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ACHIEVABLE RELATORS

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Abstract. The principal result of this paper is a reduction of the word problem for one relator semigroups. Crudely, a relator (w, w') on the alphabet X is achievable if the letters in the relator can actively participate in sequences of transitions for the semigroup presentation $\langle X; (w, w') \rangle$. (A precise definition of achievable relators is given in section 3.) Empirically, achievable relators are rare. We prove that the word problem for a one relator semigroup presentation can be reduced to the word problem for a presentation where the relator is achievable.

1 Background and Preliminaries

A **directed planar map** (or simply **map)** is a finite collection of vertices, edges, and regions in the Euclidean plane, together with an orientation for the set of edges. Here we will require that each edge has two distinct vertices for its endpoints. We also make the usual requirements that vertices, edges, and regions are pairwise disjoint, that regions are homeomorphic to the unit disk, and that each region has a connected boundary which is a union of edges and their endpoints. The orientation of the edges distinguishes an initial endpoint and a terminal endpoint for each edge. Two maps are the same if one can be mapped onto the other by a homeomorphism of the plane which induces a one-to-one function preserving vertices, edges, regions, incidence, and orientation.

Let D be a region of any map or let $\mathcal{M}$ be a connected and simply connected map. A **boundary walk** for D or for $\mathcal{M}$ is a closed walk of minimal length which includes all of the edges on the topological boundary of D or of $\mathcal{M}$. For our purposes, we make the additional requirement that all boundary walks must be oriented counterclockwise in the plane. It follows that if π is any boundary walk for a fixed map or region, then all other boundary walks for this map or region can be regarded as cyclic permutations of the edges of π. A map or region Q is **two-sided** if it has a boundary walk of the form $\alpha_Q(\omega_Q)^{-1}$ where α_Q and ω_Q are positive walks, the respective **bottom** and **top sides** of Q.

We define $\mathcal{W}_1$ to be the map consisting of a single directed edge e_1 together with its initial vertex v_0 and its terminal vertex v_1. For $k > 1$, the map $\mathcal{W}_k$ is defined inductively by $\mathcal{W}_k = \mathcal{W}_{k-1} \cup \{v_k, e_k\}$ where v_k is a vertex and e_k is an edges with initial vertex v_{k-1} and terminal vertex v_k. It is easy to see that each $\mathcal{W}_k$ is two-sided and a tree.

Let $\mathbf{F}_0 = \{\mathcal{W}_k | k \geq 1\}$. Assume, for induction, that a set $\mathbf{F}_j$ of two-sided maps having two-sided regions has been defined and that each map in $\mathbf{F}_j$ has j regions. Let I_j be an index set of triples $(k, \mathcal{M}, \sigma)$ where k is a natural number, $\mathcal{M} \in \mathbf{F}_j$, and σ is a nontrivial segment of the top side of $\mathcal{M}$ having initial endpoint u_0 and terminal endpoint u_t. Given such a triple, we define a map $\mathcal{F}(k, \mathcal{M}, \sigma)$ to be

$$\mathcal{M} \cup \{v_1, v_2, \ldots, v_{k-1}, e_1, \ldots, e_k, D_{j+1}\}$$

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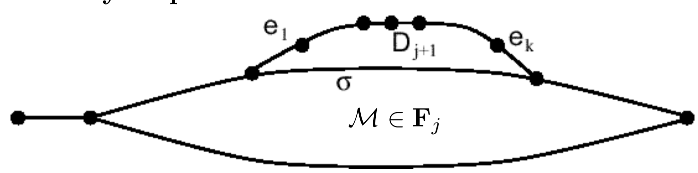
where each v_i is a vertex, each e_i is an edge with initial vertex v_{i-1} and terminal vertex v_i, (Let $v_0 = u_0$ and $v_k = u_t$.) and D_{j+1} is a region having bottom side σ and top side $e_1 e_2 \ldots e_k$. See the following illustration. Then $\mathcal{F}(k, \mathcal{M}, \sigma)$ and its $j+1$ regions are two-sided. Let $\mathbf{F}_{j+1} = \{\mathcal{F}(k, \mathcal{M}, \sigma) \mid (k, \mathcal{M}, \sigma) \in I_j\}$ and let $\mathbf{F} = \cup_{j=0}^{\infty} \mathbf{F}_j$. To facilitate a brief discussion in this paper, we will call a map in $\mathbf{F}$ a **feather** or a **feathery map** .



A vertex v in a map $\mathcal{M}$ is a **source** if for every other vertex w of $\mathcal{M}$, there is a positive walk in $\mathcal{M}$ from v to w. Dually, v is a **sink** if there is a positive walk from every such w to v. A vertex is a **transmitter** is it has indegree 0 and a **receiver** is it has outdegree 0. The set $\mathbf{R}$ of maps is defined by $\mathbf{R} = \{\mathcal{M} \mid \mathcal{M}$ is two-sided, each region of $\mathcal{M}$ is two-sided, and no interior vertex of $\mathcal{M}$ is a transmitter or a receiver$\}$. It can be shown that for maps in $\mathbf{R}$, a vertex is a transmitter if and only if it is a source and a vertex is a receiver if and only if it is a sink. John Remmers introduced the set $\mathbf{R}$ in his thesis [10] where the maps of $\mathbf{R}$ were called monotone maps. In a published part of Remmers' thesis [11], the maps of $\mathbf{R}$ are called regular maps. In both [10] and [11], Remmers proves the essential details of the following theorem. For a more recent exposition of this and other properties of regular maps, the reader should see Higgins' book [2].

Theorem 1.1. $\mathbf{R} = \mathbf{F}$.

Corollary 1.2. *If $\mathcal{M} \in \mathbf{F}$ and $\mathcal{N}$ is a two-sided submap of $\mathcal{M}$, then $\mathcal{N} \in \mathbf{F}$.*

Hence the maps that we have called feathers above are the same as the maps which have previously been called monotone maps [4,10] or regular maps [5,11]. Higgins [2] like Howie and Pride [3] bypasses the need to assign any name for these maps by proceeding directly to the closely related notion of a diagram (See below.). For the purposes of this paper, it is essential to discuss feathers or regular maps as well as diagrams. The usefulness of these maps and the related diagrams has also been discovered independently by Kasincev [6,7,8]. More recently, they have also been discovered by Power [9] who calls them pasting schemes; unlike the other authors who work with semigroups, Power uses these maps for a construction in category theory.

Given that $\mathbf{R} = \mathbf{F}$, the next three lemmas are easily seen to be true for $\mathbf{F}$ and hence for $\mathbf{R}$ also. As well as being useful observations, they are instrumental in the proof that $\mathbf{R} \subseteq \mathbf{F}$. For proofs, see [10], [11,pp. 286,287] or [2,pp. 75,76]

Lemma 1.3 (Remmers). *If $\mathcal{M}$ is a map in $\mathbf{R}$, then there are no positive closed walks in $\mathcal{M}$.*

Lemma 1.4 (Remmers). *Suppose that $\mathcal{M}$ is a map in $\mathbf{R}$. If v is a vertex of $\mathcal{M}$, then there is a positive walk from the initial vertex of $\mathcal{M}$ to the terminal vertex of $\mathcal{M}$ which contains v. If e is an edge of $\mathcal{M}$, then there is a positive walk from the initial vertex of $\mathcal{M}$ to the terminal vertex of $\mathcal{M}$ which contains e.*

A region D of the map $\mathcal{M}$ is **appended on the top side of** $\mathcal{M}$ if ω_D is a segment of $\omega_{\mathcal{M}}$ and is **appended on the bottom side of** $\mathcal{M}$ if α_D is a segment of $\alpha_{\mathcal{M}}$. A region D is an **appended region of** $\mathcal{M}$ if it is appended on either side of $\mathcal{M}$.

Lemma 1.5 (Remmers). *If $\mathcal{M} \in \mathbf{R}$ and has at least one region, then there are regions B and T of $\mathcal{M}$ which are appended on the bottom and top sides of $\mathcal{M}$, respectively. If $\mathcal{M}$ has at least two regions, then we can choose regions B and T which are distinct.*

Suppose that $\mathcal{M}$ is a feathery map and that D is an appended region on the bottom side of $\mathcal{M}$. Write $\alpha_{\mathcal{M}} = \lambda_0 \alpha_D \lambda_1$. We allow λ_0 and/or λ_1 to be empty. We define $\mathcal{M} - D$ to be the feathery submap of $\mathcal{M}$ which has boundary walk $\lambda_0 \omega_D \lambda_1 \omega_{\mathcal{M}}^{-1}$. Equivalently, $\mathcal{M} - D$ is the feathery submap of $\mathcal{M}$ which consists of all of the regions of $\mathcal{M}$ <u>except</u> D, all of the edges of $\mathcal{M}$ <u>except</u> those on α_D, and all of the vertices of $\mathcal{M}$ <u>except</u> those which are interior vertices on α_D.

Similarly, if D is appended on the top side of $\mathcal{M}$, but is not appended on the bottom side of $\mathcal{M}$, write $\omega_{\mathcal{M}} = \mu_0 \omega_D \mu_1$. Then $\mathcal{M} - D$ is the feathery submap of $\mathcal{M}$ which has boundary walk $\alpha_{\mathcal{M}}(\mu_0 \alpha_D \mu_1)^{-1}$. The reader should note that we have made a rather arbitrary choice about which side of D to include as part of $\mathcal{M} - D$ in the case where D is appended on both sides of $\mathcal{M}$.

More generally, suppose that $S = \{D_1, D_2, \ldots, D_q\}$ is any set of appended regions of $\mathcal{M}$. Write α_i and ω_i for α_{D_i} and ω_{D_i}. We may renumber so that

1. For $0 \le p \le q$, $D_1, D_2, \ldots, D_p$ are appended on the bottom side of $\mathcal{M}$,
2. $D_{p+1}, \ldots, D_q$ are appended on the top side of $\mathcal{M}$, but not on the bottom side of $\mathcal{M}$,
3. $\alpha_{\mathcal{M}} = \lambda_0 \alpha_1 \lambda_1 \alpha_2 \ldots \alpha_p \lambda_p$, and
4. $\omega_{\mathcal{M}} = \mu_p \omega_{p+1} \mu_{p+1} \ldots \omega_q \mu_q$

where the λ_i and μ_i are positive or empty walks. Then we define $\mathcal{M} - S$ to be the feathery submap of $\mathcal{M}$ which has boundary walk

$$(\lambda_0 \omega_1 \lambda_1 \omega_2 \ldots \omega_p \lambda_p)(\mu_p \alpha_{p+1} \mu_{p+1} \ldots \alpha_q \mu_q)^{-1}.$$

Observe that when $S = \{D\}$, then $\mathcal{M} - S = \mathcal{M} - D$.

If $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k$ are feathers, then we define $\mathcal{M}_1 \mathcal{M}_2 \ldots \mathcal{M}_k$ to be the feather that is obtained by identifying the terminal vertex of $\mathcal{M}_i$ with the initial vertex of $\mathcal{M}_{i+1}$ for $1 \le i \le k-1$. We will also use the notation $\vee_{i=1}^{k} \mathcal{M}_i$ for $\mathcal{M}_1 \mathcal{M}_2 \ldots \mathcal{M}_k$, and the notation $\mathcal{M}_1 \vee \mathcal{M}_2$ for $\mathcal{M}_1 \mathcal{M}_2$. In some applications, we may have feathers $\mathcal{M}_\tau$ indexed by a set whose order is irrelevant and we will write $\bigvee_\tau \mathcal{M}_\tau$ and take this to mean any one of several possible feathers depending upon the order chosen.

The vertex v in the feather $\mathcal{M}$ is a **separating vertex in** $\mathcal{M}$ if $\mathcal{M} - v$ is disconnected. Every separating vertex in $\mathcal{M}$ must be on both the top and bottom side of $\mathcal{M}$. A feather is **nonseparable** if it has no separating vertices. A **block** of a feather is a maximal nonseparable submap. A block of a feather is itself a feather. For any feather $\mathcal{M}$, we may write $\mathcal{M} = \mathcal{M}_1 \mathcal{M}_2 \ldots \mathcal{M}_k$ where the $\mathcal{M}_i$ are all of the blocks of $\mathcal{M}$ and their common vertices are all of the separating vertices of $\mathcal{M}$.

For any map $\mathcal{M}$, the set W^+ of positive walks in $\mathcal{M}$ is a partial groupoid (i.e. concatenation is a partially defined associative binary operation) generated by the edges of $\mathcal{M}$. If S is any semigroup, a function ϕ from W^+ to S is a **labelling of $\mathcal{M}$ with values in** S when ϕ is a partial homomorphism from W^+ to S. (I.e. $\phi(\pi\sigma) = \phi(\pi)\phi(\sigma)$ whenever the concatenation $\pi\sigma$ is defined.) Of course, ϕ is determined once it is defined on the edges which generate W^+ and any choice of labels for the edges will produce a well-defined labelling of $\mathcal{M}$ with values in S. A **diagram over the semigroup** S is a pair $(\mathcal{M}, \phi)$ where $\mathcal{M}$ is a map and ϕ is a labelling with values in S. If $\mathcal{N}$ is a submap of $\mathcal{M}$, it will not result in any confusion if we also use the notation ϕ for the restriction, $\phi_{|\mathcal{N}}$, of ϕ to $\mathcal{N}$. For this paper, S will always be the free semigroup F_X on some set X, and we will refer to $(\mathcal{M}, \phi)$ as a **diagram over** X. Often the function ϕ is obvious from context and we refer simply to a diagram $\mathcal{M}$. Having suppressed ϕ, we then write $\overline{\sigma}$ for the label $\phi(\sigma)$ on a positive walk σ. A vertex in a map $\mathcal{M}$ is **superfluous** if its indegree and its outdegree are both 1. Throughout this paper, the maps used will have enough superfluous vertices that we can label each edge by a letter of X rather than a longer word of F_X. We use $|w|$ for the length of a word in the free semigroup F_X, and we use $|\sigma|$ for the length of a positive walk in $\mathcal{M}$. Since we label each edge by a letter, we will always have $|\sigma| = |\overline{\sigma}|$. This convention on superfluous vertices is useful in this paper. In several of the other papers using derivation diagrams, it is useful to assume instead that maps have <u>no</u> superfluous vertices and conclude only that $|\sigma| \leq |\overline{\sigma}|$.

If Q is a two-sided diagram or if Q is a two-sided region in a diagram, then the **boundary label of** Q is the ordered pair $(\overline{\alpha}_Q, \overline{\omega}_Q)$ of labels of the sides of Q.

Suppose that $\langle X; R\rangle$ is a semigroup presentation, where X is an alphabet, F_X is the free semigroup on X, and $R \subseteq F_X \times F_X$ is a set of defining relators. Let the feather $\mathcal{M}$ be a diagram over X. Then $\mathcal{M}$ is a **derivation diagram over** $\langle X; R\rangle$ for $(\overline{\alpha}_{\mathcal{M}}, \overline{\omega}_{\mathcal{M}})$ if at least one of $(\overline{\alpha}_D, \overline{\omega}_D)$ or $(\overline{\omega}_D, \overline{\alpha}_D)$ is in R for each region D of $\mathcal{M}$.

If $\mathcal{M}$ is a feather or $(\mathcal{M}, \phi)$ is a diagram, we use the notation $|\mathcal{M}|$ for the number of regions in $\mathcal{M}$. Let $\langle X; R\rangle$ be some fixed semigroup presentation. A derivation diagram $(\mathcal{M}, \phi)$ over $\langle X; R\rangle$ is a **minimal** derivation diagram over $\langle X; R\rangle$ for $(\overline{\alpha}_{\mathcal{M}}, \overline{\omega}_{\mathcal{M}})$ if $|\mathcal{M}| \leq |\mathcal{N}|$ whenever $(\mathcal{N}, \xi)$ is also a derivation diagram over $\langle X; R\rangle$ for $(\overline{\alpha}_{\mathcal{M}}, \overline{\omega}_{\mathcal{M}})$.

Theorem 1.6 (Remmers). *Suppose that $(\mathcal{M}, \phi)$ is a minimal derivation diagram over $\langle X; R\rangle$. If the submap $\mathcal{N}$ of $\mathcal{M}$ is in $\mathbf{R}$, then $(\mathcal{N}, \phi)$ is a minimal derivation diagram over $\langle X; R\rangle$ for $(\phi(\alpha_{\mathcal{N}}), \phi(\omega_{\mathcal{N}}))$.*

Proof. If $(\mathcal{N}, \phi)$ is not minimal, there is a derivation diagram $(\mathcal{N}', \xi)$ over $\langle X; R\rangle$ having the same boundary label as $(\mathcal{N}, \phi)$. We could then replace the submap $\mathcal{N}$ of $\mathcal{M}$ by $\mathcal{N}'$ and contradict the minimality of $(\mathcal{M}, \phi)$.

A **directed walk** in a map $\mathcal{M}$ is a walk which is either a positive walk or the empty walk at some vertex of $\mathcal{M}$.

If $\mathcal{M}$ is any map in $\mathbf{R}$, Remmers has described the following natural partial orders on the vertices of $\mathcal{M}$ and on the edges of $\mathcal{M}$:

(1) For vertices u and v, $u < v$ if there is a positive walk in $\mathcal{M}$ from u to v.
(2) For edges e and f, $e < f$ if there is a directed walk from the terminal endpoint of e to the initial endpoint of f.

It is straightforward to verify, by induction on the number of regions, that any two vertices u and v in a feather $\mathcal{M}$ have both a greatest lower bound, $\mathrm{glb}_{\mathcal{M}}(u,v)$, and a least upper bound, $\mathrm{lub}_{\mathcal{M}}(u,v)$. We shall make use of these orders in the following sections. For convenience of reference, we will refer to these orders on the vertices and edges as Remmers' orders.

A derivation diagram $(\mathcal{M},\phi)$ is **quasiminimal** if every block $\mathcal{N}$ of $\mathcal{M}$ is a minimal derivation diagram for $(\phi(\alpha_{\mathcal{N}}),\phi(\omega_{\mathcal{N}}))$. A derivation diagram $(\mathcal{M},\phi)$ is **strongly quasiminimal** if $(\mathcal{M},\phi)$ is quasiminimal and for each block $\mathcal{N}$ of $\mathcal{M}$, every quasiminimal derivation diagram $(\mathcal{N}',\phi')$ for $(\phi(\alpha_{\mathcal{N}}),\phi(\omega_{\mathcal{N}}))$ has only one block. In the following theorem, the equivalence of i), ii), and iii) is due to Remmers.

Theorem 1.7. *Suppose that $\langle X;R\rangle$ presents a semigroup S and that w and w' are words on the alphabet X. Then the following are equivalent:*

- i) *The words w and w' represent the same element of S.*
- ii) *There is a derivation diagram over $\langle X;R\rangle$ for (w,w').*
- iii) *There is a minimal derivation diagram over $\langle X;R\rangle$ for (w,w').*
- iv) *There is a quasiminimal derivation diagram over $\langle X;R\rangle$ for (w,w').*
- v) *There is a strongly quasiminimal derivation diagram over $\langle X;R\rangle$ for (w,w').*

Proof. i) $\Rightarrow$ ii) If w and w' represent the same element of S, we may use any sequence of elementary transitions from w to w' to construct a map in the set $\mathbf{F}$ labelled as a derivation diagram over $\langle X;R\rangle$ for (w,w').

ii) $\Rightarrow$ iii) If the derivation diagram $(\mathcal{M},\phi)$ is not minimal, then there is a derivation diagram $(\mathcal{N},\xi)$ for (w,w') with $|\mathcal{N}|<|\mathcal{M}|$. If $(\mathcal{N},\xi)$ is not minimal, we repeat this: since $\mathcal{M}$ has only a finite number of regions, we must eventually reach a minimal diagram.

iii) $\Rightarrow$ iv) By Theorem 1.6, the blocks of a minimal derivation diagram are minimal.

iv) $\Rightarrow$ v) Let k be the smaller of $|w|$ and $|w'|$. Then no derivation diagram over $\langle X;R\rangle$ for (w,w') can have more than k blocks. Suppose $(\mathcal{M},\phi)$ is a quasiminimal derivation diagram over $\langle X;R\rangle$ for (w,w'). If $(\mathcal{M},\phi)$ is not strongly quasiminimal, then for some block $\mathcal{N}$ of $\mathcal{M}$, there is a quasiminimal derivation diagram $(\mathcal{N}',\phi')$ over $\langle X;R\rangle$ for $(\phi(\alpha_{\mathcal{N}}),\phi(\omega_{\mathcal{N}}))$, where the map $\mathcal{N}'$ has more than one block. We may replace the submap $\mathcal{N}$ of $\mathcal{M}$ by $\mathcal{N}'$ to obtain a quasiminimal derivation diagram $(\mathcal{M}',\xi)$ for (w,w') where $\mathcal{M}'$ has more blocks than $\mathcal{M}$. If $(\mathcal{M}',\xi)$ is not strongly quasiminimal, we repeat this process, but we must eventually reach a strongly quasiminimal diagram since k is a bound for the number of blocks in $\mathcal{M}'$.

ii), iii), iv), or v) $\Rightarrow$ i) We may use any derivation diagram for (w,w') to construct a sequence of elementary transitions from w to w'.

