : \mathbf{F} \rightarrow \mathbb{Z}^+$ by $f(\mathcal{M}) = 2 \text{thck}_{(0,1,2)}(\mathcal{M})$. We easily see that $f$ satisfies all of the thickness axioms except the Normalization axiom. □

Lemma 4.3. The Growth axiom is independent of the other axioms.

Proof. Define a function $f : \mathbf{F} \rightarrow \mathbb{Z}^+$ by $f(\mathcal{M}) = \begin{cases} 0 & \text{if } |\mathcal{M}| = 0 \\ 1 & \text{if } |\mathcal{M}| > 0 \text{ and every region of } \mathcal{M} \text{ has at least two edges on the boundary of } \mathcal{M} \\ 2 & \text{if some region of } \mathcal{M} \text{ has at most one edge on the boundary of } \mathcal{M} \end{cases}.$

Then $f$ fails to satisfy the Growth axiom whenever $\mathcal{M}$ has a strongly interior region, but it is not hard to see that $f$ satisfies all of the other axioms. □

Lemma 4.4. The Vertex axiom is independent of the other axioms.

Proof. Define a function $f : \mathbf{F} \rightarrow \mathbb{Z}^+$ by

$$f(\mathcal{M}) = \begin{cases} 0 & \text{if } |\mathcal{M}| = 0 \\ 1 & \text{if } |\mathcal{M}| > 0 \text{ and every region of } \mathcal{M} \text{ has at least two edges on the boundary of } \mathcal{M} \\ 2 & \text{if some region of } \mathcal{M} \text{ has at most one edge on the boundary of } \mathcal{M} \end{cases}.$$

For the feather $\mathcal{M}$ in Figure [2] the region $D$ has only the edge $e$ on the boundary of $\mathcal{M}$ and hence $f(\mathcal{M}) = 2$. If we insert a superfluous vertex on the edge $e$, then both regions in $\mathcal{M}_1$ will have two edges on the boundary of $\mathcal{M}_1$ and we will have $f(\mathcal{M}_1) = 1$. Therefore, $f$ fails to satisfy the Vertex axiom. It is relatively easy to see that $f$ does satisfy all of the other axioms. □

Lemma 4.5. The Reflection axiom is independent of the other axioms.
Proof. For a feather \( \mathcal{M} \), define \( \mathcal{M} \) to be **skewthin** if for every region \( D \) in \( \mathcal{M} \), either the initial edge of \( \alpha_D \) is on \( \alpha_{\mathcal{M}} \) or else the terminal edge of \( \omega_D \) is on \( \omega_{\mathcal{M}} \).

Define a function \( f : \mathcal{F} \rightarrow \mathbb{Z}^+ \) by

\[
f(M) = \begin{cases} 
0 & \text{if } |\mathcal{M}| = 0 \\
1 & \text{if } |\mathcal{M}| > 0 \text{ but } \mathcal{M} \text{ is skewthin} \\
2 & \text{if } \mathcal{M} \text{ is not skewthin.}
\end{cases}
\]

The feather \( \mathcal{M} \) in Figure 3 is skewthin but \( \widehat{\mathcal{M}} \) is not skewthin, hence \( f(M) = 1 \), but \( f(\widehat{\mathcal{M}}) = 2 \). It is not hard to see that \( f \) satisfies all of the other axioms. \( \square \)

**Lemma 4.6.** The Reversal axiom is independent of the other axioms.

Proof. Define a function \( f : \mathcal{F} \rightarrow \mathbb{Z}^+ \) by

\[
f(\mathcal{M}) = \begin{cases} 
0 & \text{if } |\mathcal{M}| = 0 \\
1 & \text{if } |\mathcal{M}| > 0 \text{ but no block of } \mathcal{M} \text{ has more than one region} \\
2 & \text{if some block of } \mathcal{M} \text{ has at least two regions, but no vertex of } \mathcal{M} \\
& \text{has outdegree greater than 3} \\
3 & \text{if some vertex of } \mathcal{M} \text{ has outdegree at least 4.}
\end{cases}
\]

The initial vertex of the feather \( \mathcal{M} \) in Figure 4 has outdegree 4. Since no vertex in \( \mathcal{M} \) has indegree greater than 2, no vertex in \( \widehat{\mathcal{M}} \) has outdegree greater than 2. We see that \( f(\mathcal{M}) = 3 \), but that \( f(\widehat{\mathcal{M}}) = 2 \), hence this function \( f \) fails to satisfy the Reversal axiom.

It is relatively easy to see that \( f \) does satisfy the Triviality, Normalization, Growth, Vertex, Reflection, Hereditary and Joins axioms.

To see that \( f \) also satisfies the Layers axiom, suppose that \( \mu \) is a meridian of \( \mathcal{M} \), that \( \mathcal{M}_{\text{bottom}} \) has boundary cycle \( \alpha_{\mathcal{M}} \mu \mu^{-1} \) while \( \mathcal{M}_{\text{top}} \) has boundary cycle \( \mu \omega_{\mathcal{M}}^{-1} \). We show that \( f(\mathcal{M}) \leq f(\mathcal{M}_{\text{bottom}}) + f(\mathcal{M}_{\text{top}}) \) in the four cases for \( f(\mathcal{M}) \).

If \( f(\mathcal{M}) = 0 \), then \( \mathcal{M} \), \( \mathcal{M}_{\text{bottom}} \) and \( \mathcal{M}_{\text{top}} \) are all regionless and the inequality holds with both sides equal to zero.

If \( f(\mathcal{M}) = 1 \), then \( \mathcal{M} \) does have a region. At least one of \( \mathcal{M}_{\text{bottom}} \) and \( \mathcal{M}_{\text{top}} \) must then have a region and the inequality again holds.
If \( f(\mathcal{M}) = 2 \), then some block \( \mathcal{M}_f \) of \( \mathcal{M} \) must have at least two regions. If all of the regions of \( \mathcal{M}_f \) are in \( \mathcal{M}_{\text{bottom}} \), then \( \mathcal{M}_f \) is a block of \( \mathcal{M}_{\text{bottom}} \) and \( f(\mathcal{M}_{\text{bottom}}) \geq 2 \). Similarly, if all of the regions of \( \mathcal{M}_f \) are in \( \mathcal{M}_{\text{top}} \), then \( f(\mathcal{M}_{\text{top}}) \geq 2 \). If some regions from \( \mathcal{M}_f \) are in \( \mathcal{M}_{\text{bottom}} \) and some regions from \( \mathcal{M}_f \) are in \( \mathcal{M}_{\text{top}} \), then we have both \( f(\mathcal{M}_{\text{bottom}}) \geq 1 \) and \( f(\mathcal{M}_{\text{top}}) \geq 1 \), and the inequality again holds.