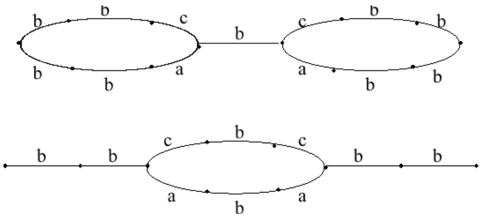
Example 1.8. Let $\langle X;R\rangle$ be the semigroup presentation

$$\langle a,b,c;\ abb=cbb,\ bba=bbc,\ aba=cbc\rangle.$$

Then the diagram $\mathcal{M}$ below is a strongly quasiminimal derivation diagram. It is not a minimal derivation diagram since the diagram $\mathcal{M}'$ is also a derivation diagram over $\langle X;R\rangle$ for $(bbababb, bbcbcbb)$.

$\mathcal{M}$

$\mathcal{M}'$



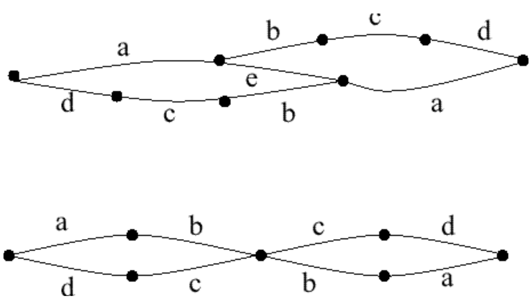
Example 1.9. Let $\langle X; R\rangle$ be the semigroup presentation

$$\langle a, b, c, d, e;\ ae = bcd,\ ea = bcd,\ ab = dc,\ cd = ba\rangle.$$

Then in following figure both $\mathcal{M}_1$ and $\mathcal{M}_2$ are minimal (and quasiminimal) derivation diagrams for $abcd = dcba$, but only $\mathcal{M}_2$ is strongly quasiminimal.

$\mathcal{M}_1$

$\mathcal{M}_2$



Lemma 1.10. *If $\mathcal{M} \in \mathbf{R}$ and $\mathcal{N}$ is a submap of $\mathcal{M}$ which is also in $\mathbf{R}$ and which contains every appended region of $\mathcal{M}$, then $\mathcal{N}$ contains every region of $\mathcal{M}$.*

Proof. Observe first that it will suffice to prove the lemma in the case where $\mathcal{M}$ has only one block. We assume that we are in this case. Let δ_1 be a positive or empty path from the initial vertex of $\mathcal{M}$ to the initial vertex of $\mathcal{N}$ and let δ_2 be a positive or empty path from the terminal vertex of $\mathcal{N}$ to the terminal vertex of $\mathcal{M}$. Then the closed walk $\delta_1\omega_{\mathcal{N}}\delta_2(\omega_{\mathcal{M}})^{-1}$ is the closed two-sided boundary for a two-sided submap $\mathcal{T}$ of $\mathcal{M}$. It follows from Corollary 1.2 that $\mathcal{T} \in \mathbf{R}$. If there are regions in $\mathcal{T}$, then by Lemma 1.5, there is a region D of $\mathcal{T}$ which is appended on the top side of $\mathcal{T}$. The region D is then also an appended region on the top side of $\mathcal{M}$. By hypothesis, D is then a region of $\mathcal{N}$, but this is impossible because $\omega_{\mathcal{N}}$ is the bottom side of $\mathcal{T}$. We conclude that $\mathcal{T}$ is regionless and that $\delta_1\omega_{\mathcal{N}}\delta_2 = \omega_{\mathcal{M}}$. A similar argument shows that $\delta_1\alpha_{\mathcal{N}}\delta_2 = \alpha_{\mathcal{M}}$. Since $\mathcal{M}$ has only one block and δ_1 and δ_2 are on both the bottom and top sides of $\mathcal{M}$, the paths δ_1 and δ_2 are empty. Then $\omega_{\mathcal{N}} = \omega_{\mathcal{M}}$ and $\alpha_{\mathcal{N}} = \alpha_{\mathcal{M}}$. We conclude that $\mathcal{N} = \mathcal{M}$ in the case where $\mathcal{M}$ has only one block.

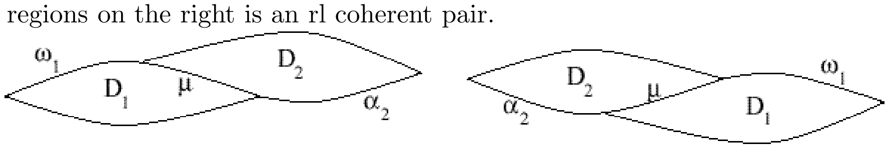
We introduce the notion of coherent pairs of regions in a feathery map. These will be useful later. Since our inductive construction of feathery maps starts at the bottom (rather than at the top) our terminology for coherent pairs will also reflect a "start-at-the-bottom" bias.

An ordered pair (D_1, D_2) of regions in a feathery map $\mathcal{M}$ is a **left-right coherent pair** if there are positive walks ω_1, μ, and α_2 in $\mathcal{M}$ such that $\omega_{D_1} = \omega_1\mu$ and

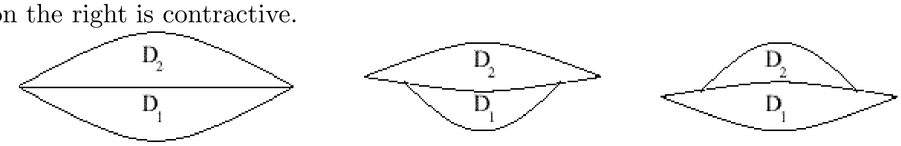
$\alpha_{D_2} = \mu\alpha_2$. Dually, an ordered pair (D_1, D_2) is a **right-left coherent pair** if there are positive walks ω_1, μ, and α_2 in $\mathcal{M}$ such that $\omega_{D_1} = \mu\omega_1$ and $\alpha_{D_2} = \alpha_2\mu$. For brevity, we will say that a left-right coherent pair is an lr pair and that a right-left coherent pair is an rl pair.

The ordered pair (D_1, D_2) of regions in a feathery map $\mathcal{M}$ is a **matched coherent pair** if $\omega_{D_1} = \alpha_{D_2}$, an **expansive coherent pair** if ω_{D_1} is a proper segment of α_{D_2}, and a **contractive coherent pair** if α_{D_2} is a proper segment of ω_{D_1}.

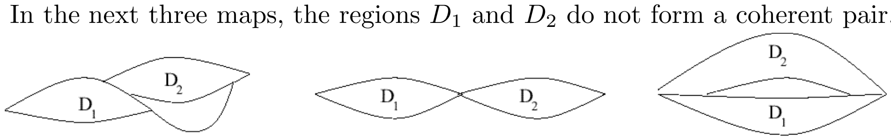
Example 1.11. The pair of regions on the left is an lr coherent pair and the pair of regions on the right is an rl coherent pair.



The pair on the left is matched, the pair in the middle is expansive, and the pair on the right is contractive.



In the next three maps, the regions D_1 and D_2 do not form a coherent pair.



A **coherent pair** of regions is an ordered pair of regions that is left-right, right-left, matched, expansive, or contractive coherent. We note that if (D_1, D_2) is a coherent pair of regions in a feathery map $\mathcal{M}$ then (1) there is a feathery submap $\mathcal{N}$ of $\mathcal{M}$ which consists of just D_1, D_2 and the edges and vertices on the boundaries of these two regions, and (2) there is at least one edge e which is in both ω_{D_1} and α_{D_2}.

Conversely, if (1) and (2) hold, then by considering the provenance of the initial and terminal endpoints of $\mathcal{N}$, we see that (D_1, D_2) must be a coherent pair.

Lemma 1.12. *Suppose that $\mathcal{M}$ is a feathery map and that D_0 is any region of $\mathcal{M}$.*

(1) *Either D_0 is appended on the bottom side of $\mathcal{M}$ or else there is a region D_- of $\mathcal{M}$ such that (D_-, D_0) is a coherent pair.*
(2) *Either D_0 is appended on the top side of $\mathcal{M}$ or else there is a region D_+ of $\mathcal{M}$ such that (D_0, D_+) is a coherent pair.*

Proof.. We prove (1) by induction on $|\mathcal{M}|$: the proof of (2) is dual. If $\mathcal{M}$ has only one region that region is necessarily appended on the bottom side of $\mathcal{M}$. Assume that $\mathcal{M} > 1$. If D_0 is appended on the bottom side of $\mathcal{M}$ we are done, so assume that D_0 is not an appended region on the bottom side of $\mathcal{M}$. By Lemma 1.5, there is a region D of $\mathcal{M}$ which is appended on the bottom side of $\mathcal{M}$. By induction, the lemma is true for D_0 as a region of $\mathcal{M} - D$. If (D_-, D_0) is a coherent pair in $\mathcal{M} - D$ then this is also a coherent pair in $\mathcal{M}$. If D_0 is appended on the bottom side of $\mathcal{M} - D$, then (D, D_0) is a coherent pair in $\mathcal{M}$.

2 Alphabetically Constructed Relators

For natural numbers $s < \ell$, an (s, ℓ) **map** is a map in $\mathbf{R}$ in which each region has a short side with s edges and a long side with ℓ edges. Given an alphabet X and words w_s and w_ℓ on X having respective lengths s and ℓ, the relator $w_s = w_\ell$ is an (s, ℓ) **relator** and the semigroup presentation $\langle X;\ w_s = w_\ell \rangle$ is an (s, ℓ) **presentation** .

Assume that X is a finite or countable, ordered alphabet: $X = \{x_1, x_2, \dots\}$, with $x_1 < x_2 \dots$. A word w on X is said to be **alphabetically constructed** if whenever x_j occurs in w and $i < j$, then at least one occurrence of x_i in w precedes the first occurrence of x_j. An (s, ℓ) relator $w_s = w_\ell$ is **alphabetically constructed** if the concatenated word $w_s w_\ell$ is.

Example 2.1. On the ordered alphabet $\{a, b, c, d, e\}$, the words $abac$, ab^2aba^3cacb, and $a^2babacbadb$ are alphabetically constructed, but $acab$, bcd, and $acad$ are not.

Proposition 2.2. *If Y is any finite alphabet and w is any word on Y, then there is an order on Y for which w is alphabetically constructed. For the letters of Y which occur in w, this order is unique.*

Proof. For letters x and y which occur in w, let $x < y$ if the first occurrence of x in w precedes the first occurrence of y in w. All letters that do not occur in w follow those that do occur.

Corollary 2.3. *If a semigroup has an (s, ℓ) presentation $\langle Y;\ w_s = w_\ell \rangle$, then there is an order on Y for which $w_s = w_\ell$ is alphabetically constructed.*

Let $x_1 < x_2 < x_3 \dots$ be a given order for the alphabet X. Then any other order $x_{\sigma 1} < x_{\sigma 2} < x_{\sigma 3} \dots$ corresponds to a permutation σ of X. The permutatuion σ in turn induces an automorphism, which we will also denote by σ, of the free semigroup F_X via $(x_{i_1} x_{i_2} \dots x_{i_n})\sigma = x_{\sigma i_1} x_{\sigma i_2} \dots x_{\sigma i_n}$.

Proposition 2.4. *If w is a word on X which is alphabetically constructed for the order $x_{\sigma 1} < x_{\sigma 2} < x_{\sigma 3} \dots$, then $w\sigma^{-1}$ is alphabetically constructed for the order $x_1 < x_2 < x_3 \dots$. If w is also alphabetically constructed for the order $x_{\tau 1} < x_{\tau 2} < x_{\tau 3} \dots$, then $w\tau^{-1} = w\sigma^{-1}$.*

Proof. Write $w\sigma^{-1} = x_{i_1} x_{i_2} \dots x_i \dots x_j \dots x_{i_n}$. Then

$$w = (w\sigma^{-1})\sigma = x_{\sigma i_1} x_{\sigma i_2} \dots x_{\sigma i} \dots x_{\sigma j} \dots x_{\sigma i_n},$$

hence x_i precedes x_j in $w\sigma^{-1}$ if and only if $x_{\sigma i}$ precedes $x_{\sigma j}$ in w. If r distinct letters occur in w, then $\sigma i \in \{\sigma 1, \sigma 2, \dots, \sigma r\}$, so $1 \le i \le r$. By Proposition 2.2, $\sigma i = \tau i$ for $1 \le i \le r$, hence

$$w = (w\sigma^{-1})\sigma = x_{\sigma i_1} x_{\sigma i_2} \dots x_{\sigma i_n} = x_{\tau i_1} x_{\tau i_2} \dots x_{\tau i_n} = (w\sigma^{-1})\tau,$$

and $w\tau^{-1} = w\sigma^{-1}$.

For a fixed order $x_1 < x_2 < x_3 \dots$ of X, define a function $\mathcal{A}$ on the set $F_X \times F_X$ by $\mathcal{A}(w, w') = (w\sigma, w'\sigma)$ for any permutation σ of X such that the word $(ww')\sigma$ is alphabetically constructed for the given order on X.

Corollary 2.5. *For any (s,ℓ) presentation $\langle X; (w_s, w_\ell)\rangle$ on an ordered alphabet X, the presentation $\langle X; \mathcal{A}(w_s, w_\ell)\rangle$ has an alphabetically constructed (s,ℓ) relator, presents the same semigroup as $\langle X; (w_s, w_\ell)\rangle$, and is effectively obtained from the presentation $\langle X; (w_s, w_\ell)\rangle$.*

The following combinatorial observation is mostly peripheral to the main results of this paper, but it will help illustrate some later remarks.

Proposition 2.6. *If $S_{n,r}$ is the number of alphabetically constructed words of length n on r distinct letters, then $S_{n,r}$ is the Stirling number of the second kind.*

Proof. Define a partition of the n positions in the word by assigning two positions to the same subset if and only if the same letter occurs in both positions. Stirling numbers of the second kind count the number of partitions that are possible.[1]

An obvious consequence of Proposition 2.6 is that for $n = s+\ell$, the number of alphabetically constructed (s,ℓ) relators is the Bell number $\sum_{r=1}^{n} S_{n,r}$. [1]

If S is any set and ρ is an equivalence relation on S, then for elements s and t of S, we use the notations $s\rho t$ and $(s,t)\in\rho$ interchangeably. We use the usual notation $[s]_\rho = \{t \mid s\rho t\}$ and $S/\rho = \{[s]_\rho \mid s \in S\}$. We write $[s]$ for $[s]_\rho$ whenever context renders this unambiguous.

When S is a finite ordered set and ρ is an equivalence relation on S, we define an induced order on S/ρ by $[x] < [y]$ if the first element in $[x]$ is less than the first element in $[y]$. Now fix natural numbers s and ℓ with $s < \ell$ and let $Z = \{z_1, z_2, \ldots, z_{s+\ell}\}$ be an auxiliary ordered set. We regard Z as a set of positions available for letters in (s,ℓ) relator. Given a relation ρ on Z, we regard Z/ρ as an initial segment of an ordered alphabet X. In more detail, suppose that $\mathbf{r} = (w_s, w_\ell)$ is an (s,ℓ) relator on X in which m distinct letters of X occur. Write $w_s = y_1y_2\ldots y_s$ and $w_\ell = y_{s+1}y_{s+2}\ldots y_{s+\ell}$ where each y_i is a letter of X. Define a relation $\rho_{\mathbf{r}}$ on Z by $\rho_{\mathbf{r}} = \{(z_j, z_k) \mid y_j \text{ and } y_k \text{ in } \mathbf{r} \text{ are the same letter of } X\}$. Then $\rho_{\mathbf{r}}$ is an equivalence relation on Z having m distinct equivalence classes. Conversely, given an equivalence relation ρ on Z, define words w_s and w_ℓ on the ordered alphabet Z/ρ by $w_s = [z_1][z_2]\ldots[z_s]$, and $w_\ell = [z_{s+1}][z_{s+2}]\ldots[z_{s+\ell}]$. Then the relator $\mathbf{r}(\rho) = (w_s, w_\ell)$ is an alphabetically constructed relator for the induced order on the alphabet Z/ρ and the distinct letters of $\mathbf{r}(\rho)$ correspond to the ρ-equivalence classes. Partly for notational convenience, we regard Z/ρ as an initial segment of the ordered alphabet X and write x_i for the i^{th} letter in the induced order on Z/ρ.

Proposition 2.7. *If ρ is an equivalence relation on Z, then $\rho = \rho_{\mathbf{r}(\rho)}$. If $\mathbf{r}$ is an (s,ℓ) relator on X, then the number of distinct letters of X that occur in $\mathbf{r}$ is the same as the number of letters in $Z/\rho_{\mathbf{r}}$. If we use the induced order on $Z/\rho_{\mathbf{r}}$ to identify $Z/\rho_{\mathbf{r}}$ with the initial segment of X, then $\mathbf{r}(\rho_{\mathbf{r}}) = \mathcal{A}(\mathbf{r})$. Thus there is a bijection between alphabetically constructed (s,ℓ) relators on X and equivalence relations on Z.*

Proof. For the first statement, $(z_i, z_j) \in \rho_{\mathbf{r}(\rho)}$ if and only if $[z_i]$ and $[z_j]$ in the relator $\mathbf{r}(\rho)$ are the same letter of the alphabet Z/ρ: the latter occurs if and only if $(z_i, z_j) \in \rho$. For the second statement, each $\rho_{\mathbf{r}}$-equivalence class corresponds to some letter of X which occurs in $\mathbf{r}$. For the third statement, suppose first that $\mathbf{r}$ is alphabetically constructed. Then the letter x_i of X is the i^{th} distinct letter to occur in $\mathbf{r}$, hence the equivalence class $[z]$ of positions in which x_i occurs will be the i^{th} class in the induced order on $Z/\rho_{\mathbf{r}}$. Since we have hypothesized that we identify this i^{th} class with x_i, we have $\mathbf{r} = \mathbf{r}(\rho_{\mathbf{r}})$ for an alphabetically constructed relator $\mathbf{r}$. For a permutation σ of X, and a (s, ℓ) relator $\mathbf{r} = (w_s, w_\ell)$, write $\mathbf{r}\sigma$ for the (s, ℓ) relator $(w_s\sigma, w_\ell\sigma)$. Then for any such σ and any $\mathbf{r}$, $\rho_{\mathbf{r}} = \rho_{\mathbf{r}\sigma}$, since $\rho_{\mathbf{r}}$ records only the positions at which the same letters occur. In particular, $\rho_{\mathbf{r}} = \rho_{\mathcal{A}(\mathbf{r})}$, so $\mathbf{r}(\rho_{\mathbf{r}}) = \mathbf{r}(\rho_{\mathcal{A}(\mathbf{r})}) = \mathcal{A}(\mathbf{r})$.

Suppose that $\mathbf{r}$ is an alphabetically constructed (s, ℓ) relator, that ρ is the equivalence relation on Z which corresponds to $\mathbf{r}$, and that x is a letter of X which occurs in $\mathbf{r}$. Using reasonably standard notation, we define the restriction of ρ to x by

$$\rho_{|x} = \{\, (z_i, z_j) \,|\, [z_i]_\rho = [z_j]_\rho = x \,\} \cup \{\, (z_k, z_k) \,\}$$

Then it is easy to see that $\rho_{|x}$ is an equivalence relation on Z, that $\rho_{|x} \subseteq \rho$, and that $\rho = \cup_{x \in X}\, \rho_{|x}$. We will also find it useful to define $J(x, \mathbf{r})$ to be the set of indices for letters of $\mathbf{r}$ which have value x. That is, $J(x, \mathbf{r}) = \{\, i \,|\, y_i = x\} = \{\, i \,|\, [z_i]_\rho = x\}$. $J = J(x, \mathbf{r})$ is the **index set** of x in $\mathbf{r}$.

Example 2.8. For the $(2, 5)$ relator $a^2 = bcabc$, we have $J(a, \mathbf{r}) = \{1, 2, 5\}$, $J(b, \mathbf{r}) = \{3, 6\}$, and $J(c, \mathbf{r}) = \{4, 7\}$. Also,

$$\rho_{|b} = \{(z_3, z_6), (z_6, z_3)\} \cup \{(z_k, z_k)\}_{1 \le k \le 7}, \quad \rho_{|c} = \{(z_4, z_7), (z_7, z_4)\} \cup \{(z_k, z_k)\}_{1 \le k \le 7}$$

$$\text{and } \rho_{|a} = \{\, (z_i, z_j) \,\}_{i,j \in J(a,\mathbf{r})} \cup \{\, (z_k, z_k) \,\}_{1 \le k \le 7}$$

The set of equivalence relations on Z has a standard lattice structure given by $\rho_1 \le \rho_2$ if $\rho_1 \subseteq \rho_2$, $\rho_1 \wedge \rho_2 = \rho_1 \cap \rho_2$, and $\rho_1 \vee \rho_2$ is the smallest equivalence relation that contains the set $\rho_1 \cup \rho_2$. Since we have a bijection between the equivalence relations on Z and the set of alphabetically constructed (s, ℓ) relators, we may regard the latter set as a lattice which is isomorphic to the lattice of equivalence relations.

Proposition 2.9. *For $i = 1, 2$, let $\mathbf{r}_i$ be an alphabetically constructed (s, ℓ) relator, let X_i be the initial segment of X consisting of letters that occur in $\mathbf{r}_i$, let ρ_i be the equivalence relation on Z that corresponds to $\mathbf{r}_i$, and let S_i be the semigroup presented by $\langle X_i; \mathbf{r}_i \rangle$. Identify X_i with Z/ρ_i. Then $\mathbf{r}_1 \le \mathbf{r}_2$ if and only if the rule $([z_j]_{\rho_1})f = [z_j]_{\rho_2}$ determines a well-defined homomorphism from S_1 onto S_2.*

Proof. Observe that $\mathbf{r}_1 \le \mathbf{r}_2 \;\Leftrightarrow\; \rho_1 \subseteq \rho_2 \;\Leftrightarrow\; f$ is a well-defined function on the alphabet Z/ρ_1. Once f is well-defined on Z/ρ_1, it is clear that $\mathbf{r}_1 f = \mathbf{r}_2$ and that f is onto.

Example 2.10. If S_1 is presented by $\langle a, b, c\,;\, ab = abcab \rangle$, and S_2 is presented by $\langle a, b, c\,;\, a^2 = a^2ba^2 \rangle$, then $\mathbf{r}_1 \le \mathbf{r}_2$ and setting $af = a, bf = a$, and $cf = b$ determines a well-defined homomorphism from S_1 onto S_2.

3 Achievable Relators

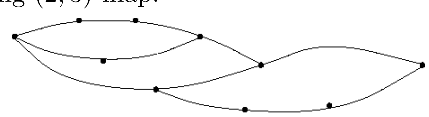
We wish to define, for each (s,ℓ) map $\mathcal{M}$, a naturally occurring alphabetically constructed (s,ℓ) relator induced by $\mathcal{M}$. This relator will be called the relator **achieved by $\mathcal{M}$.** Basically, we regard the map as a template to which we fit the most intricate relator possible.

An edge of a feathery map $\mathcal{M}$ is an **interior** edge of $\mathcal{M}$ if it is on the boundary of two regions of $\mathcal{M}$. Write $\mathcal{I}$ or when necessary, $\mathcal{I}_{\mathcal{M}}$, for the set of interior edges of the feathery map $\mathcal{M}$.

Suppose that $\mathcal{M}$ is an (s,ℓ) map, that D is a region in $\mathcal{M}$, and that e is an edge of D. Using Remmers' ordering of the edges of $\mathcal{M}$, we define the **index of e with respect to D** to be i if e is the i^{th} edge on the short side of D and to be $s+i$ if e is the i^{th} edge on the long side of D. This merely mirrors our order for the set Z of positions for letters in (s,ℓ) relators.

Let $\mathcal{E}$ be any subset of $\mathcal{I}$. We define a relation $\mathcal{R}_{\mathcal{E}}$ on Z by $\mathcal{R}_{\mathcal{E}} = \{(z_i, z_j) \mid$ There are regions D and D' of $\mathcal{M}$ having a common edge $e \in \mathcal{E}$ such that the index of e with respect to D is i and the index of e with respect to D' is $j\}$. Let $\rho_{\mathcal{E}}$ be the equivalence relation on Z that is generated by $\mathcal{R}_{\mathcal{E}}$. Then the (s,ℓ) relator $\mathbf{r}(\rho_{\mathcal{E}})$ is the relator that is **achieved** by the subset $\mathcal{E}$ of interior edges of $\mathcal{M}$. We will abbreviate $\mathbf{r}(\rho_{\mathcal{E}})$ by $\mathbf{r}_{\mathcal{E}}$. Generally, we will be interested in the case where $\mathcal{E} = \mathcal{I}$. In this case, we will use the notation $\mathcal{R}_{\mathcal{M}}, \rho_{\mathcal{M}}$, and $\mathbf{r}_{\mathcal{M}}$. The **relator that is achieved by $\mathcal{M}$** is defined to be $\mathbf{r}_{\mathcal{M}}$. When the (s,ℓ) map $\mathcal{M}$ is clear from context, we may further abbreviate to $\mathcal{R}$, ρ, and $\mathbf{r}$. If the (s,ℓ) relator $\mathbf{r}$ is the relator that is achieved by some (s,ℓ) map, then we say that $\mathbf{r}$ is an **achievable relator.**

Example 3.1. A routine calculation shows that the relator $a^2 = a^2b$ is the relator achieved by the following $(2,3)$ map.



Let $\mathcal{M}$ be a map in $\mathbf{R} = \mathbf{F}$. An edge is **constrained** if it is on the boundary of at least one region. An edge of $\mathcal{M}$ is **unconstrained** if it is not on the boundary of any region of $\mathcal{M}$. Every unconstrained edge in $\mathcal{M}$ is a block of $\mathcal{M}$ and a block of $\mathcal{M}$ that contains a region of $\mathcal{M}$ can contain no unconstrained edges of $\mathcal{M}$.

Let $\mathcal{U}_{\mathcal{M}}$ be the set of unconstrained edges of $\mathcal{M}$, and let $\mathcal{C} = \{(D,e) \mid D$ is a region of $\mathcal{M}$ and e is on the boundary of $D\}$. A **bilabelling of $\mathcal{M}$ with values in the semigroup** S is a function ϕ from $\mathcal{U} \cup \mathcal{C}$ to S. In this paper, S will always be the free semigroup on a finite ordered alphabet X. A bilabelling is **consistent** if $\phi(D_1, e) = \phi(D_2, e)$ whenever e is an interior edge on the boundary of two regions D_1 and D_2. It is clear that every labelling of $\mathcal{M}$ induces a consistent bilabelling, and that every consistent bilabelling of $\mathcal{M}$ induces a labelling.

Let $\mathcal{M}$ be an (s,ℓ) map. Let $\mathbf{r} = (w_s, w_\ell)$ be an (s,ℓ) relator on an ordered alphabet X. Write $w_s = y_1y_2\dots y_s$ and $w_\ell = y_{s+1}y_{s+2}\dots y_{s+\ell}$. We say that a bilabelling ϕ **corresponds to r** if $\phi(D,e) = y_i$ whenever the index of e with respect to D is i. (On $\mathcal{U}$, ϕ may have arbitrary values in X.) Since the consistency of bilabelling depends only on the interior edges, if one bilabelling that corresponds to $\mathbf{r}$ is consistent, then every bilabelling that corresponds to $\mathbf{r}$ is consistent.

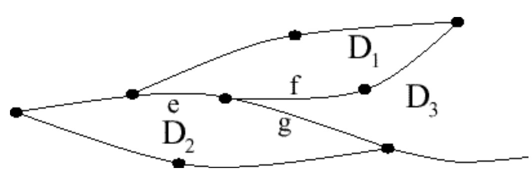
Proposition 3.2. *Suppose that $\mathcal{M}$ is an (s,ℓ) map, that $\mathbf{r}_{\mathcal{M}}$ is the relator that is achieved by $\mathcal{M}$, and that $\mathbf{r}$ is an alphabetically constructed relator which is equal to or greater than $\mathbf{r}_{\mathcal{M}}$ in the lattice of alphabetically constructed relators. Then any bilabelling of $\mathcal{M}$ which corresponds to $\mathbf{r}$ is consistent. Conversely, suppose that $\mathbf{r}$ is the least alphabetically constructed relator such that every bilabelling of $\mathcal{M}$ which corresponds to $\mathbf{r}$ is consistent. Then $\mathbf{r}$ is the relator that is achieved by $\mathcal{M}$.*

Proof. Throughout the proof we write ρ for $\rho_{\mathbf{r}}$ and $\rho_{\mathcal{M}}$ for $\rho_{\mathbf{r}_{\mathcal{M}}}$. We also write $\mathbf{r} = (w_s, w_\ell)$ where $w_s = y_1 y_2 \dots y_s$ and $w_\ell = y_{s+1} y_{s+2} \dots y_{s+\ell}$.

Suppose that $\mathbf{r}_{\mathcal{M}} \leq \mathbf{r}$. Let e be any interior edge of $\mathcal{M}$ and suppose that e has index i with respect to some region D and has index j with respect to some region D'. Then $(z_i, z_j) \in \rho_{\mathcal{M}}$. Since $\mathbf{r}_{\mathcal{M}} \leq \mathbf{r}$ by hypothesis, we have $\rho_{\mathcal{M}} \subseteq \rho$. It follows that $(z_i, z_j) \in \rho$, so that $y_i = y_j$ as required.

Recall that $\mathcal{R}_{\mathcal{M}} = \{(z_i, z_j) \mid$ There are regions D and D' of $\mathcal{M}$ having a common edge e such that the index of e with respect to D is i and the index of e with respect to D' is $j\}$. Observe that $\mathcal{R}_{\mathcal{M}} \subseteq \rho$ whenever $\mathbf{r}$ is an alphabetically constructed relator for which the corresponding bilabellings are consistent. If $\mathbf{r}$ is the least alphabetically constructed relator for which the corresponding bilabellings of $\mathcal{M}$ are consistent, then ρ is the least equivalence relation on Z containing $\mathcal{R}_{\mathcal{M}}$. Hence ρ is $\rho_{\mathcal{M}}$.

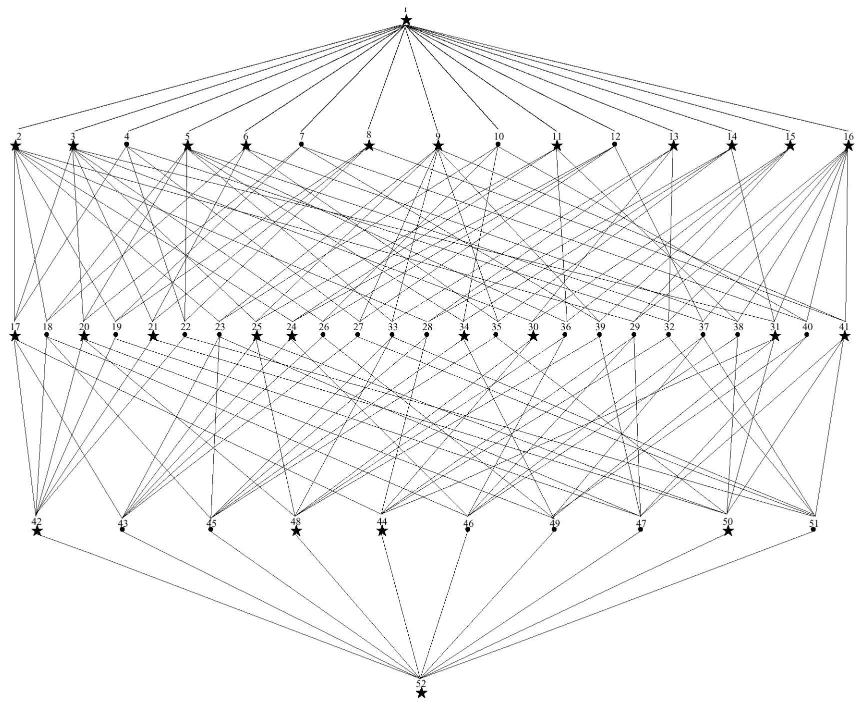
Example 3.3. The relator $a^2 = b^2 a$ is not achievable. If it were achieved by some map $\mathcal{M}$, it would induce a consistent bilabelling on $\mathcal{M}$. Since there are just the two occurrences of the letter b in the relator, there would be some edge e in $\mathcal{M}$ which has index three with respect to a region D_1 and index four with respect to a region D_2. Consider the rightmost such edge e and let f and g be the edges of D_1 and D_2, respectively, which follow e. The edge f must also have label b. Since f and g have a common initial vertex, it follows from the geometric properties of (s,ℓ) maps that f must be the initial edge on the side of some region D_3 of $\mathcal{M}$. Since f has label b, it must be the initial edge on the long side of D_3. Then f has index four with respect to D_1 and index three with respect to D_3; this contradicts our assumption that e is the rightmost such edge.



Example 3.4. There are 52 alphabetically constructed $(2,3)$ relators. We list and number these and illustrate the lattice of alphabetically constructed $(2,3)$ relators. Only 26 of these relators are achievable. The achievable relators are marked by stars. The verification that the starred relators are achievable follows as in Example 3.1 using $(2,3)$ maps having 2,3, or 4 regions. The verification that the remaining

(2, 3) relators are not achievable is generally less complicated than Example 3.3.

★1	$aa = aaa$	★14	$ab = bab$	27	$ab = acc$	40	$ab = ccb$
★2	$aa = aab$	★15	$ab = bba$	28	$ab = bac$	★41	$ab = ccc$
★3	$aa = aba$	★16	$ab = bbb$	29	$ab = bbc$	★42	$aa = bcd$
4	$aa = abb$	★17	$aa = abc$	★30	$ab = bca$	43	$ab = acd$
★5	$aa = baa$	18	$aa = bac$	★31	$ab = bcb$	★44	$ab = bcd$
★6	$aa = bab$	19	$aa = bbc$	32	$ab = bcc$	45	$ab = cad$
7	$aa = bba$	★20	$aa = bca$	33	$ab = caa$	46	$ab = cbd$
★8	$aa = bbb$	★21	$aa = bcb$	★34	$ab = cab$	47	$ab = ccd$
★9	$ab = aaa$	22	$aa = bcc$	35	$ab = cac$	★48	$ab = cda$
10	$ab = aab$	23	$ab = aac$	36	$ab = cba$	49	$ab = cdb$
★11	$ab = aba$	★24	$ab = abc$	37	$ab = cbb$	★50	$ab = cdc$
12	$ab = abb$	★25	$ab = aca$	38	$ab = cbc$	51	$ab = cdd$
★13	$ab = baa$	26	$ab = acb$	39	$ab = cca$	★52	$ab = cde$.



The lattice of alphabetically constructed (2, 3) relators

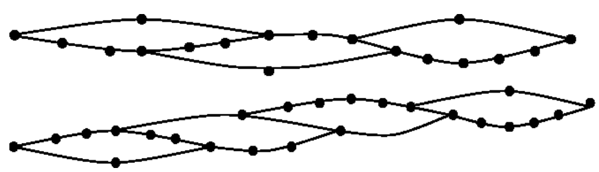
Lemma 3.5. *Suppose that* $\mathcal{M}$ *and* $\mathcal{M}'$ *are* (s, ℓ) *maps and that* $\mathcal{M} \subseteq \mathcal{M}'$. *Then* $\mathbf{r}_{\mathcal{M}} \leq \mathbf{r}_{\mathcal{M}'}$. *Conversely, suppose that* $\mathbf{r}$ *and* $\mathbf{r}'$ *are achievable* (s, ℓ) *relators with* $\mathbf{r} \leq \mathbf{r}'$ *and that* $\mathbf{r} = \mathbf{r}_{\mathcal{M}}$. *Then there is an* (s, ℓ) *map* $\mathcal{M}'$ *with* $\mathbf{r}' = \mathbf{r}_{\mathcal{M}'}$ *and* $\mathcal{M} \subseteq \mathcal{M}'$.

Proof. We use again the definition $\mathcal{R}_{\mathcal{M}} = \{(z_i, z_j) \mid$ There are regions D and D' of $\mathcal{M}$ having a common edge e such that the index of e with respect to D is i and the index of e with respect to D' is $j\}$. Since $\mathcal{M} \subseteq \mathcal{M}'$, we have $\mathcal{R}_{\mathcal{M}} \subseteq \mathcal{R}_{\mathcal{M}'}$, and $\rho_{\mathcal{M}} \subseteq \rho_{\mathcal{M}'}$. Conversely if $\mathbf{r} = \mathbf{r}_{\mathcal{M}}$ and $\mathbf{r}'$ is achievable with $\mathbf{r} \leq \mathbf{r}'$, then there is an (s, ℓ) map $\mathcal{N}$ with $\mathbf{r}' = \mathbf{r}_{\mathcal{N}}$. Let $\mathcal{M}'$ be $\mathcal{M}\mathcal{N}$, the (s, ℓ) map constructed by identifying the initial vertex of $\mathcal{N}$ with the terminal vertex of $\mathcal{M}$. Since $\mathbf{r} \leq \mathbf{r}'$, we will have that $\mathcal{R}_{\mathcal{M}} \subseteq \rho_{\mathcal{M}} = \rho_{\mathbf{r}} \subseteq \rho_{\mathbf{r}'} = \rho_{\mathcal{N}}$. It follows that $\mathcal{R}_{\mathcal{M}} \cup \mathcal{R}_{\mathcal{N}}$ is contained in $\rho_{\mathcal{N}}$, hence $\rho_{\mathcal{M}'} = \rho_{\mathcal{N}}$, and $\mathbf{r}_{\mathcal{M}'} = \mathbf{r}'$.

Example 3.6. Given $\mathbf{r} \leq \mathbf{r}'$ where $\mathbf{r}$ and $\mathbf{r}'$ are achievable (s, ℓ) relators and an (s, ℓ) map $\mathcal{M}'$ with $\mathbf{r}' = \mathbf{r}_{\mathcal{M}'}$, we cannot, in general, find a submap $\mathcal{M}$ of $\mathcal{M}'$ with $\mathbf{r} = \mathbf{r}_{\mathcal{M}}$. With $\mathbf{r} = (ab, cdc^2dc)$ and $\mathbf{r}' = (a^2, bcb^2cb)$, it is easy to see that $\mathbf{r}$ and $\mathbf{r}'$ are achieved by the following maps $\mathcal{M}$ and $\mathcal{M}'$, that $\mathbf{r} \leq \mathbf{r}'$, but that $\mathbf{r}$ is not achieved by any submap of $\mathcal{M}'$.

$\mathcal{M}$

$\mathcal{M}'$



Proposition 3.7. *If* $\mathbf{r}$ *and* $\mathbf{r}'$ *are achievable* (s, ℓ) *relators, then their join,* $\mathbf{r} \vee \mathbf{r}'$ *is also achievable.*

Proof. Suppose that $\mathbf{r}$ is achieved by $\mathcal{M}$ and that $\mathbf{r}'$ is achieved by $\mathcal{M}'$. Then $\rho_{\mathbf{r} \vee \mathbf{r}'}$ is the smallest equivalence relation on Z which contains $\rho_{\mathcal{M}} \cup \rho_{\mathcal{M}'}$, and hence is the smallest equivalence relation on Z which contains $\mathcal{R}_{\mathcal{M}} \cup \mathcal{R}_{\mathcal{M}'}$. Let $\mathcal{M}_{\vee}$ be $\mathcal{M}\mathcal{M}'$, the (s, ℓ) map constructed by identifying the initial vertex of $\mathcal{M}'$ with the terminal vertex of $\mathcal{M}$. Then $\mathcal{R}_{\mathcal{M}_{\vee}}$ is $\mathcal{R}_{\mathcal{M}} \cup \mathcal{R}_{\mathcal{M}'}$, so $\mathbf{r} \vee \mathbf{r}'$ is achieved by $\mathcal{M}_{\vee}$.

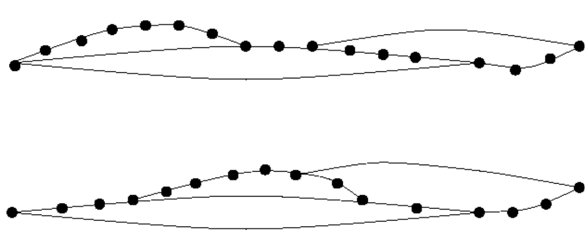
Example 3.8. While the join of two achievable relators is achievable, their meet is not in general an achievable relator. If $\mathbf{r}$ is the $(1, 7)$ relator $(a, abcabca)$ and $\mathbf{r}'$ is the $(1, 7)$ relator $(a, bcbcabc)$, then $\mathbf{r}$ and $\mathbf{r}'$ are achieved by the respective maps $\mathcal{M}$ and $\mathcal{M}'$ below. A straightforward calculation of $\rho_{\mathbf{r}}$, $\rho_{\mathbf{r}'}$, and $\rho_{\mathbf{r}} \cap \rho_{\mathbf{r}'}$ shows that

$$\rho_{\mathbf{r}} \cap \rho_{\mathbf{r}'} = \{\, (z_4, z_7), (z_7, z_4), (z_5, z_8), (z_8, z_5) \} \cup \{\, (z_i, z_i) \,\}$$

so that $\mathbf{r} \wedge \mathbf{r}'$ is $a = bcdefde$ which is easily shown to be not achievable.

$\mathcal{M}$

$\mathcal{M}'$



If $\mathbf{r}$ is an alphabetically constructed (s, ℓ) relator, let $L_{\mathbf{r}}$ be the set of achievable (s, ℓ) relators which are equal to or less than $\mathbf{r}$. We define the **paradigm** of $\mathbf{r}$, $\mathbf{P}(\mathbf{r})$ to be the join of the finite set $L_{\mathbf{r}}$. Since the relators of $L_{\mathbf{r}}$ are achievable, $\mathbf{P}(\mathbf{r})$ is an achievable relator. Since $\mathbf{r}$ itself is clearly an upper bound for $L_{\mathbf{r}}$, we also have that $\mathbf{P}(\mathbf{r}) \le \mathbf{r}$. Every achievable relator is its own paradigm.