If \( f(\mathcal{M}) = 3 \), write \( v \) for a vertex of \( \mathcal{M} \) which has outdegree at least 4. Then \( v \) is the initial vertex for at least 3 regions of \( \mathcal{M} \). If all of the regions of \( \mathcal{M} \) having \( v \) as an initial vertex are in \( \mathcal{M}_{\text{bottom}} \), then \( v \) has outdegree at least 4 in \( \mathcal{M}_{\text{bottom}} \) and \( f(\mathcal{M}_{\text{bottom}}) = 3 \). Similarly, if all of the regions of \( \mathcal{M} \) having \( v \) as an initial vertex are in \( \mathcal{M}_{\text{top}} \), then \( f(\mathcal{M}_{\text{top}}) = 3 \). If \( \mathcal{M}_{\text{bottom}} \) contains at least 2 of the regions having \( v \) as an initial vertex and \( \mathcal{M}_{\text{top}} \) also contains a region having \( v \) as an initial vertex, then \( f(\mathcal{M}_{\text{bottom}}) \geq 2 \) and \( f(\mathcal{M}_{\text{top}}) \geq 1 \), hence \( f(\mathcal{M}) \leq f(\mathcal{M}_{\text{bottom}}) + f(\mathcal{M}_{\text{top}}) \). Similarly, if \( \mathcal{M}_{\text{top}} \) contains at least 2 of the regions having \( v \) as an initial vertex and \( \mathcal{M}_{\text{bottom}} \) also contains a region having \( v \) as an initial vertex, then \( f(\mathcal{M}_{\text{top}}) \geq 2 \) and \( f(\mathcal{M}_{\text{bottom}}) \geq 1 \).

Define \( \mathcal{M} \) to be \( \text{thin}_\sigma \), if every region of \( \mathcal{M} \) is an appended region of \( \mathcal{M} \). If \( \mathcal{M} \) is not \( \text{thin}_\sigma \), then there is a region \( D \) of \( \mathcal{M} \) which is not an appended region of \( \mathcal{M} \). This region cannot be an appended region of the block of \( \mathcal{M} \) in which it occurs. We see that if \( \mathcal{M} \) is not \( \text{thin}_\sigma \), then some block of \( \mathcal{M} \) is not \( \text{thin}_\sigma \).

**Lemma 4.7.** The Hereditary axiom is independent of the other axioms.

**Proof.** If \( \mathcal{M} \) is a feather with bottom side \( \alpha_{\mathcal{M}} \) and top side \( \omega_{\mathcal{M}} \), define \( \rho(\mathcal{M}) \) to be the number of regions in \( \mathcal{M} \) which have an edge on \( \alpha_{\mathcal{M}} \) or on \( \omega_{\mathcal{M}} \). Observe that \( \rho(\mathcal{M}) = \rho(\mathcal{M}_j) \) and \( \rho(\mathcal{M}) = \rho(\mathcal{M}) \). If \( e \) is an edge in \( \mathcal{M} \), then \( \rho(\mathcal{M}_j) = \rho(\mathcal{M}) \) where \( \mathcal{M}_j \) is obtained from \( \mathcal{M} \) by inserting a superfluous vertex on \( e \). If \( \mathcal{M} \) is not \( \text{thin}_\sigma \), define \( \psi(\mathcal{M}) \) to be the maximum value for \( \rho(\mathcal{M}_j) \) over those blocks \( \mathcal{M}_j \) of \( \mathcal{M} \) which are not \( \text{thin}_\sigma \) and observe that we will have \( \psi(\mathcal{M}) \geq 2 \). Define a function \( f : \mathcal{F} \rightarrow \mathbb{Z}^+ \) by

\[
    f(\mathcal{M}) = \begin{cases} 
        0 & \text{if } |\mathcal{M}| = 0 \\
        1 & \text{if } |\mathcal{M}| > 0 \text{ but no block of } \mathcal{M} \text{ has more than one region} \\
        2 & \text{if some block of } \mathcal{M} \text{ has more than one region but } \mathcal{M} \text{ is } \text{thin}_\sigma \\
        2 & \text{if } \mathcal{M} \text{ is not } \text{thin}_\sigma \text{ and } \psi(\mathcal{M}) = 2 \\
        3 & \text{if } \mathcal{M} \text{ is not } \text{thin}_\sigma \text{ and } \psi(\mathcal{M}) \geq 3.
    \end{cases}
\]

For the feather \( \mathcal{M} \) in Figure 4 and the subfeather \( \mathcal{M} = \mathcal{M} - D \) of \( \mathcal{M} \), we see that \( f(\mathcal{M}) = 2 \), but that \( f(\mathcal{M}) = 3 \), hence this function \( f \) fails to satisfy the Hereditary axiom.
Figure 5. A feather $\mathcal{M}$ with an appended region $D$ such that $f(\mathcal{M}) = \psi(\mathcal{M}) = \rho(\mathcal{M}) = 2$, but $f(\mathcal{M} - D) = \psi(\mathcal{M} - D) = \rho(\mathcal{M} - D) = 3$. The region $D^\circ$ is not an appended region of $\mathcal{M}$ and is not an appended region of $\mathcal{M} - D$.

It is relatively easy to see that $f$ does satisfy the Triviality, Normalization, Growth, Vertex, Reflection, Reversal and Joins axioms.

To see that $f$ also satisfies the Layers axiom, suppose that $\mu$ is a meridian of $\mathcal{M}$, that $\mathcal{M}_{\text{bottom}}$ has boundary cycle $\alpha_{\mathcal{M}_{\text{bottom}}}^{-1}$, while $\mathcal{M}_{\text{top}}$ has boundary cycle $\mu \omega_{\mathcal{M}}^{-1}$. We show that we have $f(\mathcal{M}) \leq f(\mathcal{M}_{\text{bottom}}) + f(\mathcal{M}_{\text{top}})$ in the four cases corresponding to the values for $f(\mathcal{M})$.

The arguments for the cases where $f(\mathcal{M}) < 3$ are essentially the same as the arguments used in the proof of Lemma 4.6 and we omit those cases here.

If $f(\mathcal{M}) = 3$, then there is a block $\mathcal{M}_J$ of $\mathcal{M}$ with $\psi(\mathcal{M}_J) \geq 3$. Further, $\mathcal{M}_J$ is not thin and there is a region, $D^\circ$, of $\mathcal{M}_J$ which is not an appended region of $\mathcal{M}_J$. If all of the regions of $\mathcal{M}_J$ are in $\mathcal{M}_{\text{bottom}}$, then $f(\mathcal{M}_{\text{bottom}}) = 3$ and the inequality holds. Similarly, if all of the regions of $\mathcal{M}_J$ are in $\mathcal{M}_{\text{top}}$, then $f(\mathcal{M}_{\text{top}}) = 3$ and the inequality holds. If some regions from $\mathcal{M}_J$ are in $\mathcal{M}_{\text{bottom}}$ and some regions from $\mathcal{M}_J$ are in $\mathcal{M}_{\text{top}}$, we consider the region, $D^\circ$, of $\mathcal{M}_J$ which is not an appended region. If $D^\circ$ is contained in $\mathcal{M}_{\text{bottom}}$, then the block of $\mathcal{M}_{\text{bottom}}$ which contains $D^\circ$ must contain at least one more region: otherwise, $D^\circ$ would be appended to $\mathcal{M}_{\text{bottom}}$ on the bottom side of $\mathcal{M}_{\text{bottom}}$ and hence appended to $\mathcal{M}$ on the bottom side of $\mathcal{M}$. We then have $f(\mathcal{M}_{\text{bottom}}) \geq 2$. Since $\mathcal{M}_{\text{top}}$ also contains at least one region, $f(\mathcal{M}_{\text{top}}) \geq 1$ and the inequality again holds. The case where $D^\circ$ is in $\mathcal{M}_{\text{top}}$ is similar. $\square$

Lemma 4.8. The Joins axiom is independent of the other axioms.

Proof. Define a function $f : \mathbf{F} \to \mathbb{Z}^+$ by $f(\mathcal{M}) = |\mathcal{M}|$. Then $f$ fails to satisfy the Joins axiom whenever $\mathcal{M}$ has more than one nontrivial block, but it is not hard to see that $f$ satisfies all of the other axioms. $\square$

Lemma 4.9. The Layers axiom is independent of the other axioms.

Proof. Define a function $f : \mathbf{F} \to \mathbb{Z}^+$ by

$$f(\mathcal{M}) = \begin{cases} 0 & \text{if } |\mathcal{M}| = 0 \\ 1 & \text{if } |\mathcal{M}| > 0 \text{ but no block of } \mathcal{M} \text{ has more than one region} \\ 3 & \text{otherwise.} \end{cases}$$

For the feather $\mathcal{M}$, illustrated in Figure 6, we have $f(\mathcal{M}) = 3$. Let $\mu$ be the meridian $e_1 e_2 e_3$ in $\mathcal{M}$. In the notation of the Layers axiom, $f(\mathcal{M}_{\text{bottom}}) = f(\mathcal{M}_{\text{top}}) = 1$. This
function \( f \) therefore fails to satisfy the Layers axiom. It is not hard to see that \( f \) satisfies all of the other axioms. \( \square \)

The following lemma shows that the inverse of the Triviality axiom is not independent of the other axioms.

**Lemma 4.10.** For any feather \( \mathcal{M} \) and thickness function \( \text{thck}(\cdot) \), if \( |\mathcal{M}| > 0 \), then \( \text{thck}(\mathcal{M}) > 0 \).

**Proof.** If \( |\mathcal{M}| > 0 \), then \( \mathcal{M} \) contains some region \( D \). We let \( \mathcal{N} \) be the subfeather of \( \mathcal{M} \) which contains only \( D \) and the edges and vertices on \( D \). By the Normalization axiom, \( \text{thck}(\mathcal{N}) = 1 \) and by the Hereditary axiom \( \text{thck}(\mathcal{M}) \geq \text{thck}(\mathcal{N}) \). \( \square \)

For the Vertex axiom, we have made an arbitrary choice between requiring that \( \text{thck}(\mathcal{M}^\uparrow) = \text{thck}(\mathcal{M}) \) or instead requiring that \( \text{thck}(\mathcal{M}^\downarrow) = \text{thck}(\mathcal{M}) \).

**Lemma 4.11.** If \( v \) is a superfluous vertex in a feather \( \mathcal{M} \), and \( \text{thck}(\cdot) \) is any thickness function, then \( \text{thck}(\mathcal{M}^\downarrow) = \text{thck}(\mathcal{M}) \).

**Proof.** This follows from just the Vertex axiom applied to \( \mathcal{M}^\downarrow \). We return to \( \mathcal{M} \) if we remove the vertex \( v \) and then reinsert \( v \) on the new edge, so \( \mathcal{M} = (\mathcal{M}^\downarrow)^\uparrow \). \( \square \)

### 5. Consequences of the Axioms

**Lemma 5.1.** If \( \text{thck}(\cdot) \) is a thickness function, then \( \text{thck}(\mathcal{M}) \leq |\mathcal{M}| \) for every feather \( \mathcal{M} \).

**Proof.** We use induction on \( |\mathcal{M}| \) and the conclusion follows from the Triviality and Normalization axioms when \( |\mathcal{M}| \leq 1 \). Assume then that \( |\mathcal{M}| > 1 \) and that \( D \) is a region that is appended to \( \mathcal{M} \) on the top side of \( \mathcal{M} \). Write \( \omega_\mathcal{M} = \omega_0\omega_D\omega_1 \) where \( \omega_0 \) and \( \omega_1 \) are (possibly empty) paths. Let \( \mu \) be the meridian \( \omega_0\alpha_D\omega_1 \) of \( \mathcal{M} \). Using \( \mathcal{M}_{\text{top}} \) and \( \mathcal{M}_{\text{bottom}} \) as in the Layers axiom, we see that \( \mathcal{M}_{\text{top}} \) has only one region, \( D \), hence \( \text{thck}(\mathcal{M}_{\text{top}}) = 1 \) by the Normalization axiom. We also observe that \( \mathcal{M}_{\text{bottom}} = \mathcal{M} - D \) and hence \( |\mathcal{M}_{\text{bottom}}| = |\mathcal{M}| - 1 \). By the induction hypothesis, \( \text{thck}(\mathcal{M}_{\text{bottom}}) \leq |\mathcal{M}| - 1 \) and then by the Layers axiom, we have \( \text{thck}(\mathcal{M}) \leq \text{thck}(\mathcal{M}_{\text{bottom}}) + \text{thck}(\mathcal{M}_{\text{top}}) \leq |\mathcal{M}| - 1 + 1 = |\mathcal{M}| \). \( \square \)

**Lemma 5.2.** If \( \mathcal{M} \) is a feather, \( D \) is an appended region of \( \mathcal{M} \), and \( \text{thck}(\cdot) \) is any thickness function, then \( \text{thck}(\mathcal{M} - D) \) is either \( \text{thck}(\mathcal{M}) \) or \( \text{thck}(\mathcal{M}) - 1 \).

**Proof.** We treat the case where \( D \) is appended to \( \mathcal{M} \) on the bottom side of \( \mathcal{M} \). The case where \( D \) is appended to \( \mathcal{M} \) only on the top side of \( \mathcal{M} \) is very similar. Since \( \mathcal{M} - D \) is a subfeather of \( \mathcal{M} \), we have \( \text{thck}(\mathcal{M} - D) \leq \text{thck}(\mathcal{M}) \) by the Hereditary axiom. Write \( \omega_\mathcal{M} \) as \( \lambda_0\alpha_D\lambda_1 \) where \( \lambda_0 \) and/or \( \lambda_1 \) might be empty. Then \( \mu = \lambda_0\omega_D\lambda_1 \) is a meridian of \( \mathcal{M} \). Write \( \mathcal{M}_{\text{bottom}} \) for the layer of \( \mathcal{M} \) having boundary cycle \( \omega_\mathcal{M}(\mu)^{-1} \) and write \( \mathcal{M}_{\text{top}} \) for the layer of \( \mathcal{M} \) having boundary cycle \( \mu(\omega_\mathcal{M})^{-1} \).
Then $\mathcal{M} - D = \mathcal{M}_{top}$ and the only region in $\mathcal{M}_{bottom}$ is $D$. By the Normalization axiom, $\text{thck}(\mathcal{M}_{bottom}) = 1$ and then $\text{thck}(\mathcal{M}) \leq 1 + \text{thck}(\mathcal{M} - D)$ by the Layers axiom.