For further illustrations of the paradigm for a relator, we return to the $(2,3)$ relators of Example 3.4. There, we claimed that $\mathbf{r}_4 : aa = abb$ is not achievable. Inspecting the lattice, we see that $L_{\mathbf{r}_4} = \{\mathbf{r}_{17}, \mathbf{r}_{42}, \mathbf{r}_{44}, \mathbf{r}_{52}\}$, so that $\mathbf{P}(\mathbf{r}_4)$ is $\mathbf{r}_{17} : aa = abc$. Similarly, for $\mathbf{r}_{43} : ab = acd$, we have $L(\mathbf{r}_{43}) = \{\mathbf{r}_{52}\}$ so that $\mathbf{P}(\mathbf{r}_{43})$ is $\mathbf{r}_{52} : ab = cde$.

Theorem 3.9. *Suppose that* $\mathbf{r}$ *is an alphabetically constructed* (s,ℓ) *relator and that* $\mathbf{P}(\mathbf{r})$ *is the paradigm of* $\mathbf{r}$. *If* $\mathcal{M}$ *is an* (s,ℓ) *map and some bilabelling of* $\mathcal{M}$ *which corresponds to* $\mathbf{r}$ *is consistent, than any bilabelling of* $\mathcal{M}$ *which corresponds to* $\mathbf{P}(\mathbf{r})$ *is consistent.*

Proof. Let $\mathbf{r}_{\mathcal{M}}$ be the (s,ℓ) relator which is achieved by $\mathcal{M}$. Since a bilabelling of $\mathcal{M}$ which corresponds to $\mathbf{r}$ is consistent, $\mathbf{r}_{\mathcal{M}} \le \mathbf{r}$, by Proposition 3.2. Therefore, the achievable relator $\mathbf{r}_{\mathcal{M}}$ is in the set $L_{\mathbf{r}}$ of achievable relators which are equal to or less than $\mathbf{r}$. It follows that $\mathbf{r}_{\mathcal{M}} \le \mathbf{P}(\mathbf{r})$, since $\mathbf{P}(\mathbf{r})$ is the join of $L_{\mathbf{r}}$. Using Proposition 3.2 again, bilabellings of $\mathcal{M}$ which correspond to $\mathbf{P}(\mathbf{r})$ are consistent.

Theorem 3.10. *Suppose that* $\mathbf{r}$ *and* $\mathbf{r}'$ *are alphabetically constructed* (s,ℓ) *relators with* $\mathbf{r}' \le \mathbf{r}$. *Suppose that* $\mathcal{M}$ *is an* (s,ℓ) *map having no unconstrained edges and that* ϕ *and* ϕ' *are labels on* $\mathcal{M}$ *such that* $(\mathcal{M},\phi)$ *is a derivation diagram over* $\langle X;\mathbf{r}\rangle$ *for the pair* (u,v) *of words on* X *and* $(\mathcal{M},\phi')$ *is a derivation diagram over* $\langle X;\mathbf{r}'\rangle$ *for the pair* (u',v'). *If* $(\mathcal{M},\phi)$ *is a minimal derivation diagram, then* $(\mathcal{M},\phi')$ *is a minimal derivation diagram also.*

Proof. We use proof by contradiction. Suppose that $(\mathcal{M},\phi')$ is not minimal. Then there is a derivation diagram $(\mathcal{N},\zeta')$ over $\langle X;\mathbf{r}'\rangle$ for (u',v') with $|\mathcal{N}| < |\mathcal{M}|$. It will suffice to show that we can define a label ζ on $\mathcal{N}$ such that $(\mathcal{N},\zeta)$ is a derivation diagram over $\langle X;\mathbf{r}\rangle$ for (u,v), since this will contradict the minimality of $(\mathcal{M},\phi)$.

Since $(\mathcal{N},\zeta')$ is a diagram over $\langle X;\mathbf{r}'\rangle$, ζ' is induced by a consistent bilabelling which corresponds to $\mathbf{r}'$. By Proposition 3.5, $\mathbf{r}' \ge \mathbf{r}_{\mathcal{N}}$. Since $\mathbf{r} \ge \mathbf{r}'$, every bilabelling of $\mathcal{N}$ which corresponds to $\mathbf{r}$ will be consistent also. Suppose first that $\mathcal{N}$ has no unconstrained edges. Then there is only one bilabelling of $\mathcal{N}$ which corresponds to $\mathbf{r}$ and it induces a label ζ on $\mathcal{N}$. We need to show that $(\mathcal{N},\zeta)$ is a derivation diagram for (u,v). By symmetry, it will suffice to show that $\zeta(\alpha_{\mathcal{N}}) = u$. Because $\phi'(\alpha_{\mathcal{M}}) = u' = \zeta'(\alpha_{\mathcal{N}})$, $\alpha_{\mathcal{M}}$ and $\alpha_{\mathcal{N}}$ are paths of the same length, and we only need to show that $\zeta(f) = \phi(e)$ if f is the k^{th} edge on $\alpha_{\mathcal{N}}$ and e is the k^{th} edge on $\alpha_{\mathcal{M}}$. Suppose that the edge e has index i with respect to some region D_1 of $\mathcal{M}$ and that f has index j with respect to some region D_2 of $\mathcal{N}$. Because $\phi'(\alpha_{\mathcal{M}}) = u' = \zeta'(\alpha_{\mathcal{N}})$, we have that $\phi'(e)$ and $\zeta'(f)$ are the same letter. Both $(\mathcal{M},\phi')$ and $(\mathcal{N},\zeta')$ are derivation diagrams over $\langle X;\mathbf{r}'\rangle$, hence $(z_i,z_j) \in \rho_{\mathbf{r}'}$. Since $\mathbf{r}' \le \mathbf{r}$, $\rho_{\mathbf{r}'} \subseteq \rho_{\mathbf{r}}$, and $(z_i,z_j) \in \rho_{\mathbf{r}}$. This insures that the labels of $\phi(e)$ and $\zeta(f)$ are the same letter of X.

When $\mathcal{N}$ has unconstrained edges, we still induce ζ from a consistent bilabelling which corresponds to $\mathbf{r}$, but we must specify how ζ is defined on the unconstrained edges of $\mathcal{N}$. If f is an unconstrained edge of $\mathcal{N}$, then f occurs on both the bottom side, $\alpha_{\mathcal{N}}$, and the top side, $\omega_{\mathcal{N}}$, of $\mathcal{N}$. Suppose f is the j^{th} edge which occurs on $\alpha_{\mathcal{N}}$, and is the k^{th} edge which occurs on $\omega_{\mathcal{N}}$. Let e_b be the j^{th} edge on the bottom side $\alpha_{\mathcal{M}}$ of $\mathcal{M}$, and let e_t be the k^{th} edge on the top side $\omega_{\mathcal{M}}$ of $\mathcal{M}$. Then $\phi'(e_b) = \zeta'(f)$ since $\phi'(\alpha_{\mathcal{M}}) = u' = \zeta'(\alpha_{\mathcal{N}})$, and similarly, $\phi'(e_t) = \zeta'(f)$ since $\phi'(\omega_{\mathcal{M}}) = v' = \zeta'(\omega_{\mathcal{N}})$. Using $\phi'(e_b) = \phi'(e_t)$ and $\mathbf{r}' \leq \mathbf{r}$, we argue as above that $\phi(e_b) = \phi(e_t)$. We may then define $\zeta(f) = \phi(e_b) = \phi(e_t)$, and it will follow that $\zeta(\alpha_{\mathcal{N}}) = u$ and $\zeta(\omega_{\mathcal{N}}) = v$.

Corollary 3.11. *Let $\mathbf{r}$ and $\mathbf{r}'$ be alphabetically constructed (s, ℓ) relators with $\mathbf{r}' \leq \mathbf{r}$. Suppose that $(\mathcal{M}, \phi)$ is a derivation diagram over $\langle X; \mathbf{r}\rangle$ for the pair (u, v) of words on X and $(\mathcal{M}, \phi')$ is a derivation diagram over $\langle X; \mathbf{r}'\rangle$ for the pair (u', v'). If $(\mathcal{M}, \phi)$ is a quasiminimal derivation diagram, then $(\mathcal{M}, \phi')$ is a quasiminimal derivation diagram also.*

Proof. We may write $\mathcal{M} = \mathcal{M}_1\mathcal{M}_2 \ldots \mathcal{M}_k$, where each $\mathcal{M}_i$ is a block of $\mathcal{M}$. Each $\mathcal{M}_i$ is either a single unconstrained edge of $\mathcal{M}$, or else $\mathcal{M}_i$ contains no unconstrained edges and $(\mathcal{M}_i, \phi)$ is a minimal derivation diagram over $\langle X; \mathbf{r}\rangle$. When $\mathcal{M}_i$ is a single edge, clearly $(\mathcal{M}_i, \phi')$ is a minimal derivation diagram over $\langle X; \mathbf{r}'\rangle$. By Theorem 3.10, $(\mathcal{M}_i, \phi')$ is also minimal over $\langle X; \mathbf{r}'\rangle$ when $(\mathcal{M}_i, \phi)$ is minimal over $\langle X; \mathbf{r}\rangle$ and $\mathcal{M}_i$ contains no unconstrained edges.

Theorem 3.12. *Let $\mathbf{r}$ be an alphabetically constructed (s, ℓ) relator on an ordered alphabet X. If we know the paradigm $\mathbf{P}(\mathbf{r})$ for $\mathbf{r}$ and we have an algorithm which solves the word problem for the semigroup presentation $\langle X\ ;\ \mathbf{P}(\mathbf{r})\rangle$ then the word problem for the semigroup presentation $\langle X\ ; \mathbf{r}\rangle$ is also solvable.*

Proof. Suppose that u and v are words on the alphabet X. Write S for the semigroup presented by $\langle X\ ; \mathbf{r}\rangle$ and write P for the semigroup presented by $\langle X\ ;\ \mathbf{P}(\mathbf{r})\rangle$. By Theorem 1.7, it will suffice to exhibit an algorithm which determines whether or not there is a quasiminimal derivation diagram over $\langle X\ ; \mathbf{r}\rangle$ for (u, v).

We may assume here that the alphabet X is finite. We list all pairs of words (u', v') on X such that $|u'| = |u|$ and $|v'| = |v|$. The number of such pairs is finite. For each such pair, we may use the algorithm which solves the word problem for $\langle X\ ;\ \mathbf{P}(\mathbf{r})\rangle$ to determine whether or not u' and v' represent the same element of P. If u' and v' do represent the same element of P, we may use the algorithm for $\langle X\ ;\ \mathbf{P}(\mathbf{r})\rangle$ to find all possible quasiminimal derivation diagrams $(\mathcal{M}, \phi')$ over $\langle X\ ;\ \mathbf{P}(\mathbf{r})\rangle$ for (u', v'). For any such $(\mathcal{M}, \phi')$, we have $\mathbf{r}_{\mathcal{M}} \leq \mathbf{P}(\mathbf{r}) \leq \mathbf{r}$, so by Proposition 3.2, every bilabelling of $\mathcal{M}$ which corresponds to $\mathbf{r}$ will be consistent. For each $(\mathcal{M}, \phi')$ we obtain derivation diagrams over $\langle X\ ; \mathbf{r}\rangle$ using the bilabelling of $\mathcal{M}$ which corresponds to $\mathbf{r}$ and using all possible labels on any unconstrained edges of $\mathcal{M}$.

We claim that there is a quasiminimal derivation diagram over $\langle X\ ; \mathbf{r}\rangle$ for (u, v) if and only if some diagram in the list of the preceding paragraph is a derivation diagram for (u, v). Clearly if one of the preceding diagrams is a derivation diagram for (u, v) then there are quasiminimal derivation diagrams for (u, v) and we may effectively find one. Conversely, suppose that there is a quasiminimal derivation diagram $\mathcal{M}$ over $\langle X\ ; \mathbf{r}\rangle$ for (u, v). Then by Theorem 3.9, the bilabelling of $\mathcal{M}$ which corresponds to $\mathbf{P}(\mathbf{r})$ is consistent and we will obtain a derivation diagram $(\mathcal{M}, \phi')$ over $\langle X\ ;\ \mathbf{P}(\mathbf{r})\rangle$ for some pair of words (u', v') with $|u'| = |u|$ and $|v'| = |v|$. By Corollary 3.11, this is a quasiminimal derivation diagram, hence the quasiminimal derivation diagram for (u, v) will occur in the list of the previous paragraph.

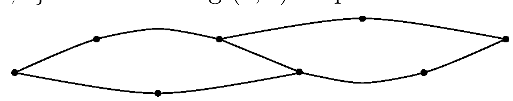
We wish to improve upon Theorem 3.12 by showing that we always can effectively compute the paradigm $\mathbf{P}(\mathbf{r})$ of an alphabetically constructed (s,ℓ) relator $\mathbf{r}$.

4. Local Computations: The Base Case

If $\mathbf{r}$ is an achievable (s,ℓ) relator, let $\rho = \rho_{\mathbf{r}}$ be the corresponding equivalence relation on the auxiliary alphabet Z. Let x represent some fixed ρ-equivalence class. Then the (s,ℓ) map $\mathcal{M}$ is an x-section for $\mathbf{r}$ if

$$(\ddagger) \qquad \rho_{|x} \subseteq \rho_{\mathcal{M}} \subseteq \rho$$

Example 4.1. The relator $\mathbf{r} : a^2 = bcb$ is an achievable $(2,3)$ relator where $b = [z_3]_\rho = [z_5]_\rho$ and $J(b,\mathbf{r}) = \{3,5\}$. The following $(2,3)$ map is a b-section for $\mathbf{r}$.



Remarks. (1) The inclusion $\rho_{|x} \subseteq \rho_{\mathcal{M}}$ in $(\ddagger)$ requires that $[z_i]_{\rho_{\mathcal{M}}} = [z_j]_{\rho_{\mathcal{M}}}$ whenever $x = [z_i]_\rho = [z_j]_\rho$. In this sense, an x-section for $\mathbf{r}$ "achieves the x-part of $\mathbf{r}$. " Note however, that if $[z_i]_\rho = x$, the equivalence class $[z_i]_{\rho_{\mathcal{M}}}$ will, in general, be represented by some element of the ordered alphabet X which follows x, rather than by x itself. In the example above, $\mathbf{r}_{\mathcal{M}}$ is $ab = cdc$ and $[z_3]_{\rho_{\mathcal{M}}} = [z_5]_{\rho_{\mathcal{M}}} = c$.

(2) The inclusion $\rho_{\mathcal{M}} \subseteq \rho$ is equivalent to $\mathbf{r}_{\mathcal{M}} \leq \mathbf{r}$. By Proposition 3.2, this is equivalent to requiring that $\mathcal{M}$, with the natural labelling induced by $\mathbf{r}$, is always a derivation diagram over the presentation $\langle X; \mathbf{r}\rangle$. Thus an x-section for $\mathbf{r}$ is a derivation diagram over $\langle X; \mathbf{r}\rangle$ which achieves the x-part of $\mathbf{r}$.

An x-section $\mathcal{M}$ for $\mathbf{r}$ is x-**inceptive** for $\mathbf{r}$ if $|\mathcal{M}| \leq |\mathcal{N}|$ whenever $\mathcal{N}$ is also an x-section for $\mathbf{r}$.

If an x-inceptive map $\mathcal{M}$, with $\mathcal{M} > 0$, has unconstrained edges, then we can concatenate those blocks of $\mathcal{M}$ which do contain interior edges to construct a map which is x-inceptive for $\mathbf{r}$ and which has no unconstrained edges. In the case where $|J(x,\mathbf{r})| = 1$, all x-inceptive maps for $\mathbf{r}$ are regionless, and any walk $\mathcal{W}_k \in \mathbf{F}_0$ is x-inceptive for $\mathbf{r}$.

In this section we show (Theorem 4.20) that if $|J(x,\mathbf{r})| = 2$ and $\mathcal{M}$ is an x-inceptive map for $\mathbf{r}$, then $|\mathcal{M}| \leq \ell$. This is the base case for an induction on $|J(x,\mathbf{r})|$. In the next section, we show that if $|J(x,\mathbf{r})| = j$ and $\mathcal{M}$ is an x-inceptive map for $\mathbf{r}$, then $|\mathcal{M}| \leq (j-1)\ell$.

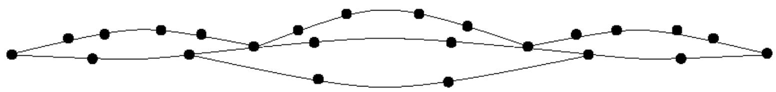
Lemma 4.2. *Suppose that* $\mathbf{r}$ *is an achievable* (s,ℓ) *relator and let* $X_2 = \{x \in X \ : \ |J(x,\mathbf{r}) \geq 2\}$. *For each* $x \in X_2$, *let* $\mathcal{M}_x$ *be an* x*-section for* $\mathbf{r}$. *Then the concatenation* $\vee_{x\in X_2}\mathcal{M}_x$ *is an* (s,ℓ) *map which achieves* $\mathbf{r}$.

Proof. It will suffice to show that $\rho = \cup_{x\in X_2}\rho_{\mathcal{M}_x}$. Since each $\mathcal{M}_x$ is an x-section for $\mathbf{r}$, we have $\rho_{\mathcal{M}_x} \subseteq \rho$ for each $x \in X_2$, and $\cup_{x\in X_2}\rho_{\mathcal{M}_x} \subseteq \rho$. Earlier (at the end of section 2), it was remarked that $\rho = \cup_{x\in X}\rho_{|x}$. Since $\rho_{|x}$ is the identity relation on Z when $|J(x,\mathbf{r})| = 1$, we can strengthen this to $\rho = \cup_{x\in X_2}\rho_{|x}$. Then $\rho = \cup_{x\in X_2}\rho_{|x} \subseteq \cup_{x\in X_2}\rho_{\mathcal{M}_x}$.

Lemma 4.3. *If* $\mathbf{r}$ *is an achievable* (s,ℓ) *relator,* x *occurs in* $\mathbf{r}$*, and* $\mathcal{M}$ *is an* x*-inceptive* (s,ℓ) *map for* $\mathbf{r}$*, then* $\mathcal{M}$ *is an inceptive* (s,ℓ) *map for* $\mathbf{r}_{\mathcal{M}}$*.*

Proof. Let $\mathcal{N}$ be any (s,ℓ) map that achieves $\mathbf{r}_{\mathcal{M}}$. We need to show that $|\mathcal{M}| < |\mathcal{N}|$. Since $\mathbf{r}_{\mathcal{N}} = \mathbf{r}_{\mathcal{M}}$, we have $\rho_{\mathcal{N}} = \rho_{\mathcal{M}}$ and $\rho_{|x} \subseteq \rho_{\mathcal{N}} \subseteq \rho$ because $\rho_{|x} \subseteq \rho_{\mathcal{M}} \subseteq \rho$. Thus $\mathcal{N}$ is an x-section for $\mathbf{r}$ and the conclusion follows.

Example 4.4. By Lemma 4.3, every (s,ℓ) map $\mathcal{M}$ which is x-inceptive (for some x and some $\mathbf{r}$) is inceptive for the relator which it achieves. Being x-inceptive is more restrictive than being inceptive. The following map is an inceptive $(3,5)$ map for the relator $abc = cabca$, but it is not x-inceptive for $x = a, b$, or c.

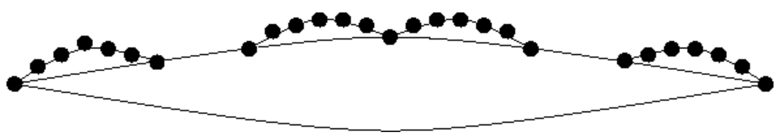


Lemma 4.5. *Suppose that* $\mathbf{r}$ *is an achievable* (s,ℓ) *relator, that* x *occurs in* $\mathbf{r}$ *and that* $\mathcal{M}$ *is* x*-inceptive for* $\mathbf{r}$*. Let* $i \in J(x,\mathbf{r})$ *and* $\bar{x} = [z_i]_{\rho_{\mathcal{M}}}$*. Then* $\mathcal{M}$ *is also* $\bar{x}$*-inceptive for* $\mathbf{r}_{\mathcal{M}}$*.*

Proof. Let $\mathcal{N}$ be any $\bar{x}$-section for $\mathbf{r}_{\mathcal{M}}$. It will suffice to show that $\mathcal{N}$ is also an x-section for $\mathbf{r}$. Since $\mathcal{M}$ is an x-section for $\mathbf{r}$, we have that $\rho_{\mathcal{M}} \subseteq \rho$. A routine argument shows that $\rho_{\mathbf{r}|x} = \rho_{\mathbf{r}_{\mathcal{M}}|\bar{x}}$. Since $\mathcal{N}$ is an $\bar{x}$-section for $\mathbf{r}_{\mathcal{M}}$, we have

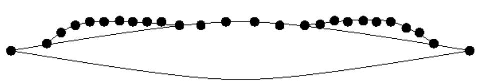
$$\rho_{\mathbf{r}|x} = \rho_{\mathbf{r}_{\mathcal{M}}|\bar{x}} \subseteq \rho_{\mathcal{N}} \subseteq \rho_{\mathcal{M}} \subseteq \rho.$$

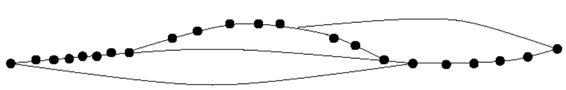
Example 4.6. The following map is an a-section for the achievable $(1,6)$-relator $a = (aba)^2$ and it is a-inceptive for $\mathbf{r}_{\mathcal{M}}: \ a = abaaca$, but it is not inceptive or a-inceptive for $a = (aba)^2$ which can be achieved by $(1,6)$ maps with only four regions.



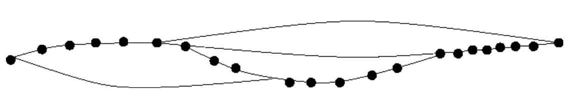
Example 4.7. The following three $(1,9)$ maps $\mathcal{M}_1, \mathcal{M}_2$, and $\mathcal{M}_3$ are all a-inceptive for the $(1,9)$ relator $a = baccdbbac$. Since $\mathbf{r}_{\mathcal{M}_1}$ is $a = bacdefgah$, $\mathbf{r}_{\mathcal{M}_2}$ is $a = baccdebac$, and $\mathbf{r}_{\mathcal{M}_3}$ is $a = bacdebbac$, we see that different x-inceptive maps may achieve the x-part of $\mathbf{r}$ in distinctly different ways.

$\mathcal{M}_1$





$\mathcal{M}_2$



$\mathcal{M}_3$

Let $\mathcal{M}$ be an (s,ℓ) map and e an interior edge of $\mathcal{M}$. The edge e is an $\{i,j\}$**-edge** if e has index i with respect to D_1 and has index j with respect to D_2, where e is on the common boundary of regions D_1 and D_2. We allow the possibility that $i = j$. If e is an $\{i,i\}$-edge for some i, we will say that e is **reflective.** (In a slightly different context, Remmers [10], see [11, p.293] or [2], defines such an edge to be an interface.) If $\mathbf{r}$ is an achievable (s,ℓ) relator and $i \in J(x,\mathbf{r})$, we will say that an $\{i,i\}$-edge e is x-**reflective.**

Lemma 4.8. *Suppose that* $\mathbf{r}$ *is an achievable* (s,ℓ) *relator, that the letter* x *occurs in* $\mathbf{r}$, *that* $J(x,\mathbf{r}) = \{i_1,i_2\}$, *where* $i_1 \neq i_2$, *and that* $\mathcal{M}$ *is an* x*-inceptive* (s,ℓ) *map for* $\mathbf{r}$. *Then*

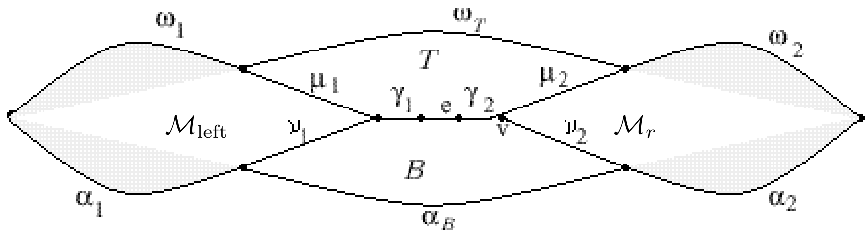
1. $\mathcal{M}$ *has exactly two appended regions and every* $\{i_1,i_2\}$*-edge of* $\mathcal{M}$ *is on their common boundary, and*
2. *there is exactly one interior edge of* $\mathcal{M}$ *which is an* $\{i_1,i_2\}$*-edge.*

Proof. By the definition of an x-section, $\mathcal{M}$ must have at least one edge which is an $\{i_1,i_2\}$-edge. Since this edge is an interior edge, we must have $|\mathcal{M}| \geq 2$, and hence by Lemma 1.5, $\mathcal{M}$ has at least two appended regions. Suppose that T is any region which is appended on the top side of $\mathcal{M}$. If $(z_{i_1}, z_{i_2}) \in \rho_{\mathcal{M}-T}$, then $\mathcal{M}-T$ is also an x-section for $\mathbf{r}$: this would contradict our hypothesis that $\mathcal{M}$ is x-inceptive for $\mathbf{r}$. Thus every $\{i_1,i_2\}$-edge of $\mathcal{M}$ must occur on the bottom side of every region which is appended on the top side of $\mathcal{M}$. This is impossible if there is more than one region that is appended on the top side of $\mathcal{M}$. Similarly, there is exactly one appended region on the bottom of $\mathcal{M}$ and every $\{i_1,i_2\}$-edge of $\mathcal{M}$ occurs on the top side of this region. Thus the first claim of the lemma is verified.

Suppose then that e and f are distinct $\{i_1,i_2\}$-edges in $\mathcal{M}$. We show that $i_1 < i_2$ and $i_2 < i_1$, a contradiction. Let T be appended on the top side of $\mathcal{M}$ and let B be appended on the bottom side of $\mathcal{M}$, so that both e and f occur on both α_T and ω_B. Assume, without loss of generality, that e precedes f on α_T and hence on ω_B as well. Choose notation for the indices so that e has index i_1 with respect to B. Then f, by default, must have index i_2 with respect to B. Since e precedes f on ω_B, we have $i_1 < i_2$. Since e and f are both $\{i_1,i_2\}$-edges, e has index i_2 with respect to T and f has index i_1 with respect to T. Since e precedes f on α_T, we have $i_2 < i_1$.

Several of the next lemmas will consider the size and structure of an x-inceptive (s,ℓ) map, $\mathcal{M}$, for $\mathbf{r}$ when $|J(x,\mathbf{r})| = 2$. We find bounds for $|\mathcal{M}|$ depending upon the values of i_1 and i_2. It will be efficient to use the conclusion of Lemma 4.8 and to extend the notation in its proof as a common starting point for all of these lemmas. Let $\mathbf{r}$ be an achievable (s,ℓ) relator and x a letter that occurs exactly twice in $\mathbf{r}$. Let $J(x,\mathbf{r})| = \{\, i_1, i_2 \,\}$, where $i_1 < i_2$. Let $\mathcal{M}$ be an x-inceptive (s,ℓ) map for $\mathbf{r}$. By Lemma 4.8, let e be the unique $\{i_1,i_2\}$-edge in $\mathcal{M}$ and let B and T be the regions of $\mathcal{M}$ which are appended on the bottom and top sides, respectively. Write $\alpha_T = \mu_1\gamma_1 e\gamma_2\mu_2$ and $\omega_B = \nu_1\gamma_1 e\gamma_2\nu_2$, where $\gamma_1 e\gamma_2$ is the largest common walk on the boundary of B and T containing e. Let v be the terminal vertex of $\gamma_1 e\gamma_2$. Express $\alpha_{\mathcal{M}}$ as $\alpha_1\alpha_B\alpha_2$ and express $\omega_{\mathcal{M}}$ as $\omega_1\omega_T\omega_2$. We allow the possibility that, for $i = 1, 2$, any of $\alpha_i, \omega_i, \mu_i, \nu_i$, or γ_i might be an empty walk. Let $\mathcal{M}_r$ (for $\mathcal{M}_{\text{right}}$) be the feathery submap of $\mathcal{M}$ which is bounded by $\nu_2\alpha_2(\mu_2\omega_2)^{-1}$. Note that v

is the initial vertex of $\mathcal{M}_r$. Let $\mathcal{M}_{\text{left}}$ be the feathery submap of $\mathcal{M}$ bounded by $\alpha_1\nu_1(\omega_1\mu_1)^{-1}$. All of our arguments will treat $\mathcal{M}_r$. The corresponding arguments for $\mathcal{M}_{\text{left}}$ are dual.



Observe that if μ_2 is empty, then $\mathcal{M}_r$ can have no regions. Otherwise, some region of $\mathcal{M}_r$ that is appended on the top side of $\mathcal{M}_r$ would also be appended on the top side of $\mathcal{M}$, contradicting Lemma 4.8. Similarly, if ν_2 is empty, then $|\mathcal{M}_r| = 0$, and if either μ_1 or ν_1 is empty, then $|\mathcal{M}_{\text{left}}| = 0$.

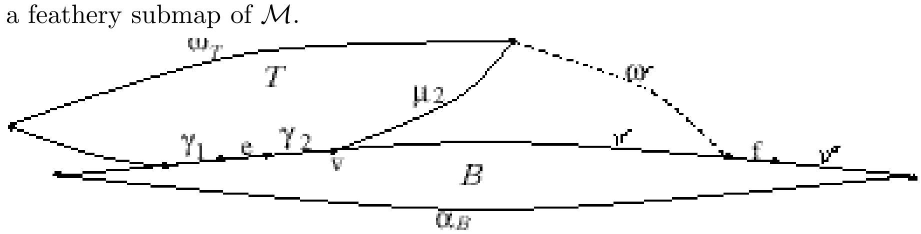
Lemma 4.9. *Suppose that* $\mathbf{r}$ *is an achievable* (s, ℓ) *relator, that the letter* x *occurs in* $\mathbf{r}$, *that* $J(x, \mathbf{r}) = \{ i_1, i_2 \}$, *where* $i_1 < i_2$, *and that* $\mathcal{M}$ *is an* x*-inceptive* (s, ℓ) *map for* $\mathbf{r}$.

(1) *If* $i_2 \leq s$, *then* $|\mathcal{M}| = 2$.
(2) *If* $i_1 > s$, *and* $i_2 - i_1 \leq s$, *then* $|\mathcal{M}| = 2$.

Proof. Assume without loss of generality that e has index i_1 with respect to B and has index i_2 with respect to T. We show that μ_2 is empty and hence $|\mathcal{M}_r| = 0$. A dual argument shows that ν_1 is empty also. If e has index i_2 with respect to B, then the same line of reasoning shows that both μ_1 and ν_2 are empty.

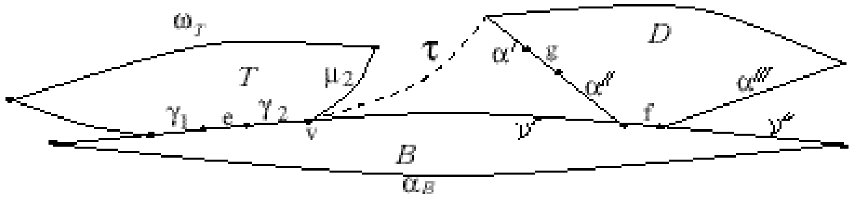
Let f be the edge on the top side of B which has index i_2 with respect to B. Since e has index i_2 with respect to T (and $|J(x, \mathbf{r})| = 2$), f cannot be an edge of γ_2. Write $\nu_2 = \nu' f \nu''$. In case (1), both e and f are on the short side of B, and ν' is a segment of ω_B containing neither e nor f, hence $|\nu'| \leq s - 2$. In case (2), we have $|\nu'| \leq |\gamma_2\nu'| \leq i_2 - i_1 - 1 < s$.

Suppose first that the edge f is on the top side of $\mathcal{M}$. Let ω' be the segment of $\omega_{\mathcal{M}}$ from the terminal vertex of T to the initial vertex of f. Then both ν' and $\mu_2\omega'$ are directed walks from v to the initial vertex of f. Hence $\nu'(\mu_2\omega')^{-1}$ bounds a feathery submap of $\mathcal{M}$.



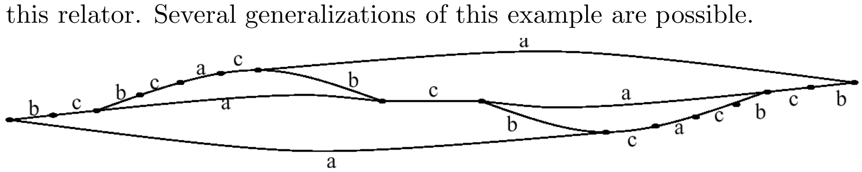
Since $|\nu'| < s$, this submap cannot contain any regions, hence $\nu' = \mu_2\omega'$. By our choice of γ_2 ($\gamma_1 e \gamma_2$ is a maximal walk common to ω_B and α_T containing e), any initial edges of μ_2 and $\nu_2 = \nu' f \nu''$ must be distinct. We conclude that μ_2 is empty.

Now suppose that f is an interior edge of $\mathcal{M}$. We show that this must lead to a contradiction. If f is an interior edge of $\mathcal{M}$, then f occurs on the bottom side of some region D of $\mathcal{M}$. Because e is the unique $\{i_1, i_2\}$-edge of $\mathcal{M}$, f must be an $\{i_2, i_2\}$-edge. Write $\alpha_D = \alpha' g \alpha'' f \alpha'''$ where the edge g has index i_1 with respect to D. Since e and f occur on ω_B and $|J(x, \mathbf{r})| = 2$, g cannot be an edge of ν' which is a segment of ω_B.



By Lemma 1.4, we can find a directed walk, τ in the map $\mathcal{M}_r$ from the initial vertex v of $\mathcal{M}_r$ to the initial vertex of D. Then $\nu'(\tau\alpha' g \alpha'')^{-1}$ bounds a feathery submap of $\mathcal{M}_r$. Since $|\nu'| < s$, this submap is regionless and g is an edge of ν'. This contradiction shows that f cannot be an interior edge of $\mathcal{M}$.

Example 4.10. Case (2) in the preceding lemma was included because it required little more than the argument for case (1). We will consider below the general case of an x-inceptive map $\mathcal{M}$ for $\mathbf{r}$ when $J(x, \mathbf{r}) = \{i_1, i_2\}$ and $s < i_1 < i_2$. For now, we give a fairly simple example which shows that $\mathcal{M}$ can have more than two regions if we omit the hypothesis that $i_2 - i_1 < s$. The $(1, 5)$ relator $a = bcacb$ is an achievable relator on the alphabet $\{a, b, c\}$. The map below is c-inceptive and is inceptive for this relator. Several generalizations of this example are possible.

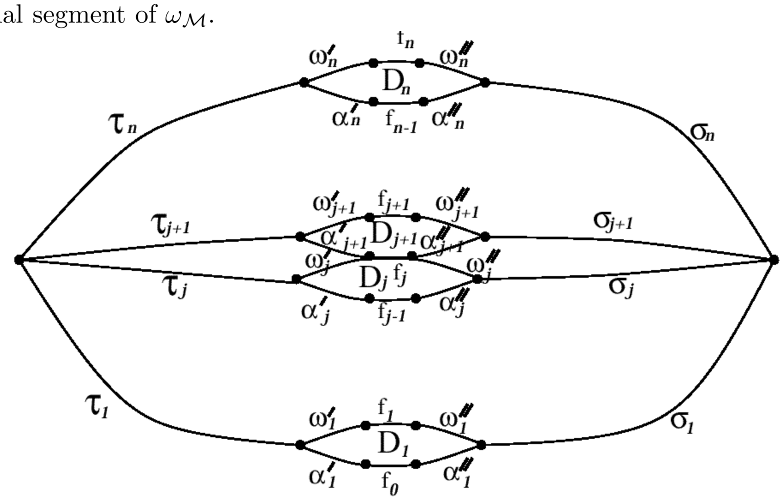


Lemma 4.11. *Suppose that* $\mathbf{r}$ *is an achievable* (s, ℓ) *relator, that the letter* x *occurs in* $\mathbf{r}$*, that* $J(x, \mathbf{r}) = \{\, i_1, i_2 \,\}$*, where* $i_1 \le s < i_2$*, and that* $\mathcal{M}$ *is an* x*-inceptive* (s, ℓ) *map for* $\mathbf{r}$*. Then no edge of* $\mathcal{M}$ *is* x*-reflective.*

Proof. We may assume without loss of generality that e has index i_1 with respect to B and has index i_2 with respect to T, so that e is on the short side of B (since $i_1 \le s$) and is on the long side of T (since $s < i_2$). Here, an x-reflective edge can be only an $\{i_1, i_1\}$-edge or an $\{i_2, i_2\}$-edge. No edge on B can be x-reflective: e is an $\{i_1, i_2\}$-edge and the edge which has index i_2 with respect to B is on the bottom side of $\mathcal{M}$ and is not an interior edge. Similarly, no edge on T can be x-reflective. Since interior edges of $\mathcal{M}$ are either interior edges of $\mathcal{M}_r$ or interior edges of $\mathcal{M}_{\text{left}}$, or else occur on α_T or ω_B, by symmetry, it will suffice to show that no interior edge of $\mathcal{M}_r$ is x-reflective. (In the next lemma, using the current hypotheses, we will show that $\mathcal{M}_r$ has no interior edges.)

We want to regard the edges of $\mathcal{M}$ as partially ordered using the order from section 1 where $g < f$ if there is a directed walk in $\mathcal{M}$ from the terminal vertex of g to the initial vertex of f. Say that an x-reflective edge, f, of $\mathcal{M}_r$ is left-primitive if there is no x-reflective edge g of $\mathcal{M}_r$ which precedes f in $\mathcal{M}_r$. If $\mathcal{M}_r$ contains any x-reflective edge f, then we can always obtain a left-primitive edge by repeatedly replacing f by such a predecessor g until we obtain an x-reflective edge which is left-primitive. We will assume that there is at least one x-reflective edge in $\mathcal{M}_r$ and we will contradict our hypothesis that $\mathcal{M}_r$ is x-inceptive.

For a (momentarily indeterminant) natural number i, let f_i be a left-primitive edge in $\mathcal{M}_r$ which occurs on the top side of a region D_{i-1} of $\mathcal{M}_r$ and on the bottom side of a region D_i of $\mathcal{M}_r$. Inductively, for $j < i$, let f_j be the edge on the bottom side of D_{j+1} which has index in $J(x, \mathbf{r})$. At some point, the edge f_j must occur on the bottom side of $\mathcal{M}$: when this occurs, we determine i so that $j = 0$ and f_0 occurs on the bottom side of $\mathcal{M}$. When f_j is not on the bottom side of $\mathcal{M}$, f_j is on the top side of some region D_j of $\mathcal{M}_r$ and we choose f_{j-1} to be the edge on the bottom side of D_j which has index in $J(x, \mathbf{r})$. Similarly, for $j \geq i$, let f_{j+1} be the edge on the top side of D_j which has index in $J(x, \mathbf{r})$. If f_{j+1} is not on the top side of $\mathcal{M}_r$ (and of $\mathcal{M}$), then f_{j+1} is on the bottom side of some region D_{j+1} of $\mathcal{M}$. Eventually, we must arrive at an edge f_n which is on the top side of $\mathcal{M}_r$. We have edges f_j for $0 \leq j \leq n$, and for some i with $0 < i < n$, the edge f_i is left-primitive. Write the bottom side, α_{D_j}, of D_j as $\alpha'_j f_{j-1} \alpha''_j$, and write the top side, ω_{D_j}, of D_j as $\omega'_j f_j \omega''_j$. For $1 \leq j \leq n$, let τ_j be a directed walk in $\mathcal{M}_r$ from the initial vertex v of $\mathcal{M}_r$ to the initial vertex of D_j. Observe that we can do this is such a way that $(\tau_j \omega'_j)(\tau_{j+1}\alpha'_{j+1})^{-1}$ is a counterclockwise walk in $\mathcal{M}_r$. (That is, if necessary, we can exchange segments of τ_j with segments of τ_{j+1} so that $(\tau_j \omega'_j)(\tau_{j+1}\alpha'_{j+1})^{-1}$ is a counterclockwise walk in $\mathcal{M}_r$, and we can start this with τ_0 and τ_1 and work our way up.) Similarly, for $1 \leq j \leq n$, let σ_j be a directed walk in $\mathcal{M}_r$ from the terminal vertex of D_j to the terminal vertex of $\mathcal{M}_r$, chosen so that $(\omega''_j \sigma_j)(\alpha''_{j+1}\sigma''_{j+1})^{-1}$ is a counterclockwise walk. Let $\hat{\sigma}_1$ denote the terminal segment of $\alpha_{\mathcal{M}}$ whose initial vertex is the terminal vertex of f_0. then $\hat{\sigma}_1 = \alpha''_1 \sigma_1$: otherwise $\hat{\sigma}_1(\alpha''_1\sigma_1)^{-1}$ would contain a region of $\mathcal{M}$ appended on the bottom side of $\mathcal{M}$. Similarly, σ_n must be a terminal segment of $\omega_{\mathcal{M}}$.



We show concurrently, by induction on j for $i \leq j < n$, that $\alpha'_{j+1} = \omega'_j$ and that f_j is left-primitive. A similar induction, from i backwards to $j = 0$, shows that $\alpha'_{j+1} = \omega'_j$ and that f_j is left-primitive for $0 < j \leq i$.

For the base step of the induction, we are given that f_i is left-primitive. We show that:

(a) given that f_j is left-primitive, we have $\alpha'_{j+1} = \omega'_j$ and $\tau_j = \tau_{j+1}$, and

(b) given that f_j is left-primitive and $\alpha'_{j+1} = \omega'_j$, then f_{j+1} is left-primitive.