**Lemma 5.3.** If $\mathcal{M}$ is a feather, $\{\mu_0 = \alpha_{\mathcal{M}}, \mu_1, \mu_2, \ldots, \mu_k = \omega_{\mathcal{M}}\}$ is a layering of $\mathcal{M}$ with layers $\mathcal{N}_i$ for $1 \leq i \leq k$, and $\text{thck}(\cdot)$ is any thickness function, then

$$\text{thck}(\mathcal{M}) \leq \sum_{i=1}^{k} \text{thck}(\mathcal{N}_i)$$

**Proof.** This follows by induction on $k$ using the meridians $\mu_i$ in the Layers axiom. For the base step, when $k = 1$, the conclusion says that $\text{thck}(\mathcal{M}) \leq \text{thck}(\mathcal{M})$.

To emphasize that we have selected some particular, but arbitrary, thickness function, we will write $\text{thck}_\eta(\cdot)$ rather than $\text{thck}(\cdot)$.

Reversing the logical order used in [1], we will define a feather $\mathcal{M}$ to be thin$_\eta$ if $\text{thck}_\eta(\mathcal{M}) \leq 1$. Define the layered $\eta$-thickness, $K_\eta(\cdot)$, by $K_\eta(\mathcal{M}) = 0$ if $|\mathcal{M}| = 0$ and $K_\eta(\mathcal{M}) = \min\{k \mid \text{there is a layering of } \mathcal{M} \text{ with } k \text{ nontrivial } \text{thin}_\eta \text{ layers}\}$ if $|\mathcal{M}| > 0$. Following some lemmas, we will prove in Theorem 5.10 below that $K_\eta$ is a thickness function.

**Lemma 5.4.** If $\text{thck}_\eta(\cdot)$ is thickness function, then $K_\eta(\mathcal{M}) \geq \text{thck}_\eta(\mathcal{M})$ for every feather $\mathcal{M}$.

**Proof.** To simplify subscripts, write $k$ for $K_\eta(\mathcal{M})$. When $|\mathcal{M}| = 0$, both $\text{thck}_\eta(\mathcal{M})$ and $K_\eta(\mathcal{M})$ are 0, so assume that $|\mathcal{M}| > 0$.

Choose a layering, $\{\mu_0 = \alpha_{\mathcal{M}}, \mu_1, \mu_2, \ldots, \mu_k = \omega_{\mathcal{M}}\}$ of $\mathcal{M}$ with $k$ nontrivial thin$_\eta$ layers $\mathcal{N}_i$. Since the layers $\mathcal{N}_i$ are nontrivial, we have $\text{thck}_\eta(\mathcal{N}_i) > 0$ for $1 \leq i \leq k$ by Lemma 4.10 and then $\text{thck}_\eta(\mathcal{N}_i) = 1$ for $1 \leq i \leq k$ by the definition for thin$_\eta$. It follows from Lemma 5.3 that $\text{thck}_\eta(\mathcal{M}) \leq k$.

If $\mu$ is a positive path in the feather $\mathcal{M}$, we will write $\hat{\mu}$ for the positive path in $\widehat{\mathcal{M}}$ that is the image of $\mu$. If $e$ is an edge in $\mathcal{M}$ from $v_1$ to $v_2$, we will write $\overrightarrow{e}$ for the corresponding edge from the image of $v_2$ to the image of $v_1$ in $\widehat{\mathcal{M}}$. If $\mu = e_1 e_2 \ldots e_m$ is a positive path of length $m$ in the feather $\mathcal{M}$, we will write $\overrightarrow{\mu}$ for the corresponding positive path $\overrightarrow{e_m} \overrightarrow{e_2} \overrightarrow{e_1}$ in $\widehat{\mathcal{M}}$ from the image of the final vertex of $\mu$ to the image of the initial vertex of $\mu$.

**Lemma 5.5.** If $\text{thck}_\eta(\cdot)$ is a thickness function and $\mathcal{M}$ is a feather, then $K_\eta(\widehat{\mathcal{M}}) = K_\eta(\mathcal{M})$ and $K_\eta(\widehat{\mathcal{M}}) = K_\eta(\mathcal{M})$.

**Proof.** Write $k$ for $K_\eta(\mathcal{M})$ and let $\mathcal{L} = \{\alpha_{\mathcal{M}} = \mu_0, \mu_1, \ldots, \mu_k = \omega_{\mathcal{M}}\}$ be a layering of $\mathcal{M}$ with $k$ nontrivial thin$_\eta$ layers, $\mathcal{N}_i$.

From $\mathcal{L}$, we have a layering $\widehat{\mathcal{L}} = \{\overrightarrow{\mu_0}, \ldots, \overrightarrow{\mu_1}, \overrightarrow{\mu_0}\}$ of $\widehat{\mathcal{M}}$ for which the $i$th layer of $\widehat{\mathcal{M}}$ is the reflection of the $(k - i + 1)^{th}$ layer from $\mathcal{L}$ and hence is nontrivial and is thin$_\eta$ by the Reflection axiom. We see that $\text{thck}_\eta(\widehat{\mathcal{M}}) \leq \text{thck}_\eta(\mathcal{M})$. Applying this to $\widehat{\mathcal{M}}$, and using $\widehat{\widehat{\mathcal{M}}} = \widehat{\mathcal{M}}$, we see that $\text{thck}_\eta(\mathcal{M}) \leq \text{thck}_\eta(\widehat{\mathcal{M}})$.

Similarly, we have a layering $\overleftarrow{\mathcal{L}} = \{\overleftarrow{\mu_0}, \ldots, \overleftarrow{\mu_1}, \overleftarrow{\mu_0}\}$ of $\overleftarrow{\mathcal{M}}$ for which the $i$th layer of $\overleftarrow{\mathcal{M}}$ is the reversal of the $(k - i + 1)^{th}$ layer from $\mathcal{L}$ and hence is nontrivial and is thin$_\eta$ by the Reversal axiom. We see that $\text{thck}_\eta(\overleftarrow{\mathcal{M}}) \leq \text{thck}_\eta(\mathcal{M})$. Applying this to $\overleftarrow{\mathcal{M}}$, and using $\overleftarrow{\overleftarrow{\mathcal{M}}} = \overleftarrow{\mathcal{M}}$, we see that $\text{thck}_\eta(\mathcal{M}) \leq \text{thck}_\eta(\overleftarrow{\mathcal{M}})$.
Lemma 5.6. If $e$ is some edge in the feather $M$ and $M_1$ is the feather obtained from $M$ by inserting a superfluous vertex on $e$, then $K_\eta(M_1) = K_\eta(M)$.