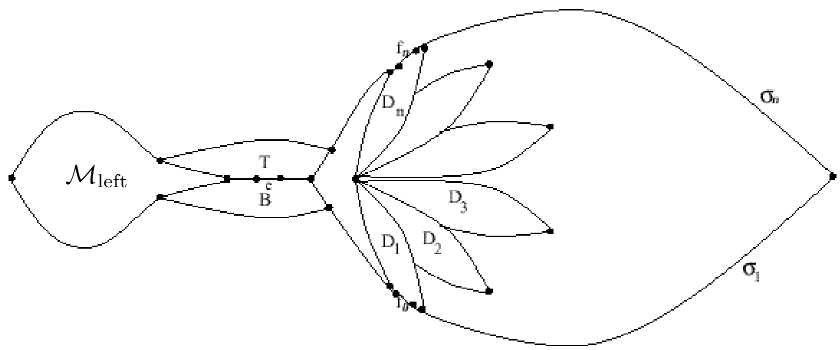
For (a), let $\mathcal{N}_j$ be the feathery submap of $\mathcal{M}_r$ bounded by $(\tau_j\omega'_j)(\tau_{j+1}\alpha'_{j+1})^{-1}$. It will suffice to show that $|\mathcal{N}_j| = 0$ since this guarantees that $\tau_j\omega'_j = \tau_{j+1}\alpha'_{j+1}$. Because f_{j+1} has the same index with respect to both D_j and D_{j+1}, $|\omega'_j| = |\alpha'_{j+1}|$. It thus follows from $|\mathcal{N}_j| = 0$, that $\omega'_j = \alpha'_{j+1}$ and $\tau_j = \tau_{j+1}$. We will assume that $|\mathcal{N}_j| > 0$ and find a region E of $\mathcal{M}_r$ which is appended to $\mathcal{M}$ on the top side of $\mathcal{M}$, contradicting Lemma 4.8.

If $|\mathcal{N}_j| > 0$, let D be a region of $\mathcal{N}_j$ which is appended on the top side of $\mathcal{N}_j$. Let g be the edge on the top side of D whose index with respect to D is in $J(x, \mathbf{r})$. The edge g must occur as an edge of τ_{j+1} rather than in α'_{j+1}: the edge f_j is the edge on the bottom side of D_{j+1} whose index with respect to D_{j+1} is in $J(x, \mathbf{r})$. Further, g cannot be an $\{i_1, i_2\}$-edge by the uniquenes of e, and g cannot be x-reflective because f_j is left-primitive and g precedes f_j in the positive walk $\tau_{j+1}\alpha'_{j+1}f_j$. Since g cannot be an interior edge of $\mathcal{M}$, it must occur on the to side of $\mathcal{M}$. Let λ be the terminal segment of $\omega_{\mathcal{M}}$ whose initial vertex is the terminal vertex of g. Let τ' be the terminal segment of τ_{j+1} whose initial vertex is the terminal vertex of g. Let $\mathcal{P}_j$ be the feathery submap of $\mathcal{M}_r$ bounded by $\tau'\alpha_{D_{j+1}}\sigma_{j+1}\lambda^{-1}$. Then $|\mathcal{P}_j| > 0$, because D_{j+1} is appended to $\mathcal{P}_j$ on its bottom side. Then any region E which is appended on the top side of $\mathcal{P}_j$ is also appended on the top side of $\mathcal{M}$, producing the expected contradiction.

For (b), we assume by induction that f_j is left-primitive and that $\alpha'_{j+1} = \omega'_j$. We need to show that f_{j+1} is left-primitive. We show that if we suppose that f_{j+1} is not left-primitive, then we can contradict Lemma 4.8 by finding a region E_2 of $\mathcal{M}_r$ which is appended on the bottom side of $\mathcal{M}$.

Suppose that g is an x-reflective edge of $\mathcal{M}_r$ which precedes f_{j+1}. Let ξ_1 be a directed walk in $\mathcal{M}_r$ from the initial vertex v of $\mathcal{M}_r$ to the initial vertex of g and let ξ_2 be a directed walk in $\mathcal{M}_r$ from the terminal vertex of g to the initial vertex of f_{j+1}. Because f_{j+1} is left-primitive, the walk ξ_2 cannot intersect $\tau_{j+1}\omega'_{j+1}$. We can choose the walk ξ_1 so that $\tau_{j+1}\omega'_{j+1}(\xi_1 g\xi_2)^{-1}$ is a counterclockwise walk in $\mathcal{M}_r$. Let $\mathcal{Q}_j$ be the feathery submap of $\mathcal{M}_r$ that is bounded by this walk. Then $|\mathcal{Q}_j| > 0$, because g occurs on its topside but cannot occur on its bottom side. Let D be a region of $\mathcal{Q}_j$ which is appended to the bottom side of $\mathcal{Q}_j$ and let g_2 be the edge on the bottom side of D whose index with respect to D is in $J(x, \mathbf{r})$. Then g_2 can be neither an $\{i_1, i_2\}$-edge nor an x-reflective edge, so g_2 must occur on the bottom side of $\mathcal{M}$. Arguing as in part (a), let τ'' be the terminal segment of τ_{j+1} whose initial endpoint is the terminal endpoint of g_2, and let λ_2 be the terminal segment of $\alpha_{\mathcal{M}}$ whose initial endpoint is the terminal endpoint of g. Then the submap of $\mathcal{M}_r$ bounded by $\lambda_2(\tau''\omega_{D_{j+1}}\sigma_{j+1})^{-1}$ must contain a region E_2 of $\mathcal{M}_r$ which is appended to $\mathcal{M}$ on the bottom side of $\mathcal{M}$. This completes the inductive proof of (a) and (b).

We have shown that if there were x-reflective edges in $\mathcal{M}_r$, then we would have, for some $n \geq 2$, left-primitive edges f_j, for $0 < j < n$. Structurally, these edges occur vertically between an edge f_0 on the bottom side of $\mathcal{M}$ and an edge f_n on the top side of $\mathcal{M}$. Further, each left-primitive f_j would be on the top side of a region D_j and on the bottom side of a region D_{j+1}, where all of these regions have a common initial vertex. Let $\mathcal{N}'$ be the submap of $\mathcal{M}_r$ bounded by $\alpha'_1 f_0\alpha''_1\sigma_1(\omega'_n f_n\omega''_n\sigma_n)^{-1}$.



If n is odd, then ω_n' is an initial segment of the long side of D_n when α_1' is an initial segment of the short side of D_1 and ω_n' is an initial segment of the short side of D_n when α_1' is an initial segment of the long side of D_1. Suppose without loss of generality that α_1' is an initial segment on the long side of D_1. We replace the submap $\mathcal{N}'$ (having at least three regions D_1, D_2, D_3) in $\mathcal{M}$ by a single region D'. We identify the initial segment of length $|\alpha_1'|$ on the long side of D' with α_1' and we identify the initial segment of length $|\omega_1'$ on the short side of D' with ω_1'. Let $\mathcal{M}'$ be the result of thus replacing $\mathcal{N}'$ in $\mathcal{M}$ with D'. The indices of edges on α_1' and ω_1' are not changed by this substitution, so $\rho_{\mathcal{M}'} \subseteq \rho_{\mathcal{M}} \subseteq \rho_{\mathbf{r}}$. Since the edge e also occurs in $\mathcal{M}'$, $\mathcal{M}'$ is an x-section for $\mathbf{r}$. This contradicts our choice of $\mathcal{M}$ as x-inceptive.

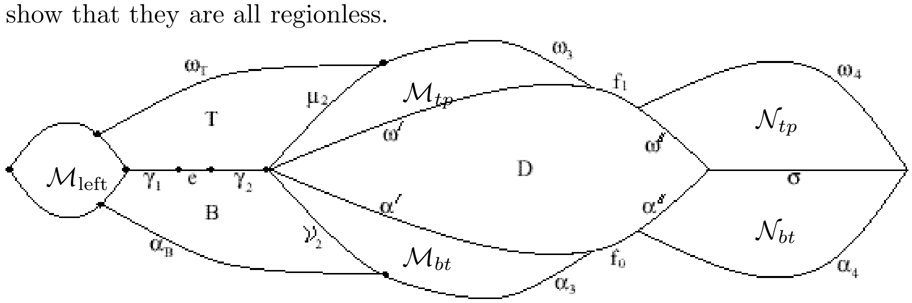
If $\mathcal{N}$ is even, then ω_n' is an initial segment of the long side of D_n when α_1' is an initial segment of the long side of D_1, and ω_n' is an initial segment of the short side of D_n when α_1' is an initial segment of the short side of D_1. We obtain a map $\mathcal{M}'$ by deleting $\mathcal{N}'$ and identifying α_1' with ω_1'. Suppose that e_1 is an edge on α_1' which is identified with an edge e_n on ω_n' in $\mathcal{M}'$. If the index of e_1 with respect to D_1 in $\mathcal{M}$ is j_0, then j_0 is also the index of e_n with respect to D_n in $\mathcal{M}$. If e_1 and e_n are both interior edges of $\mathcal{M}$, then e_1 has index j_1 with respect to some region and e_n has index j_n with respect to some region. Since (z_{j_0}, z_{j_1}) and (z_{j_0}, z_{j_n}) are in the equivalence relation $\rho_{\mathcal{M}}$, (z_{j_1}, z_{j_n}) is in $\rho_{\mathcal{M}}$ also. If either e_1 or e_n is a boundary edge in $\mathcal{M}$, then their identification contributes nothing to $\rho\mathcal{M}'$. We again have $\rho_{\mathcal{M}'} \subseteq \rho_{\mathcal{M}} \subseteq \rho_{\mathbf{r}}$ and that $\mathcal{M}'$ is also an x-section for $\mathbf{r}$, contradicting our choice of $\mathcal{M}$.

Lemma 4.12. *Suppose that* $\mathbf{r}$ *is an achievable* (s, ℓ) *relator, that the letter* x *occurs in* $\mathbf{r}$, *that* $J(x, \mathbf{r}) = \{ i_1, i_2 \}$, *where* $i_1 \leq s < i_2$, *and that* $\mathcal{M}$ *is an* x*-inceptive* (s, ℓ) *map for* $\mathbf{r}$. *Then* $|\mathcal{M}| \leq 4$.

It will suffice to prove that $\mathcal{M}_r$ (and dually, $\mathcal{M}_{\text{left}}$) has at most one region. For this draft, we provide two different proofs.

First Proof. Assume that $|\mathcal{M}_r| \geq 2$ and that B_r and T_r are distinct regions of $\mathcal{M}_r$ which are appended on the bottom and top sides of $\mathcal{M}_r$, respectively. Let f be the edge on the top side of B_r whose index with respect to B_r is in $J(x, \mathbf{r})$. Let g be the edge on the bottom side of T_r whose index with respect to T_r is in $J(x, \mathbf{r})$. By Lemma 4.8, the edges f and g cannot be $\{i_1, i_2\}$-edges. By Lemma 4.11, they cannot be x-reflective. Hence f must occur on the top side of $\mathcal{M}$ and g must occur on the bottom side of $\mathcal{M}$. If f precedes ω_{T_r} on $\omega_{\mathcal{M}_r}$, then T_r is appended to the top side of $\mathcal{M}$, contradicting Lemma 4.8. Similarly α_{B_r} must precede g on $\alpha_{\mathcal{M}_r}$. Let θ_1 be a directed walk on $\alpha_{\mathcal{M}_r}$ from the terminal vertex of α_{B_r} to the initial vertex of g and let θ_2 be a directed walk on $\omega_{\mathcal{M}_r}$ from the terminal vertex of ω_{T_r} to the initial vertex of f. Write $\alpha_{T_r} = \alpha' g \alpha''$ and $\omega_{B_r} = \omega' f \omega''$. Then $\alpha'' \theta_2 f \omega'' \theta_1 g$ is a positive closed walk in $\mathcal{M}$ which is impossible by Lemma 1.3.

Second Proof. Let D be any region of $\mathcal{M}_r$ whose initial vertex is the initial vertex v of $\mathcal{M}_r$. (By our choice of maximality of $\gamma_1 e \gamma_2$, there must be such a region D if $|\mathcal{M}_r| > 0$.) Let f_0 and f_1 be the edges on the bottom and top sides of D whose indices with respect to D are in $J(x, \mathbf{r})$. Write $\alpha_D = \alpha' f_0 \alpha''$ and $\omega_D = \omega' f_1 \omega''$. By Lemmas 4.8 and 4.11, f_0 must occur on the bottom side of $\mathcal{M}$ as an edge of α_2 in $\alpha_{\mathcal{M}} = \alpha_1 \alpha_B \alpha_2$ and f_1 must occur on the top side of $\mathcal{M}$ as an edge of ω_2 in $\omega_{\mathcal{M}} = \omega_1 \omega_T \omega_2$. Write $\alpha_2 = \alpha_3 f_0 \alpha_4$ and $\omega_2 = \omega_3 f_1 \omega_4$. Let σ be any directed walk from the terminal vertex of D to the terminal vertex of $\mathcal{M}_r$. Recall that $\mathcal{M}_r$ is bounded by $\nu_2 \alpha_2 (\mu_2 \omega_2)^{-1}$, where ν_2 and μ_2 are terminal segments of ω_B and α_T, respectively. Let $\mathcal{M}_{bm}, \mathcal{M}_{tp}, \mathcal{N}_{bm}$, and $\mathcal{N}_{tp}$ be the feathery submaps of $\mathcal{M}_r$ bounded by $\nu_2 \alpha_3 (\alpha')^{-1}, \omega' (\mu_2 \omega_3)^{-1}, \alpha_4 (\alpha'' \sigma)^{-1}$, and $\omega'' \sigma (\omega_4)^{-1}$, respectively. Then any region of $\mathcal{M}_r$, other than D, must be in one of these four submaps. We show that they are all regionless.

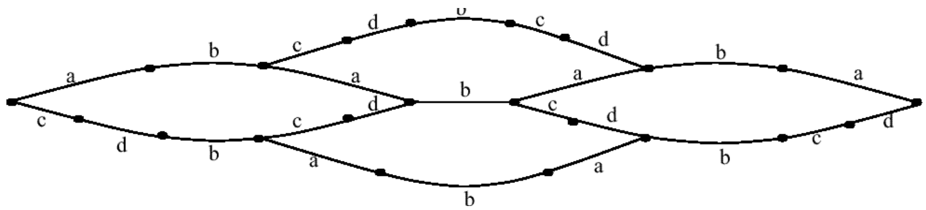


If $|\mathcal{M}_{bt}| > 0$, then some region of $\mathcal{M}_{bt}$ is appended to $\mathcal{M}_{bt}$ on the top side of $\mathcal{M}_{bt}$. Some edge on the top side of this appended region will have its index with respect to the appended region in $J(x, \mathbf{r})$. But f_0 is the only edge on the bottom side of D whose index with respect to D is in $J(x, \mathbf{r})$. We conclude that $|\mathcal{M}_{bt}| = 0$, and similarly that $|\mathcal{M}_{tp}| = 0$.

If $|\mathcal{N}_{bt}| > 0$, then some region of $\mathcal{N}_{bt}$ would be appended to $\mathcal{N}_{bt}$ on the bottom side of $\mathcal{N}_{bt}$ and appended to $\mathcal{M}$ on the bottom side of $\mathcal{M}$. This contradicts Lemma 4.8, so $|\mathcal{N}_{bt}| = 0$. Similarly, $|\mathcal{N}_{tp}| = 0$.

We define 'up'-words, ${}_i^j u$, on the ordered alphabet $X = \{x_1, x_2, \ldots, x_n\}$. For $0 \leq i < j \leq n$, let ${}_i^j u = x_{i+1} x_{i+2} \ldots x_j$. If $j \leq i$, then ${}_i^j u$ is the empty word. Observe that $|{}_i^j u| = j - i$ if $i \leq j$ and that $({}_i^j u)({}_j^k u) = {}_i^k u$ if $i \leq j \leq k$.

Example 4.13. The following $(3, 5)$ map is both inceptive and b-inceptive for the $(3, 5)$ relator $aba = cdbcd$. If D is either of the appended regions of $\mathcal{M}$, then $\mathcal{M} - D$ is not inceptive. Since $|\mathcal{M}| = 4$, we cannot, in general, improve on the bound given in the lemma above. If we replace a, b, c, and d with upwords on an ordered alphabet X, we obtain similar (s,ℓ) maps with $\ell - s$ an even number and $|s| \geq 3$, where $\mathcal{M}$ is x_i-inceptive for every letter x_i that occurs in the word b.



For an alphabetically contructed (s,ℓ) relator $\mathbf{r}$ and a letter x that occurs in $\mathbf{r}$, say that x is **restricted to the long side** of $\mathbf{r}$ if $i \in J(x,\mathbf{r}) \;\Rightarrow\; i > s$, and that x is **restricted to the short side** of $\mathbf{r}$ if $i \in J(x,\mathbf{r}) \;\Rightarrow\; i \leq s$. Typically, a letter might occur in both sides of a relator rather than being restricted to one side or the other.

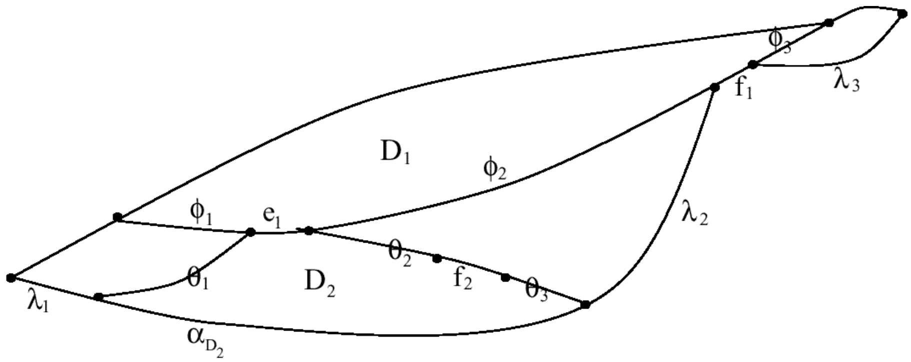
Lemma 4.14. *Suppose that* $\mathbf{r}$ *is an alphabetically constructed* (s,ℓ) *relator, that the letter* x *occurs in* $\mathbf{r}$, *and that* x *is retricted to either the short side or the long side of* $\mathbf{r}$. *Let* $i_1, i_2 \in J(x,\mathbf{r})$ *with* $i_1 \neq i_2$ *and let* $\mathcal{N}$ *be a derivation diagram over* $\langle X;\mathbf{r}\rangle$ *which contains no* $\{i,j\}$*-edges for* $i,j \in J(x,\mathbf{r})$ *except possibly* x*-reflective ones.*

Let e_1 *be an* $\{i_1,i_1\}$*-edge in* $\mathcal{N}$ *which occurs on the bottom side of a region* D_1 *of* $\mathcal{N}$ *and on the top side of a region* D_2 *of* $\mathcal{N}$. *Then the edge on the bottom side of* D_1 *which has index* i_2 *with respect to* D_1 *is also the edge on the top side of* D_2 *which has index* i_2 *with respect to* D_2.

Proof. We may assume without loss of generality that $i_1 < i_2$. We use induction on $|\mathcal{N}|$. When $|\mathcal{N}| = 1$, there are no $\{i_1,i_1\}$-edges and the lemma is vacuously true. When $|\mathcal{N}| = 2$, either there are no $\{i_1,i_1\}$-edges or else we must have $\alpha_{D_1} = \omega_{D_2}$ in order that $\mathcal{N}$ is two-sided.

Assume then that $|\mathcal{N}| > 2$. If $\mathcal{N}$ has an appended region D which is neither D_1 nor D_2, then the lemma is true by induction for $\mathcal{N} - D$ and is true then also for $\mathcal{N}$. By this, we may assume that D_1 is the only appended region on the top side of $\mathcal{N}$ and that D_2 is the only appended region on the bottom side of $\mathcal{N}$. Let f_1 be the edge of D_1 which has index i_2 with respect to D_1 and let f_2 be the edge of D_2 which has index i_2 with respect to D_2. We need to show that f_1 is f_2. A consequence is that this edge is an interior edge. As a step in the proof, we next show that at least one of f_1 or f_2 must be an interior edge.

Assume, without loss of generality, that f_1 is not an interior edge. Then f_1 must occur on the bottom side of $\mathcal{N}$. An easy argument shows that f_1 must occur after α_{D_2} on $\alpha_{\mathcal{N}}$. (Otherwise, we will have a directed walk from f_1 to e_1 as well as a directed walk from e_1 to f_1). Write $\alpha_{\mathcal{N}} = \lambda_1\alpha_{D_2}\lambda_2 f_1\lambda_3$, $\alpha_{D_1} = \phi_1 e_1\phi_2 f_1\phi_3$ and $\omega_{D_2} = \theta_1 e_1\theta_2 f_2\theta_3$.



Let $\mathcal{S}$ be the feathery submap of $\mathcal{N}$ bounded by $\theta_2 f_2 \theta_3 \lambda_2 \phi_2^{-1}$. Since $\mathcal{N}$ is a derivation diagram over $\langle X; \mathbf{r}\rangle$ and no edges of ϕ_2 are labelled by x, the edge f_2 cannot occur on the top side of $\mathcal{S}$. Hence, f_2, occurs on the bottom side of some region of $\mathcal{S}$ and is an interior edge of $\mathcal{N}$, as required.