Proof. Write $k$ for $K_\eta(M)$ and let $L = \{\alpha_M = \mu_0, \mu_1, \ldots, \mu_k = \omega_M\}$ be a layering of $M$ with $k$ nontrivial thin$_\eta$ layers, $M_i$. Then the edge $e$ occurs in one or more of the layers $M_i$. For each such layer, we have $\text{thck}_\eta(M_i) = \text{thck}(M_i) = 1$ applying the Vertex axiom for $\text{thck}_\eta()$ to $M_i$. Thus, we obtain a layering of $M_1$ with $k$ nontrivial thin$_\eta$ layers and $K_\eta(M_1) \leq k$.

Conversely, write $\nu$ for the inserted vertex and now write $k$ for $K_\eta(M_1)$ and let $L = \{\mu_0, \mu_1, \ldots, \mu_k\}$ be a layering of $M_1$ with $k$ nontrivial thin$_\eta$ layers, $M_i$. Observe that, when deleting this inserted vertex, $(M_1)_i = M_i$. The vertex $v$ occurs in one or more of the layers $M_i$. For each such layer, we have $\text{thck}_\eta(M_i) = \text{thck}_\eta(M_i) = 1$ applying Lemma 4.11 for $\text{thck}_\eta()$ to $M_i$. Thus, we obtain a layering of $M_1$ with $k$ nontrivial thin$_\eta$ layers and $K_\eta(M_1) \leq k$.

Lemma 5.7. If $\text{thck}_\eta()$ is a thickness function, $M$ is a feather and $D$ is an appended region of $M$, then $K_\eta(M - D) \leq K_\eta(M)$.

Proof. By Lemma 5.5 we may assume that $D$ is appended to $M$ on the bottom side of $M$. Write $k$ for $K_\eta(M)$ and choose a layering $\{\mu_0, \mu_1, \ldots, \mu_k\}$ of $M$ with $k$ nontrivial thin$_\eta$ layers, $M_i$. Then $D$ is in some layer $M_i$ of $M$. By Lemma 2.11 we can choose a path $\lambda_0$ in $M_j$ from the initial vertex of $M$ to the initial vertex of $D$ and choose a path $\lambda_1$ in $M_i$ from the terminal vertex of $D$ to the terminal vertex of $M$. For $0 < i < J$, we can apply Lemma 2.3 to any edge on $\alpha_D$ with $\phi_0 = \mu_0, \phi_1 = \mu_i$ and $\phi_2 = \lambda_0 \alpha_D \lambda_1$ and then conclude that $\alpha_D$ must occur as a segment of $\mu_i$ for $0 \leq i < J$. For $0 \leq i < J$, obtain $\mu_i'$ from $\mu_i$ by replacing the segment $\alpha_D$ by $\omega_D$. Then $\{\mu_0', \mu_1', \ldots, \mu_{i-1}', \mu_i, \ldots, \mu_k\}$ is a layering of $M - D$ with layers $M_i'$. For $i > J$, $M_i' = M_i$ and hence is nontrivial and thin$_\eta$. For $1 \leq i < J$, the nontrivial blocks of $M_i'$ are the same as the nontrivial blocks of $M_i$ and hence $M_i'$ is nontrivial and thin$_\eta$. Observe that $M_i' = M_j - D$. Since $M_i'$ is thin$_\eta$, $\text{thck}_\eta(M_i') = 1$ and then $\text{thck}_\eta(M_i') \leq 1$. If $M_j$ has no regions, then $\mu_{i-1}' = \mu_i$ and we obtain a layering of $M - D$ with $k - 1$ nontrivial thin$_\eta$ layers. Otherwise, we have a layering of $M - D$ with $k$ nontrivial thin$_\eta$ layers and $K_\eta(M - D) \leq k$.

Lemma 5.8. Suppose that $M$ is a feather and that $M = M_1 M_2 \ldots M_l$ where the $M_j$ are all of the blocks of $M$. If $\text{thck}_\eta()$ is any thickness function, then

$$K_\eta(M) = \max_{1 \leq j \leq l} \{K_\eta(M_j)\}.$$ 

Proof. This proof is essentially an abstraction of the proof of Lemma 2.16 in [1]. We show that $K_\eta(M_j) \leq K_\eta(M)$ for each $j$ and then that $K_\eta(M) \leq \max_{1 \leq j \leq l} \{K_\eta(M_j)\}$.

Suppose that $K_\eta(M_j) = k$ and let $L = \{\mu_0, \mu_1, \ldots, \mu_k\}$ be a layering of $M$ where all of the layers $M_i$, for $1 \leq i \leq k$, are thin$_\eta$. For $1 \leq j \leq t$ and for $0 \leq i \leq k$, let $\mu_{i,j}$ be the segment of the meridian $\mu_i$ which occurs in the block $M_j$. For each $j$, $L_j = \{\mu_{0,j}, \mu_{1,j}, \ldots, \mu_{k,j}\}$ is a layering of $M_j$. Each layer $M_{i,j}$ of $M_j$ is a subfeather of the layer $M_i$ of $M$. Since $M_i$ is thin$_\eta$, by definition, $\text{thck}_\eta(M_i) \leq 1$. Then $\text{thck}_\eta(M_{i,j}) \leq 1$ by the Hereditary axiom for $\text{thck}_\eta()$. Thus the layers $M_{i,j}$ are thin$_\eta$. They will not, in general, be nontrivial layers, but we can eliminate any trivial layers in each block $M_j$ to see that $K_\eta(M_j) \leq k$.

Conversely, suppose that $K = \max_{1 \leq j \leq l} \{K_\eta(M_j)\}$. For each $j$, let $k_j = K_\eta(M_j)$ and let $L_j = \{\mu_{0,j}, \mu_{1,j}, \ldots, \mu_{k,j}\}$ be a layering of $M_j$ where all of the layers $M_{i,j}$,
are thin. Choose a $J$ with $1 \leq J \leq t$ such that $K = k_J = K_\eta(\mathfrak{M}_J)$. If $1 \leq j \leq t$ and $k_j < K$, define $\mu_{i,j}$ to be $\mu_{k_i,j}$ for $k_i \leq i < K$ and then, for $0 \leq i < K$, define the $i^{th}$ meridian, $\mu_i$, of $\mathfrak{M}$ to be the concatenation of paths $\mu_{i,1}\mu_{i,2}\ldots\mu_{i,t}$. Let $\mathcal{L} = \{\mu_0, \mu_1, \ldots, \mu_K\}$ and write $\mathfrak{M}_1, \mathfrak{M}_2, \ldots, \mathfrak{M}_K$ for the corresponding layers of $\mathfrak{M}$. Each layer $\mathfrak{M}_i$ is nontrivial because the block of $\mathfrak{N}_i$ that is contained in the block $\mathfrak{M}_i$ of $\mathfrak{M}$ is nontrivial. Consider an arbitrary nontrivial block of some layer $\mathfrak{N}_i$ of $\mathfrak{M}$. This block must occur in some block $\mathfrak{M}_j$ of $\mathfrak{M}$ and is then a subfeather of $\mathfrak{N}_{i,j}$. By the Hereditary axiom for $\text{thck}_\eta()$, we see that this block, like $\mathfrak{N}_{i,j}$ must be thin. Since every block of $\mathfrak{N}_i$ is thin, $\mathfrak{N}_i$ is thin by the Joins axiom for $\text{thck}_\eta()$, and we see that $K_\eta(\mathfrak{M}) \leq K$.

**Lemma 5.9.** If $\text{thck}_\eta()$ is a thickness function, $\mathfrak{M}$ is a feather and $\mathfrak{N}$ is a subfeather of $\mathfrak{M}$, then $K_\eta(\mathfrak{N}) \leq K_\eta(\mathfrak{M})$.

**Proof.** By Lemma 2.1 there is a positive or an empty path in $\mathfrak{N}$ from the initial vertex of $\mathfrak{M}$ to the initial vertex of $\mathfrak{N}$ and a positive or an empty path in $\mathfrak{M}$ from the terminal vertex of $\mathfrak{M}$ to the terminal vertex of $\mathfrak{N}$. If we join these paths to $\mathfrak{N}$, we obtain a layer $\mathfrak{M}'$ of $\mathfrak{M}$. Then the nontrivial blocks of $\mathfrak{N}'$ are the same as the nontrivial blocks of $\mathfrak{N}$, hence, using Lemma 5.8, we have $K_\eta(\mathfrak{N}') = K_\eta(\mathfrak{N})$. Thus, we may assume that $\mathfrak{N}$ is a layer of $\mathfrak{M}$.

We use induction on $|\mathfrak{M}|$. For the base of the induction, if $|\mathfrak{M}| = 0$, then for any subfeather $\mathfrak{N}$ of $\mathfrak{M}$, $|\mathfrak{N}| = 0$ and $K_\eta(\mathfrak{N}) = 0$. Similarly, if $|\mathfrak{M}| = 1$, then $K_\eta(\mathfrak{M}) = 1$ and for any subfeather $\mathfrak{N}$ of $\mathfrak{M}$, either $K_\eta(\mathfrak{N}) = 1$ or $K_\eta(\mathfrak{N}) = 0$ depending upon whether or not the region of $\mathfrak{N}$ is also in $\mathfrak{N}$. Suppose that $|\mathfrak{M}| > 1$. By Lemma 2.2 $\mathfrak{M}$ has an appended region. By Lemma 2.4, if $\mathfrak{M}$ contains every appended region of $\mathfrak{M}$, then $\mathfrak{M} = \mathfrak{M}$ and hence $K_\eta(\mathfrak{M}) = K_\eta(\mathfrak{M})$. Let $D$ be an appended region of $\mathfrak{M}$ which is not in $\mathfrak{N}$. Then applying the induction hypothesis to $\mathfrak{M} - D$ with layer $\mathfrak{N}$ and using Lemma 5.7 we have $K_\eta(\mathfrak{N}) \leq K_\eta(\mathfrak{M} - D) \leq K_\eta(\mathfrak{M})$.

**Theorem 5.10.** If $\text{thck}_\eta()$ is a thickness function, then $K_\eta()$ is a thickness function.

**Proof.** $K_\eta()$ satisfies the Triviality axiom by definition. Since $K_\eta(\mathfrak{M}) = 1$ for any nontrivial feather $\mathfrak{M}$ which is thin, and this includes feathers with just one region, $K_\eta()$ also satisfies the Normalization axiom.

If a feather $\mathfrak{M}$ contains a strongly interior region, then $\text{thck}_\eta(\mathfrak{M}) > 1$ by the Growth axiom for $\text{thck}_\eta()$ and then $K_\eta(\mathfrak{M}) \geq \text{thck}_\eta(\mathfrak{M}) > 1$, using Lemma 5.4. We see that $K_\eta()$ satisfies the Growth axiom.

By Lemma 5.6 $K_\eta()$ satisfies the Vertex axiom. By Lemma 5.5 $K_\eta()$ satisfies the Reflection and Reversal axioms. By Lemma 5.9 $K_\eta()$ satisfies the Hereditary axiom, and by Lemma 5.8 $K_\eta()$ satisfies the Joins axiom.

To see that $K_\eta()$ satisfies the Layers axiom, suppose that $\mathfrak{M}$ is a feather, $\mu$ is a meridian of $\mathfrak{M}$, $\mathfrak{M}_{\text{bottom}}$ is the layer of $\mathfrak{M}$ having boundary cycle $\alpha_{\mathfrak{M}}(\mu)^{-1}$ and $\mathfrak{M}_{\text{top}}$ is the layer of $\mathfrak{M}$ having boundary cycle $\mu(\omega_{\mathfrak{M}})^{-1}$. Write $k_b$ for $K_\eta(\mathfrak{M}_{\text{bottom}})$ and $k_t$ for $K_\eta(\mathfrak{M}_{\text{top}})$. If either of $\mathfrak{M}_{\text{bottom}}$ or $\mathfrak{M}_{\text{top}}$ is without regions, then the other must be all of $\mathfrak{M}$ and the Layers axiom for $K_\eta()$ holds in this case. Choose a layering $\{\alpha_{\mathfrak{M}} = \mu_0, \ldots, \mu_{k_b} = \mu\}$ of $\mathfrak{M}_{\text{bottom}}$ with $k_b$ nontrivial thin layers and choose a layering $\{\mu = \mu'_0, \ldots, \mu'_{k_t} = \omega_{\mathfrak{M}}\}$ of $\mathfrak{M}_{\text{top}}$ with $k_t$ nontrivial thin layers. Concatenating these layers, $\{\mu_0, \ldots, \mu'_1, \ldots, \mu'_t\}$ is a layering of $\mathfrak{M}$ with $k_b + k_t$ nontrivial thin layers, hence $K_\eta(\mathfrak{M}) \leq k_b + k_t$. □
If $x$ is any real number, we use $\lfloor x \rfloor$ as notation for the greatest integer that is equal to or less than $x$.

**Theorem 5.11.** For any natural number $m \geq 2$ and any thickness function $\text{thck}_\eta()$, define the function $\text{thck}_{m\#\eta}()$ by $\text{thck}_{m\#\eta}(\mathcal{M}) = 1 + \lfloor \frac{\text{thck}_\eta(\mathcal{M}) - 1}{m+1} \rfloor$. Then $\text{thck}_{m\#\eta}()$ is a thickness function.

**Proof.** To see that $\text{thck}_{m\#\eta}()$ satisfies the Triviality, Normalization, Vertex, Reflection and Reversal axioms, we need only routine calculations using the hypothesis that $\text{thck}_\eta()$ is a thickness function.

Define $g(x, y) = \frac{xy - 1}{x + 1}$, so that $\text{thck}_{m\#\eta}(\mathcal{M}) = 1 + \lfloor g(m, \text{thck}_\eta(\mathcal{M})) \rfloor$. From multivariable calculus, we see that the partial derivatives $g_x(x, y) = \frac{y + 1}{(x + 1)^2}$ and $g_y(x, y) = \frac{x}{x + 1}$ are positive for $x > 0, y > -1$ hence for positive $x$ and $y$, $g$ is an increasing function of $x$ and of $y$. It follows that, for $m \geq 2$, $\text{thck}_{m\#\eta}()$ is a nondecreasing function in $m$ and in $\text{thck}_\eta(\mathcal{M})$.