Now, using that f_2 is an interior edge of $\mathcal{N}$, either f_2 is also an interior edge of $\mathcal{N} - D_1$ or else f_2 is on the top side of $\mathcal{N} - D_1$. In the latter case, f_2 must coincide with some edge on the bottom side of D_1 and the index requirements force $f_1 = f_2$. If the former case were to occur, by induction we could apply the lemma to the edge f_2 as an $\{i_2, i_2\}$-edge of the derivation diagram $\mathcal{N} - D_1$. A contradiction occurs since e_1 must then occur on the bottom side of some region of $\mathcal{N} - D_1$.

Lemma 4.15. *Suppose that $\mathcal{N}$ is an (s, ℓ) map, that n_1 regions of $\mathcal{N}$ have their long side on the top and that n_2 regions of $\mathcal{N}$ have their long side on the bottom. Then $|\omega_{\mathcal{N}}| = |\alpha_{\mathcal{N}}| + n_1(\ell - s) + n_2(s - \ell)$.*

Proof. This follows by a routine induction on $|\mathcal{N}|$.

Lemma 4.16. *Suppose that $\mathbf{r}$ is an alphabetically constructed (s, ℓ) relator, that the letter x occurs in $\mathbf{r}$, and that x is retricted to either the short side or the long side of $\mathbf{r}$. Let $\mathcal{N}$ be a derivation diagram over $\langle X; \mathbf{r}\rangle$ such that $\mathcal{N}$ contains no $\{i, j\}$-edges for $i \neq j$ and such that x occurs in neither $\overline{\alpha}_{\mathcal{N}}$ nor $\overline{\omega}_{\mathcal{N}}$. Then $|\mathcal{N}|$ is even and $|\alpha_{\mathcal{N}}| = |\omega_{\mathcal{N}}|$.*

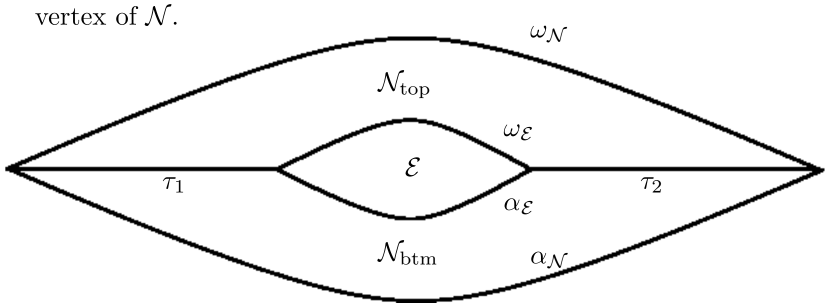
Proof. Let n_1 be the number of regions of $\mathcal{N}$ which have their long side on the top and let n_2 be the number of regions of $\mathcal{N}$ which have their long side on the bottom. By Lemma 4.14, any interior x-edge of $\mathcal{N}$ determines a pair of regions such that every x-edge which occurs on the boundary of either region must occur on the top side of one of the regions and and the bottom side of the other region. If $n_1 > n_2$, then some region with edges labelled x on the top side cannot be paired with a region where the edges labelled x are on the bottom side: when this happens, x must occur in $\overline{\omega}_{\mathcal{N}}$. Similarly, if $n_2 > n_1$, then x occurs in $\overline{\alpha}_{\mathcal{N}}$. Since by hypothesis, neither $\overline{\alpha}_{\mathcal{N}}$ nor $\overline{\omega}_{\mathcal{N}}$ contains an occurrence of x, we must have $n_1 = n_2$. Then $|\mathcal{N}| = n_1 + n_2$ is even and $|\alpha_{\mathcal{N}}| = |\omega_{\mathcal{N}}|$ by Lemma 4.15.

Lemma 4.17. *Suppose that $\mathbf{r}$ is an achievable (s, ℓ) relator, that the letter x occurs in $\mathbf{r}$, and that $J(x, \mathbf{r}) = \{\, i \,\}$, where $s < i$. Let $\mathcal{N}$ be a derivation diagram over $\langle X; \mathbf{r}\rangle$ such that x occurs in neither $\overline{\alpha}_{\mathcal{N}}$ nor $\overline{\omega}_{\mathcal{N}}$. Then $\overline{\alpha}_{\mathcal{N}} = \overline{\omega}_{\mathcal{N}}$.*

Proof. We use induction on $|\mathcal{N}|$ for $|\mathcal{N}| \geq 0$. It is obvious that $\overline{\alpha}_{\mathcal{N}} = \overline{\omega}_{\mathcal{N}}$ when $|\mathcal{N}| = 0$. If $|\mathcal{N}| = 1$, the hypothesis that x occurs in neither $\overline{\alpha}_{\mathcal{N}}$ nor $\overline{\omega}_{\mathcal{N}}$ must fail, so the conclusion follows vacuously. When $|\mathcal{N}| = 2$, the only way that the hypothesis on occurrences of x can be satisfied is if the two regions share a common long side; in this case, the conclusion is again clear. Assume then that $|\mathcal{N}| > 2$. The following sublemma will be useful. A corollary to the sublemma is that we may assume that $\mathcal{N}$ has only one block.

Sublemma 4.17.1. *If $\mathcal{N}$ contains a feathery submap $\mathcal{E}$ with $0 < |\mathcal{E}| < |\mathcal{N}|$ such that x occurs in neither $\overline{\alpha}_{\mathcal{E}}$ nor $\overline{\omega}_{\mathcal{E}}$, then the lemma is verified for $\mathcal{N}$.*

Proof of the first sublemma. By the induction hypothesis, we have that $\overline{\alpha}_{\mathcal{E}} = \overline{\omega}_{\mathcal{E}}$. Let τ_1 be a directed walk from the initial vertex of $\mathcal{N}$ to the initial vertex of $\mathcal{E}$ and let τ_2 be a directed walk from the terminal vertex of $\mathcal{E}$ to the terminal vertex of $\mathcal{N}$.



Let $\mathcal{N}_{\text{btm}}$ be the feathery submap of $\mathcal{N}$ bounded by $\alpha_{\mathcal{N}}(\tau_1\alpha_{\mathcal{E}}\tau_2)^{-1}$ and let $\mathcal{N}_{\text{top}}$ be the feathery submap of $\mathcal{N}$ bounded by $\tau_1\omega_{\mathcal{E}}\tau_2\omega_{\mathcal{N}}^{-1}$. The top side of $\mathcal{N}_{\text{btm}}$ has the same label as the bottom side of $\mathcal{N}_{\text{top}}$, so we obtain a derivation diagram $\mathcal{N}'$ over $\langle X; \mathbf{r}\rangle$ when we identify the top side of $\mathcal{N}_{\text{btm}}$ with the bottom side of $\mathcal{N}_{\text{top}}$. Because $|\mathcal{E}| > 0$, we have $|\mathcal{N}'| < |\mathcal{N}|$. By induction, the lemma is true for $\mathcal{N}'$. Since $\alpha\mathcal{N}' = \alpha_{\mathcal{N}}$ and $\omega_{\mathcal{N}'} = \omega_{\mathcal{N}}$, the lemma is verified for $\mathcal{N}$ also. Sublemma 4.17.1

We will call a pair of regions in $\mathcal{N}$ which share an edge labelled by x an **x-pair.** Every region of $\mathcal{N}$ is a member of exactly one x-pair. The **top** region of an x-pair has the edge which is labelled by x on its bottom side; the **bottom** region of an x-pair has the edge which is labelled by x on its top side. We will write x-pairs as ordered pairs $\gg D, E \ll$ where D is the bottom region of the pair and E is the top region. An x-pair is **coinitial** if the two regions have the same initial vertex; when this occurs, we will call the common initial vertex the **coinitial vertex** of the pair. An x-pair is coterminal if the two regions have the same terminal vertex. By the sublemma, we may assume that no x-pair in $\mathcal{N}$ is both coinitial and coterminal.

Sublemma 4.17.2. *With the hypotheses of the lemma, if $\mathcal{N}$ contains an x-pair, then $\mathcal{N}$ contains at least one coinitial x-pair and at least one coterminal x-pair.*

Proof of the second sublemma. If f is a leftmost x-reflective edge, the the x-pair which contains f is coinitial. If f is a rightmost x-reflective edge, then the x-pair which contains f is coterminal. Sublemma 4.17.2

If $\gg D_1, E_1 \ll$ and $\gg D_2, E_2 \ll$ are x-pairs in $\mathcal{N}$, then the number of **coherence bonds** between $\gg D_1, E_1 \ll$ and $\gg D_2, E_2 \ll$ is the number of coherence pairs in $\mathcal{N}$ among the following:
$(D_1, D_2), (D_1, E_2), (E_1, D_2), (E_1, E_2), (D_2, D_1), (E_2, D_1), (D_2, E_1)$, and (E_2, E_1).

Sublemma 4.17.3. *If $\mathcal{N}$ contains no proper feathery submaps with x-free boundary and $\gg D_1, E_1 \ll$ and $\gg D_2, E_2 \ll$ are x-pairs in $\mathcal{N}$, then there are at most three coherence bonds between $\gg D_1, E_1 \ll$ and $\gg D_2, E_2 \ll$. If there are three coherence bonds between $\gg D_1, E_1 \ll$ and $\gg D_2, E_2 \ll$ then by duality and change of notation, we may assume that one of the following two cases occurs:*

case *(a)* (D_1, D_2) *is left-right coherent,*
(D_2, E_1) *is right-left coherent, and*
(E_1, E_2) *is left-right coherent, or*

case *(b)* (D_1, D_2) *is contractive,*
(D_2, E_1) *is right-left coherent, and*
(E_1, E_2) *is expansive.*

Proof of the third sublemma. For any regions A and B, the definition for the ordered pair (A, B) to be a coherent pair requires that A and B share a common edge which is on the top side of A and the bottom side of B. It follows that at most one of (A, B) and (B, A) can be a coherent pair. (For a proof, consider the appended regions of the smallest feathery submap of $\mathcal{N}$ containing both A and B.)

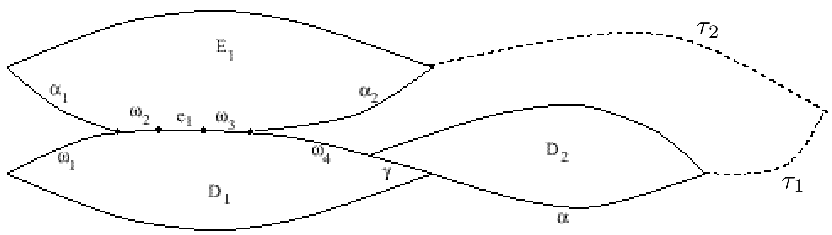
By the paragraph above, it is clear that there can be at most four coherence bonds between $\gg D_1, E_1 \ll$ and $\gg D_2, E_2 \ll$. We need to show that when we do have at least three coherence bonds that the fourth cannot occur. In doing this the structural assertions in the case where there are exactly three coherence bonds will be justified. Suppose then that we do have at least three coherence bonds. At least one of the coherence bonds must come either from one of (D_1, E_2) and (E_2, D_1) or else from (E_1, D_2) and (D_2, E_1). If only one coherence bond comes from these, then a second of the three coherence bonds must come from (D_1, D_2) or (D_2, D_1) and the third from (E_1, E_2) or (E_2, E_1). By vertical duality, we may assume that one bond is contributed by (D_1, D_2) or (D_2, D_1) and another is contributed by (D_2, E_1) or (E_1, D_2). If it were the case that coherence bonds were formed both among (D_1, E_2) and (E_2, D_1) and also among (D_2, E_1) and (E_1, D_2), then any third or fourth bond would be between either (D_1, D_2) and (D_2, D_1) or else between (E_1, E_2) and (E_2, E_1). Using vertical duality again, we can assume in this case also that one bond is contributed by (D_1, D_2) or (D_2, D_1) and another is contributed by (D_2, E_1) or (E_1, D_2): that is, using vertical duality we may assume that this happens in any case when there are at least three coherence bonds.

Since we may re-index the x-pairs by switching subscripts, we may assume that (D_1, D_2) rather than (D_2, D_1) is a coherent pair. Using horizontal duality, we may assume either that

(a) (D_1, D_2) is a left-right coherent pair or else that

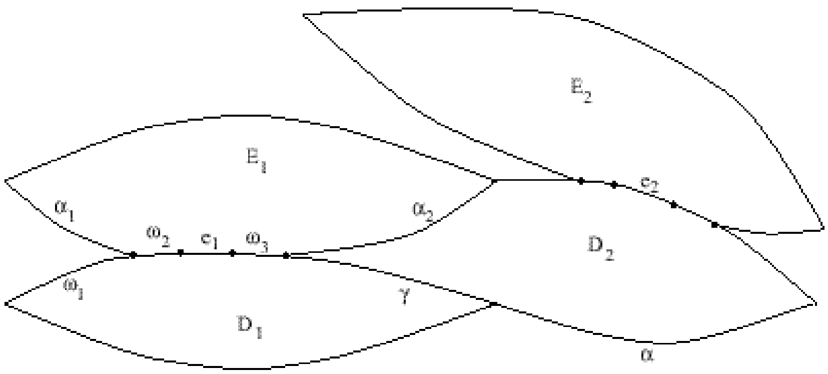
(b) (D_1, D_2) is a contractive coherent pair in which the edges on α_{D_2} occur to the right of the edge on ω_{D_1} that is labelled x. (In this case, it is also clear that α_{D_2} is the short side of D_2.) We separate the further analysis of cases (a) and (b).

case (a). We have used duality above to ensure that one of (D_2, E_1) or (E_1, D_2) is a coherent pair in $\mathcal{N}$. We want to next verify that, with our choices, (D_2, E_1) is a right-left coherent pair. Since (D_1, D_2) is a left-right coherent pair, we may write $\omega_{D_1} = \omega\gamma$ and $\alpha_{D_2} = \gamma\alpha$ for some positive paths γ and α. Since $\gg D_1, E_1 \ll$ is an x-pair, we may write $\omega = \omega_1\omega_2 e_1\omega_3\omega_4$ where e_1 is the common edge of ω_{D_1} and α_{E_1} which has label x and $\omega_2 e_1 \omega_3$ is the longest common segment of ω_{D_1} and α_{E_1} which contains e_1.



Let α_1 be the initial segment of α_{E_1} which has the same terminal vertex as ω_1 and let α_2 be the terminal segment of α_{E_1} which has the same initial vertex as ω_4. Let τ_1 be any directed path in $\mathcal{N}$ from the terminal vertex of D_2 to the terminal vertex of $\mathcal{N}$. Observe that there is at least one directed path τ_2 in $\mathcal{N}$ from the terminal vertex of E_1 to the terminal vertex of $\mathcal{N}$ and that we may choose one of these so that the path $(\omega_4\alpha_{D_2}\tau_1)(\alpha_2\tau_2)^{-1}$ is counter-clockwise and bounds a feathery submap $\mathcal{N}_1$ of $\mathcal{N}$. Since α_{D_2} is on the bottom side of $\mathcal{N}_1$ and edges of α_{E_1} occur on the top side of $\mathcal{N}_1$, we can see that (D_2, E_1) rather than (E_1, D_2) must be a coherent pair.

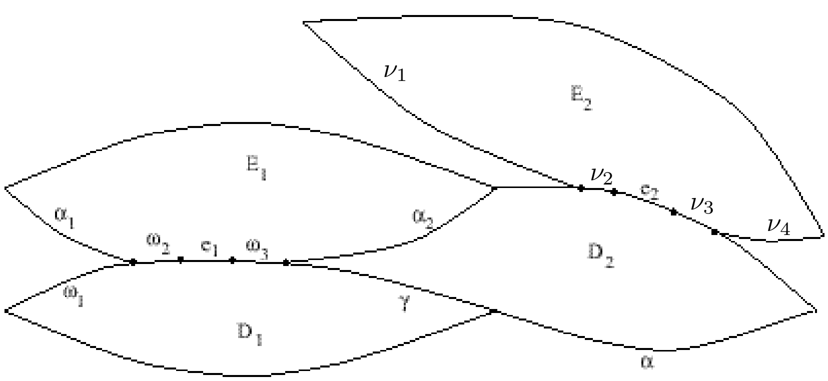
Next we claim that no edges of ω_{D_2} can be edges on α_1: if an edge f occurred on both α_1 and on ω_{D_2}, we could construct a positive closed path in $\mathcal{N}$ from f to e_1 along α_{E_1} and from e_1 to f along ω_{D_1} and ω_{D_2}. Thus, the coherent pair (D_2, E_1) cannot be a left-right coherent pair. Since e_1 is on α_{E_1} and is on ω_{D_1} rather than ω_{D_2}, the coherent pair (D_2, E_1) cannot be matched or contractive. Since D_2 is the bottom region of the $\gg D_2, E_2 \ll$, the side ω_{D_2} has the same length as α_{E_1} and (D_2, E_1) cannot be expansive. The remaining possibility is that (D_2, E_1) is a right-left coherent pair as required. Notice that this forces ω_4 to be an empty path since $\mathcal{N}$ contains no feathery submaps with x-free boundary.



From the illustration above, it is clear that neither (D_1, E_2) nor (E_2, D_1) can now be a coherent pair in $\mathcal{N}$. For a slightly more formal argument, first note that (E_2, D_1) cannot be a coherent pair in $\mathcal{N}$. If that happened, edges of ω_{E_2} would then be on the bottom side of the feathery submap of $\mathcal{N}$ bounded by $(\alpha_{D_1}\alpha)(\omega_1\omega_2 e_1\omega_3\omega_{D_2})^{-1}$ and the edge labelled by x on the bottom side of E_2 would occur on the top side of this same submap. (D_1, E_2) cannot be a matched, expansive, or contractive coherent pair because ω_{D_1} and α_{E_2} have the same length and share edges labelled by x with other regions. (D_1, E_2) cannot be a left-right coherent pair because (D_1, D_2) is a left-right coherent pair. Let e_2 be the edge common to ω_{D_2} and α_{E_2} that is labelled by x. (D_1, E_2) cannot be a right-left coherent pair because this would force a positive closed path in $\mathcal{N}$ (along ω_{D_1} and ω_{D_2} from e_1 to e_2 and then along α_{E_2} and ω_{D_1} from e_2 to e_1.)

Next, we need to show that (E_1, E_2) rather than (E_2, E_1) is the third coherent pair and that (E_1, E_2) is a left-right coherent pair.

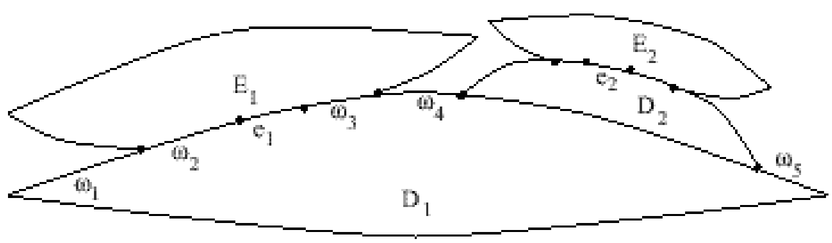
To see that (E_2, E_1) cannot be a coherent pair, first observe that, except for the edges on α_1, the edges on α_{E_1} are already shared with ω_{D_1} or ω_{D_2}. If the terminal edge f of ω_{E_2} were to occur on α_1 then there would be a positive closed path in $\mathcal{N}$ from e_2 to the terminal vertex of f along α_{E_2} and from the terminal vertex of f to e_2 along α_{E_1} and ω_{D_2}.



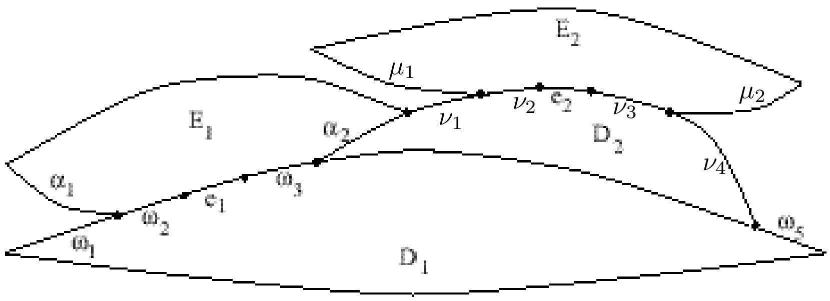
Write α_{E_2} as $\nu_1\nu_2 e_2\nu_3\nu_4$ where $\nu_2 e_2\nu_3$ is the longest common segment of ω_{D_2} and α_{E_2} which contains e_2. Because e_2 is on α_{E_2}, (E_1, E_2) cannot be a matched coherent pair or a contractive coherent pair. If any edge f of ν_4 was also an edge on ω_{E_1}, then there would be a positive closed path in $\mathcal{N}$ from e_2 to f along ν_3 and ν_4, and then from f to e_2 along ω_{E_1} and ω_{D_2}. Hence (E_1, E_2) can be neither a right-left coherent pair nor an expansive coherent pair with ω_{E_1} a segment of ν_4. We will conclude case (a) by showing that (E_1, E_2) can only be a left-right coherent pair because ν_1 is too short for (E_1, E_2) to be an expansive coherent pair with ω_{E_1} a segment of ν_1. We need to show that $|\omega_{E_1}| > |\nu_1|$.