To see that $\text{thck}_{m\#\eta}()$ satisfies the Growth axiom, suppose that $\mathcal{M}$ is a feather and that $D$ is a strongly interior region of $\mathcal{M}$. Then $\text{thck}_\eta(\mathcal{M}) \geq 2$ by the Growth axiom for $\text{thck}_\eta()$. We calculate $1 + \lfloor g(2, 2) \rfloor = 2$ and observe that 2 is then the smallest possible value for $\text{thck}_{m\#\eta}(\mathcal{M})$ since $g$ is an increasing function of $x$ and of $y$ for positive values of $x$ and $y$.

If $\mathcal{M}$ is a subfeather of $\mathcal{M}$, then $\text{thck}_\eta(\mathcal{M}) \leq \text{thck}_\eta(\mathcal{M})$ since $\text{thck}_\eta()$ satisfies the Hereditary axiom. Then $g(m, \text{thck}_\eta(\mathcal{M})) \leq g(m, \text{thck}_\eta(\mathcal{M}))$ and $\text{thck}_{m\#\eta}(\mathcal{M}) \leq \text{thck}_{m\#\eta}(\mathcal{M})$.

Write $\mathcal{M} = \mathcal{M}_t \mathcal{M}_{t+1} \cdots \mathcal{M}_t$ where $\{\mathcal{M}_j\}_{j=1}^t$ is the set of all blocks of $\mathcal{M}$. By the Joins axiom for $\text{thck}_\eta()$, we may choose a block $\mathcal{M}_j$ with $\text{thck}_\eta(\mathcal{M}) = \text{thck}_\eta(\mathcal{M}_j) = \max\{\text{thck}_\eta(\mathcal{M}_j)\}$. Then $\text{thck}_{m\#\eta}(\mathcal{M}) = \text{thck}_{m\#\eta}(\mathcal{M}_j)$. Since $\text{thck}_{m\#\eta}(\mathcal{M})$ is a nondecreasing function in $\text{thck}_\eta(\mathcal{M})$, we see that $\text{thck}_{m\#\eta}(\mathcal{M}_j) \leq \text{thck}_{m\#\eta}(\mathcal{M}_j)$ for $1 \leq j \leq t$ and $\text{thck}_{m\#\eta}(\mathcal{M}_j) = \max\{\text{thck}_{m\#\eta}(\mathcal{M}_j)\}$. Thus, $\text{thck}_{m\#\eta}()$ satisfies the Joins axiom.

To see that $\text{thck}_{m\#\eta}()$ satisfies the Layers axiom, suppose that $\mathcal{M}$ is a feather, $\mu$ is a meridian of $\mathcal{M}$, $\mathcal{M}_{\text{bottom}}$ is the layer of $\mathcal{M}$ having boundary cycle $\omega_{\mathcal{M}}(\mu)^{-1}$ and $\mathcal{M}_{\text{top}}$ is the layer of $\mathcal{M}$ having boundary cycle $\mu(\omega_{\mathcal{M}})^{-1}$. Write $\mathcal{M}$ for $\text{thck}_\eta(\mathcal{M})$, $\delta$ for $\text{thck}_\eta(\mathcal{M}_{\text{bottom}})$ and $\tau$ for $\text{thck}_\eta(\mathcal{M}_{\text{top}})$ and observe that $\mathcal{M} \leq \delta + \tau$ by the Layers axiom for $\text{thck}_\eta()$. Since $g(x, y)$ is an increasing function of $y$ for positive $y$, we have $1 + \lfloor g(m, \mathcal{M}_t) \rfloor \leq 1 + \lfloor g(m, \delta + \tau) \rfloor$ and it will suffice to prove that $1 + \lfloor g(m, \delta + \tau) \rfloor \leq 1 + \lfloor g(m, \delta) \rfloor + 1 + \lfloor g(m, \tau) \rfloor$ or equivalently $[g(m, \delta + \tau)] \leq 1 + [g(m, \delta)] + [g(m, \tau)]$. Since $m$ and $\delta$ are integers, we can divide $\delta - 1$ by $m + 1$ to obtain an integer quotient, $q_r$, and an integer remainder, $r_r$. This is, we have integers $q_r$ and $r_r$ with $\delta - 1 = (m + 1)q_r + r_r$ and $0 \leq r_r \leq m$. Similarly, we have integers $q_t$ and $r_t$ with $\tau - 1 = (m + 1)q_t + r_t$ and $0 \leq r_t \leq m$. Observe that $\lfloor g(m, \delta) \rfloor = q_r$, $\lfloor g(m, \tau) \rfloor = q_t$, and that $r_r + r_t + 1 \leq 2m + 1$, hence
\[
\left| \frac{rb + rt + 1}{m + 1} \right| \leq 1.
\]
We see that
\[
|g(m, s + \tau)| = \left| \frac{(m(s + \tau) - 1)}{m + 1} \right| = \left| \frac{(ms - 1) + (m\tau - 1) + 1}{m + 1} \right|
= \left| q_b + \frac{rb}{m + 1} + q_t + \frac{rt}{m + 1} + \frac{1}{m + 1} \right| = q_b + q_t + \left| \frac{rb + rt + 1}{m + 1} \right|
\leq |g(m, s)| + |g(m, \tau)| + 1.
\]
We have made very strong use here of the hypothesis that \( m \) is a natural number.

\hfill \Box

**Example 2.** The hypothesis that \( m \) is a natural number is generally necessary for \( \text{thck}_{m\#\eta}(\cdot) \) to satisfy the Layers axiom. Suppose that \( \mathfrak{M} \) is a feather with bottom and top layers, \( \mathfrak{M}_{\text{bottom}} \) and \( \mathfrak{M}_{\text{top}} \), and that \( \text{thck}_{\eta}(\cdot) \) is a thickness function with \( \text{thck}_{\eta}(\mathfrak{M}) = 48 \) and \( \text{thck}_{\eta}(\mathfrak{M}_{\text{bottom}}) = \text{thck}_{\eta}(\mathfrak{M}_{\text{top}}) = 24 \). Then, with \( m = 5/2 \), we see that \( 1 + |g(5/2, 48)| = 35 \), but \( 1 + |g(5/2, 24)| = 17 \).

**Lemma 5.12.** For any natural number \( m \geq 2 \), thickness function \( \text{thck}_{\eta}(\cdot) \) and feather \( \mathfrak{M} \), \( \text{thck}_{m\#\eta}(\mathfrak{M}) \leq \text{thck}_{\eta}(\mathfrak{M}) \).

**Proof.** It will suffice to prove that
\[
\left| \frac{m \cdot \text{thck}_{\eta}(\mathfrak{M}) - 1}{m + 1} \right| \leq \text{thck}_{\eta}(\mathfrak{M}) - 1.
\]
We observe
\[
\left| \frac{m \cdot \text{thck}_{\eta}(\mathfrak{M}) - 1}{m + 1} \right| = \left| \frac{m \cdot \text{thck}_{\eta}(\mathfrak{M}) + \text{thck}_{\eta}(\mathfrak{M}) - \text{thck}_{\eta}(\mathfrak{M}) - 1}{m + 1} \right|
= \left| \frac{(m + 1) \cdot \text{thck}_{\eta}(\mathfrak{M}) - \text{thck}_{\eta}(\mathfrak{M}) - 1}{m + 1} \right|
= \text{thck}_{\eta}(\mathfrak{M}) + \left| \frac{-\text{thck}_{\eta}(\mathfrak{M}) - 1}{m + 1} \right| \leq \text{thck}_{\eta}(\mathfrak{M}) - 1.
\]

\hfill \Box

**Example 3.** Fix a thickness function \( \text{thck}_{\eta}(\cdot) \). If \( \text{thck}_{\eta}(\mathfrak{M}) = 200 \) for some feather \( \mathfrak{M} \), then it is an easy calculation to see that \( \text{thck}_{2\#\eta}(\mathfrak{M}) = 134 \). If \( m \) is any natural number which is at least two and there is a feather \( \mathfrak{M} \) with \( \text{thck}_{\eta}(\mathfrak{M}) = m \), then \( \text{thck}_{m\#\eta}(\mathfrak{M}) = m \).

6. **Some Axiomatic Generalizations**

We will define a thickness function \( \text{thck}_{\eta}(\cdot) \) to be **closed** if \( \text{thck}_{\eta}(\mathfrak{M}) = K_{\eta}(\mathfrak{M}) \) for every feather \( \mathfrak{M} \). We define a set \( \mathfrak{P}_\circ \) of feathers to be **pedigreed** if whenever the feather \( \mathfrak{M} \) is in \( \mathfrak{P}_\circ \), then \( \mathfrak{M} \) is in \( \mathfrak{P}_\circ \) and every subfeather of \( \mathfrak{M} \) is in \( \mathfrak{P}_\circ \).

Our final four results are axiomatic generalizations of the final four theorems in \( \mathbb{H} \). The proofs of these results are essentially unchanged and are omitted here.

**Theorem 6.1.** Let \( \mathfrak{P}_\circ \) be a pedigreed set of feathers and suppose that \( \text{thck}_{\eta}(\cdot) \) is a closed thickness function. Assume that we have a bound \( B_\eta \geq 1 \) such that whenever \( \mathfrak{M} \) is a feather in \( \mathfrak{P}_\circ \) which is thin_{\eta}, then
\[
(\dagger) \quad \left( \frac{1}{B_\eta} \right) |\alpha_{\mathfrak{M}}| \leq |\omega_{\mathfrak{M}}| \leq (B_\eta) |\alpha_{\mathfrak{M}}|.
\]
If \( \mathfrak{M} \) is a feather in \( \mathfrak{P}_\circ \) with \( \text{thck}_{\eta}(\mathfrak{M}) = k \), bottom side \( \alpha_{\mathfrak{M}} \) and top side \( \omega_{\mathfrak{M}} \), then
Theorem 6.2. Let $\mathcal{F}_0$ be a pedigreed set of feathers and suppose that $\text{thck}_\eta()$ is a closed thickness function. Assume that we have a bound $B_\eta \geq 1$ such that whenever $\mathcal{M}$ is a feather in $\mathcal{F}_0$ which is thin$_\eta$, then

$$(\frac{1}{B_\eta})^k |\alpha_\mathcal{M}| \leq |\omega_{\mathcal{M}}| \leq (B_\eta)^k |\alpha_\mathcal{M}|.$$ 

If $\mathcal{M}$ is a feather in $\mathcal{F}_0$ with $\text{thck}_\eta(\mathcal{M}) = k$, bottom side $\alpha_\mathcal{M}$, top side $\omega_{\mathcal{M}}$ and $\mu$ is any meridian in $\mathcal{M}$, then

$$(\frac{1}{B_\eta})^k \max\{|\alpha_\mathcal{M}|, |\omega_{\mathcal{M}}|\} \leq |\mu| \leq (B_\eta)^k \min\{|\alpha_\mathcal{M}|, |\omega_{\mathcal{M}}|\}.$$ 

Theorem 6.3. Let $\mathcal{F}_0$ be a pedigreed set of feathers and suppose that $\text{thck}_\eta()$ is a closed thickness function. Assume that we have constants $B_\eta > 1, C_\eta > 0$ and $D_\eta$ such that whenever $\mathcal{M}$ is a feather in $\mathcal{F}_0$ which is thin$_\eta$, then

$$(\frac{1}{B_\eta})^k |\alpha_\mathcal{M}| \leq |\omega_{\mathcal{M}}| \leq (B_\eta)^k |\alpha_\mathcal{M}|$$

and

$|\mathcal{M}| \leq C_\eta \min\{|\alpha_\mathcal{M}|, |\omega_{\mathcal{M}}|\} + D_\eta.$

If $\mathcal{M}$ is a feather in $\mathcal{F}_0$ with $\text{thck}_\eta(\mathcal{M}) = k$, bottom side $\alpha_\mathcal{M}$ and top side $\omega_{\mathcal{M}}$ then

$|\mathcal{M}| \leq C_\eta \frac{B_\eta^k - 1}{B_\eta - 1} \min\{|\alpha_\mathcal{M}|, |\omega_{\mathcal{M}}|\} + kD_\eta.$

Define the integer parity function $\text{pr}$ by $\text{pr}(k) = 0$ if $k$ is even and $\text{pr}(k) = 1$ if $k$ is odd. Observe also that $|k/2| = k/2$ if $k$ is even and $|k/2| = \frac{k-1}{2}$ if $k$ is odd.

Theorem 6.4. Let $\mathcal{F}_0$ be a pedigreed set of feathers and suppose that $\text{thck}_\eta()$ is a closed thickness function. Assume that we have constants $B_\eta > 1, C_\eta > 0$ and $D_\eta$ such that whenever $\mathcal{M}$ is a feather in $\mathcal{F}_0$ which is thin$_\eta$, then

$$(\frac{1}{B_\eta})^k |\alpha_\mathcal{M}| \leq |\omega_{\mathcal{M}}| \leq (B_\eta)^k |\alpha_\mathcal{M}|$$

and

$|\mathcal{M}| \leq C_\eta \min\{|\alpha_\mathcal{M}|, |\omega_{\mathcal{M}}|\} + D_\eta.$

If $\mathcal{M}$ is a feather in $\mathcal{F}_0$ with $\text{thck}_\eta(\mathcal{M}) = k$, bottom side $\alpha_\mathcal{M}$ and top side $\omega_{\mathcal{M}}$ then

$|\mathcal{M}| \leq C_\eta \left[\left(\frac{B_\eta^{k/2} - 1}{B_\eta - 1}\right) + \text{pr}(k)C_\eta B_\eta^{k/2}\right] \min\{|\alpha_\mathcal{M}|, |\omega_{\mathcal{M}}|\} + kD_\eta.$

References

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