Because $\mathcal{N}$ contains no feathery submaps with x-free boundary, we can make the following observation. If (E_1, E_2) is an expansive coherent pair with ω_{E_1} a segment of ν_1, then $\nu_1 = \mu\omega_{E_1}$ for some directed path μ and the terminal vertex of E_1 is the initial vertex of ν_2. Since the initial segments of α_{E_2} and ω_{D_2} are then $\nu_1\nu_2e_2$ and $\alpha_2\nu_2e_2$ (where e_2 has the same index with respect to E_2 and D_2), we see that $|\nu_1| = |\alpha_2|$. A similar argument for $e_1\omega_3\alpha_2$ and $e_1\omega_3\gamma$ shows that $|\alpha_2| = |\gamma|$. Since (D_1, D_2) is a left-right coherent pair, with $\alpha_{D_2} = \gamma\alpha$, we have $|\alpha_{D_2}| > |\gamma|$. Both α_{D_2} and ω_{E_1} are short sides of regions, so we have $|\omega_{E_1}| = |\alpha_{D_2}| > |\gamma| = |\alpha_2| = |\nu_1|$, as required

case (b). Assume now that α_{D_2} is a segment of ω_{D_1} that occurs to the right of e_1 on ω_{D_1} and write ω_{D_1} as $\omega_1\omega_2e_1\omega_3\omega_4\alpha_{D_2}\omega_5$ where $\omega_2e_1\omega_3$ is the longest common segment of ω_{D_1} and α_{E_1} which contains e_1.



As in case (a), we have already used duality to assure that one of (D_2, E_1) or (D_2, E_1) is a coherent pair in $\mathcal{N}$. In the current case, it is easy to see that the coherent pair must be (D_2, E_1) because α_{D_2} is a segment of ω_{D_1}. Since ω_{D_2} and α_{E_1} have the same length and share edges labelled by x with other regions, (D_2, E_1) cannot be an expansive, contractive, or matched coherent pair. As in earlier argument, if (D_2, E_1) were a left-right coherent pair, we would have a positive closed path in $\mathcal{N}$. Thus (D_2, E_1) must be a right-left coherent pair. Let e_2 be the edge common to ω_{D_2} and α_{E_2} which has label x. We may write α_{E_1} as $\alpha_1\omega_2e_1\omega_3\alpha_2$ and ω_{D_2} as $\alpha_2\nu_1\nu_2e_2\nu_3\nu_4$ where $\nu_2e_2\nu_3$ is the longest common segment of ω_{D_2} and α_{E_2} containing e_2.



As in case (a), we need to show that neither (E_2, D_1) nor (D_1, E_2) can be a coherent pair. Since $\alpha_{D_1}(\omega_1\omega_2 e_1\omega_3\omega_{D_2}\omega_5)^{-1}$ bounds a feathery submap of $\mathcal{N}$ and e_2 occurs on the top side of this map, no edge of ω_{E_2} can occur on α_{D_1}. Thus, (E_2, D_1) is not a coherent pair. Since ω_{D_1} and α_{E_2} have the same length and share edges labelled by x with other regions, (D_1, E_2) cannot be a contractive, expansive or matched coherent pair. If (D_1, E_2) were either a right-left coherent pair or a left-right coherent pair, we would find positive closed paths in $\mathcal{N}$.

Next we need to show that (E_1, E_2) rather than (E_2, E_1) is a coherent pair in $\mathcal{N}$. We need only observe that if the terminal edge of ω_{E_2} were an edge on α_1, then we would have a positive closed path in $\mathcal{N}$. We are now reduced to the case that (E_1, E_2) is a coherent pair in $\mathcal{N}$. Write α_{E_2} as $\mu_1\nu_2\mathcal{E}_2\nu_3\mu_2$. To conclude the proof of this sublemma, we will show that ω_{E_1} is a segment of μ_1. Because e_2 occurs on α_{E_2}, we cannot have that (E_1, E_2) is matched or contractive. If the initial edge of ω_{E_2} occurred on μ_2, we would have a positive closed path in $\mathcal{N}$. The remaining possibilities are that (E_1, E_2) is left-right coherent or that ω_{E_1} is a segment of μ_1. In either case, we can see that (because $\mathcal{N}$ contains no proper feathery submaps with x-free boundary) ν_1 must be empty. We now need only show that $|\mu_1| \geq |\omega_{E_1}|$. With arguments like those in case (a), we can see that $|\mu_1| = |\alpha_2| = |\alpha_{D_2}\omega_5| \geq |\alpha_{D_2}| = |\omega_{e_1}|$.

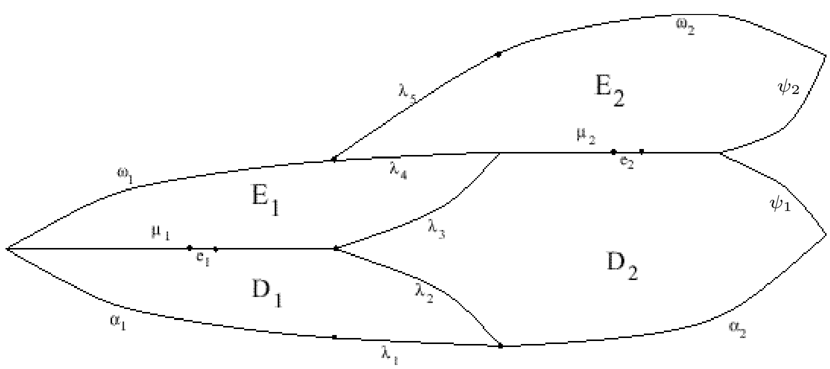
Sublemma 4.17.3

Sublemma 4.17.4. *If* $\gg D_1, E_1 \ll$ *is a coinitial x-pair in $\mathcal{N}$,* $\gg D_2, E_2 \ll$ *is an x-pair in $\mathcal{N}$, and there are three coherence bonds between* $\gg D_1, E_1 \ll$ *and* $\gg D_2, E_2 \ll$*, then the lemma is verified for $\mathcal{N}$.*

If $\gg D_1, E_1 \ll$ *is a coterminal x-pair in $\mathcal{N}$,* $\gg D_2, E_2 \ll$ *is an x-pair in $\mathcal{N}$, and there are three coherence bonds between* $\gg D_1, E_1 \ll$ *and* $\gg D_2, E_2 \ll$*, then the lemma is verified for $\mathcal{N}$.*

Proof of the fourth sublemma. The two assertions are dual, so we may assume that we are in the case where $\gg D_1, E_1 \ll$ is coinitial and then that we are in one of the two structural cases (a) and (b) in the conclusion of the previous sublemma.

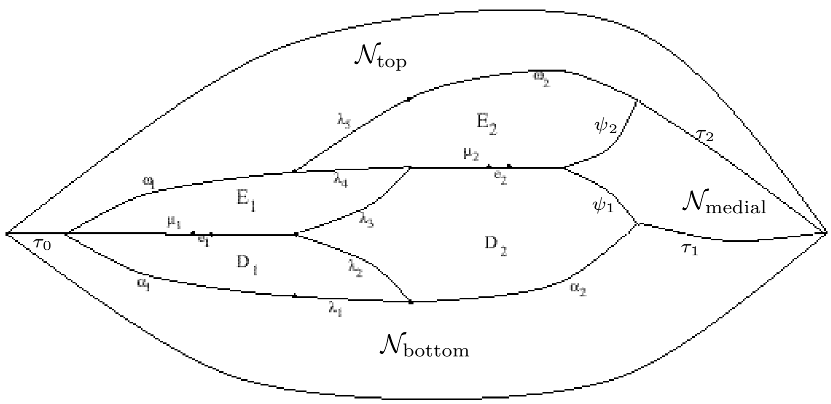
case (a) Let e_i be the x-relective edge for $\gg D_i, D_i \ll$. Since $\gg D_1, E_1 \ll$ is coinitial, we can write ω_{D_1} as $\mu_1\lambda_2$ and α_{E_1} as $\mu_1\lambda_3$ for positive paths μ_1, λ_2 and λ_3 where e_1 occurs on μ_1 and the initial edge of λ_2 is distinct from the initial edge of λ_3. Since (D_1, D_2) is a left-right coherent pair, we can write α_{D_2} as $\lambda_2\alpha_2$ for some positive path α_2. Since (D_2, E_1) is a right left-coherent pair and $\gg D_2, E_2 \ll$ is an x-pair, we can write ω_{D_2} as $\lambda_3\mu_2\psi_1$ where e_2 occurs on μ_2 and μ_2 is the longest common segment of ω_{D_2} and α_{E_2} containing e_2. Since (E_1, E_2) is a left-right coherent pair, write ω_{E_1} as $\omega_1\lambda_4$ and α_{E_2} as $\lambda_4\mu_2\psi_2$.



In the previous sublemma, we saw, with different notation, that $|\lambda_2| = |\lambda_3| = |\lambda_4|$ and that this common length is less than the length s for short sides of regions. Hence, we may also write α_{D_1} as $\alpha_1\lambda_1$ and ω_{E_2} as $\lambda_5\omega_2$ for positive walks $\lambda_1, \alpha_1, \lambda_5$ and ω_2 where $|\lambda_i| = |\lambda_j|$ for $1 \leq i, j \leq 5$.

We next argue that $\overline{\lambda_i} = \overline{\lambda_j}$ for $1 \leq i, j \leq 5$. First, $\overline{\lambda_2} = \overline{\lambda_3}$, since λ_2 and λ_3 have the same length and are terminal segments of the long sides of D_1 and E_1, respectively. Next, $\overline{\lambda_3} = \overline{\lambda_4}$ since λ_3 and λ_4 have the same length and are initial segments on the long sides of D_2 and E_2, respectively. Then, $\overline{\lambda_2} = \overline{\lambda_5}$, since λ_2 and λ_5 are initial segments on the short sides of D_2 and E_2. Finally, $\overline{\lambda_1} = \overline{\lambda_4}$, since λ_1 and λ_4 are terminal segments on the short sides of D_1 and E_1. It is similarly apparent that $\overline{\alpha_1} = \overline{\omega_1}$ and $\overline{\alpha_2} = \overline{\omega_2}$.

Choose any directed path τ_0 from the initial endpoint of $\mathcal{N}$ to the initial endpoint of α_1, μ_1, and ω_1. Let τ_1 be a directed path from the terminal endpoint of ψ_1 to the terminal endpoint of $\mathcal{N}$ and choose a directed path τ_2 from the terminal endpoint of ψ_2 to the terminal endpoint of $\mathcal{N}$ such that $(\psi_1\tau_1)(\psi_2\tau_2)^{-1}$ is counterclockwise. Let $\mathcal{N}_{\text{medial}}$ be the feathery submap of $\mathcal{N}$ that is bounded by $(\psi_1\tau_1)(\psi_2\tau_2)^{-1}$. Let $\mathcal{N}_{\text{bottom}}$ and $\mathcal{N}_{\text{top}}$ be the feathery submaps of $\mathcal{N}$ that are bounded by $\alpha_{\mathcal{N}}(\tau_0\alpha_1\lambda_1\alpha_2\tau_1)^{-1}$ and $(\tau_0\omega_1\lambda_5\omega_2\tau_2)\omega_{\mathcal{N}}^{-1}$.



Append a region D_2' to the top of $\mathcal{N}_{\text{bottom}}$, identifying the short side of D_2' with $\lambda_1\alpha_2$ to obtain a feathery map $\mathcal{N}_1$. Write the top side of D_2' as $\lambda_3'\mu_2'\psi_1'$ where $\overline{\lambda_3} = \overline{\lambda_3'}, \overline{\mu_2} = \overline{\mu_2'}$ and $\overline{\psi_1} = \overline{\psi_1'}$. Append the feathery submap $\mathcal{N}_{\text{medial}}$ to the tops side of $\mathcal{N}_1$ along $\psi_1'\tau_1$ to obtain a feathery map $\mathcal{N}_2$. Append a region E_2' to the top side of $\mathcal{N}_2$ along $\lambda_3'\mu_2'\psi_2$, to obtain a feathery map $\mathcal{N}_3$. Here, $\gg D_2', E_2' \ll$ is a coinitial x-pair. Write the top side of E_2' as $\lambda_5'\omega_2'$ where $\overline{\lambda_5} = \overline{\lambda_5'}$ and $\overline{\omega_2} = \overline{\omega_2'}$. Since $\overline{\alpha_1} = \overline{\omega_1}$ we can identify the bottom side of $\mathcal{N}_{\text{top}}$ with the top side of $\mathcal{N}_3$ to obtain a feathery map $\mathcal{N}_4$ which has $|\mathcal{N}| - 2$ regions. Then $\overline{\alpha_{\mathcal{N}_4}} = \overline{\omega_{\mathcal{N}_4}}$, by induction on $|\mathcal{N}|$, so $\overline{\alpha_{\mathcal{N}}} = \overline{\omega_{\mathcal{N}}}$.

case(b)

Sublemma 4.17.4

Sublemma 4.17.5. *If $\mathcal{N}$ contains at least two x-pairs and there are no proper feathery submaps of $\mathcal{N}$ with x-free boundary, then there is a coinitial x-pair $\gg D_1, E_1 \ll$ and an x-pair $\gg D_2, E_2 \ll$ with at least two coherence bonds between $\gg D_1, E_1 \ll$ and $\gg D_2, E_2 \ll$.*

If $\mathcal{N}$ contains at least two x-pairs and there are no proper feathery submaps of $\mathcal{N}$ with x-free boundary, then there is a coterminal x-pair $\gg D_1, E_1 \ll$ and an x-pair $\gg D_2, E_2 \ll$ with at least two coherence bonds between $\gg D_1, E_1 \ll$ and $\gg D_2, E_2 \ll$.

Proof of the fifth sublemma.

Sublemma 4.17.5

Sublemma 4.17.6. *If $\gg D_1, E_1 \ll$ is a coinitial x-pair in $\mathcal{N}$, $\gg D_2, E_2 \ll$ is an x-pair in $\mathcal{N}$, and there are two coherence bonds between $\gg D_1, E_1 \ll$ and $\gg D_2, E_2 \ll$, then the lemma is verified for $\mathcal{N}$.*

If $\gg D_1, E_1 \ll$ is a coterminal x-pair in $\mathcal{N}$, $\gg D_2, E_2 \ll$ is an x-pair in $\mathcal{N}$, and there are two coherence bonds between $\gg D_1, E_1 \ll$ and $\gg D_2, E_2 \ll$, then the lemma is verified for $\mathcal{N}$.

Proof of the sixth sublemma.

Sublemma 4.17.6

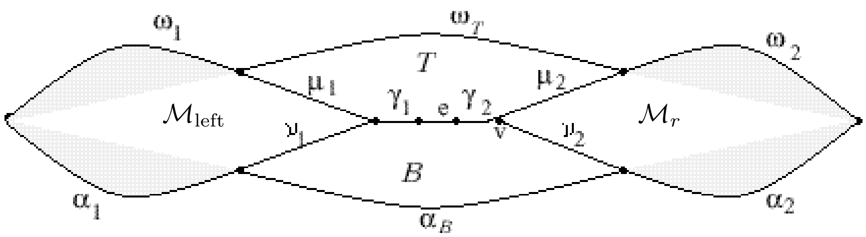
TO BE COMPLETED

Lemma 4.18. *Suppose that* $\mathbf{r}$ *is an achievable* (s,ℓ) *relator, that the letter* x *occurs in* $\mathbf{r}$, *and that* $J(x,\mathbf{r}) = \{ i_1, i_2 \}$, *where* $s < i_1 < i_2$. *Let* $\mathcal{N}$ *be a derivation diagram over* $\langle X;\mathbf{r}\rangle$ *which contains no* $\{i_1,i_2\}$*-edges. If neither* $\overline{\alpha}_{\mathcal{N}}$ *nor* $\overline{\omega}_{\mathcal{N}}$ *contains an occurrence of* x, *then* $\overline{\alpha}_{\mathcal{N}} = \overline{\omega}_{\mathcal{N}}$.

Proof. Since $J(x,\mathbf{r}) = \{ i_1, i_2 \}$ and $\mathcal{N}$ contains no $\{i_1,i_2\}$-edges, $[z_{i_1}]_{\rho_{\mathcal{N}}} \neq [z_{i_2}]_{\rho_{\mathcal{N}}}$ and $J([z_{i_1}]_{\rho_{\mathcal{N}}}, \mathbf{r}_{\mathcal{N}}) = \{ i_1 \}$. Write ϕ for the labelling on $\mathcal{N}$ which corresponds to $\mathbf{r}$ and write ϕ' for the labelling on $\mathcal{N}$ which corresponds to $\mathbf{r}_{\mathcal{N}}$. Since x occurs in neither $\phi(\alpha_{\mathcal{N}})$ nor $\phi(\omega_{\mathcal{N}})$, $[z_{i_1}]_{\rho_{\mathcal{N}}}$ cannot occur in $\phi'(\alpha_{\mathcal{N}})$ or $\phi'(\omega_{\mathcal{N}})$. By Lemma 4.17, we have $\phi'(\alpha_{\mathcal{N}}) = \phi'(\omega_{\mathcal{N}})$. By Lemma 3.2, $\mathbf{r}_{\mathcal{N}} \leq \mathbf{r}$. Define a function f from $\langle X;\mathbf{r}_{\mathcal{N}}\rangle$ to $\langle X;R\rangle$ by $f([z_j]_{\rho_{\mathcal{N}}}) = [z_j]_{\rho}$. By Proposition 2.9, this is a well-defined homomorphism. Since $f\phi'(e) = \phi(e)$ for every edge e, we have $\phi(\alpha_{\mathcal{N}}) = \phi(\omega_{\mathcal{N}})$.

Lemma 4.19. *Suppose that* $\mathbf{r}$ *is an achievable* (s,ℓ) *relator, that the letter* x *occurs in* $\mathbf{r}$, *that* $J(x,\mathbf{r}) = \{ i_1, i_2 \}$, *where* $s < i_1 < i_2$, *and that* $\mathcal{M}$ *is an* x*-inceptive* (s,ℓ) *map for* $\mathbf{r}$. *Then no edge of* $\mathcal{M}$ *is* x*-reflective.*

Proof. We return to the notation for $\mathcal{M}$ that was introduced following Lemma 4.8.



By (horizontal) symmetry, it will suffice to prove that no edge on μ_2 or ν_2 is x-reflective and that no interior edge of $\mathcal{M}_r$ is x-reflective. By (vertical) symmetry, we may assume that e has index i_1 with respect to T and has index i_2 with respect to B. Let f be the edge on the bottom side of T which has index i_2 with respect to T.

The only edges on the top side of B which can have label x are those which have index i_1 or i_2 with respect to B. Since e has label i_2 with respect to B and ν_2 follows e on ω_B, no edge of ν_2 can be x-reflective or even have label x.

The edge f must occur on μ_2 rather than on γ_2: it has label x so it cannot follow e on the top side of B. Suppose, for the sake of contradiction, that f is x-reflective in $\mathcal{M}$. Then Lemma 4.14 applies to f and e as edges of $\mathcal{M} - B$ and e must be an x-reflective edge in $\mathcal{M} - B$, also. This is impossible since e is the unique $\{i_1,i_2\}$-edge of $\mathcal{M}$.

We need finally to show that no interior edge of $\mathcal{M}_r$ can be x-reflective. We show that every edge of $\mathcal{M}_r$ which has label x must occur on the bottom side of $\mathcal{M}_r$.

TO BE COMPLETED

Lemma 4.20. *Suppose that* $\mathbf{r}$ *is an achievable* (s, ℓ) *relator, that the letter* x *occurs in* $\mathbf{r}$, *that* $J(x, \mathbf{r}) = \{ i_1, i_2 \}$, *where* $s < i_1 < i_2$, *and that* $\mathcal{M}$ *is an* x*-inceptive* (s, ℓ) *map for* $\mathbf{r}$. *Then* $|\mathcal{M}| \leq \ell$.

Theorem 4.21. *Suppose that* $\mathbf{r}$ *is an achievable* (s, ℓ) *relator, that the letter* x *occurs in* $\mathbf{r}$, *that* $|J(x, \mathbf{r})| = 2$, *and that* $\mathcal{M}$ *is an* x*-inceptive* (s, ℓ) *map for* $\mathbf{r}$. *Then* $|\mathcal{M}| \leq \ell$.

Proof. This summarizes Lemmas 4.9, 4.12, and 4.20.

5. Local Computations: The Induction Step

Theorem 5.1. *Suppose that* $\mathbf{r}$ *is an achievable* (s, ℓ) *relator, that the letter* x *occurs in* $\mathbf{r}$, *that* $|J(x, \mathbf{r})| = j$, *and that* $\mathcal{M}$ *is an* x*-inceptive* (s, ℓ) *map for* $\mathbf{r}$. *Then* $|\mathcal{M}| \leq (j-1)\ell$.

Theorem 5.2. *Suppose that* $\mathbf{r}$ *is an achievable* (s, ℓ) *relator and that* $\mathcal{M}$ *is an inceptive* (s, ℓ) *map for* $\mathbf{r}$. *Then* $|\mathcal{M}| \leq \ell(s + \ell - 2)$.

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