ACHIEVABLE RELATORS

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ABSTRACT. The principal result of this paper is a reduction of the word problem for one relator semigroups. Crudely, a relator \((w, w')\) on the alphabet \(X\) is achievable if the letters in the relator can actively participate in sequences of transitions for the semigroup presentation \(\langle X; (w, w') \rangle\). (A precise definition of achievable relators is given in section 3.) Empirically, achievable relators are rare. We prove that the word problem for a one relator semigroup presentation can be reduced to the word problem for a presentation where the relator is achievable.

1 BACKGROUND AND PRELIMINARIES

A directed planar map (or simply map) is a finite collection of vertices, edges, and regions in the Euclidean plane, together with an orientation for the set of edges. Here we will require that each edge has two distinct vertices for its endpoints. We also make the usual requirements that vertices, edges, and regions are pairwise disjoint, that regions are homeomorphic to the unit disk, and that each region has a connected boundary which is a union of edges and their endpoints. The orientation of the edges distinguishes an initial endpoint and a terminal endpoint for each edge. Two maps are the same if one can be mapped onto the other by a homeomorphism of the plane which induces a one-to-one function preserving vertices, edges, regions, incidence, and orientation.

Let \(D\) be a region of any map or let \(M\) be a connected and simply connected map. A boundary walk for \(D\) or for \(M\) is a closed walk of minimal length which includes all of the edges on the topological boundary of \(D\) or of \(M\). For our purposes, we make the additional requirement that all boundary walks must be oriented counterclockwise in the plane. It follows that if \(\pi\) is any boundary walk for a fixed map or region, then all other boundary walks for this map or region can be regarded as cyclic permutations of the edges of \(\pi\). A map or region \(Q\) is two-sided if it has a boundary walk of the form \(\alpha Q (\omega Q)^{-1}\) where \(\alpha Q\) and \(\omega Q\) are positive walks, the respective bottom and top sides of \(Q\).

We define \(W_1\) to be the map consisting of a single directed edge \(e_1\) together with its initial vertex \(v_0\) and its terminal vertex \(v_1\). For \(k > 1\), the map \(W_k\) is defined inductively by \(W_k = W_{k-1} \cup \{v_k, e_k\}\) where \(v_k\) is a vertex and \(e_k\) is an edges with initial vertex \(v_{k-1}\) and terminal vertex \(v_k\). It is easy to see that each \(W_k\) is two-sided and a tree.

Let \(F_0 = \{W_k | k \geq 1\}\). Assume, for induction, that a set \(F_j\) of two-sided maps having two-sided regions has been defined and that each map in \(F_j\) has \(j\) regions. Let \(I_j\) be an index set of triples \((k, M, \sigma)\) where \(k\) is a natural number, \(M \in F_j\), and \(\sigma\) is a nontrivial segment of the top side of \(M\) having initial endpoint \(u_0\) and terminal endpoint \(u_t\). Given such a triple, we define a map \(F(k, M, \sigma)\) to be

\[
M \cup \{v_1, v_2, \ldots, v_{k-1}, e_1, \ldots, e_k, D_{j+1}\}
\]
where each $v_i$ is a vertex, each $e_i$ is an edge with initial vertex $v_{i-1}$ and terminal vertex $v_i$. (Let $v_0 = u_0$ and $v_k = u_t$.) and $D_{j+1}$ is a region having bottom side $\sigma$ and top side $e_1 \ldots e_k$. See the following illustration. Then $F(k, M, \sigma)$ and its $j + 1$ regions are two-sided. Let $F_{j+1} = \{F(k, M, \sigma) \mid (k, M, \sigma) \in I_j\}$ and let $F = \bigcup_{j=0}^{\infty} F_j$. To facilitate a brief discussion in this paper, we will call a map in $F$ a feather or a feathery map.

A vertex $v$ in a map $M$ is a source if for every other vertex $w$ of $M$, there is a positive walk in $M$ from $v$ to $w$. Dually, $v$ is a sink if there is a positive walk from every such $w$ to $v$. A vertex is a transmitter if it has indegree 0 and a receiver is it has outdegree 0. The set $R$ of maps is defined by $R = \{M \mid M$ is two-sided, each region of $M$ is two-sided, and no interior vertex of $M$ is a transmitter or a receiver\}. It can be shown that for maps in $R$, a vertex is a transmitter if and only if it is a source and a vertex is a receiver if and only if it is a sink. John Remmers introduced the set $R$ in his thesis [10] where the maps of $R$ were called monotone maps. In a published part of Remmers’ thesis [11], the maps of $R$ are called regular maps. In both [10] and [11], Remmers proves the essential details of the following theorem. For a more recent exposition of this and other properties of regular maps, the reader should see Higgins’ book [2].

**Theorem 1.1.** $R = F$.

**Corollary 1.2.** If $M \in F$ and $N$ is a two-sided submap of $M$, then $N \in F$.

Hence the maps that we have called feathers above are the same as the maps which have previously been called monotone maps [4,10] or regular maps [5,11]. Higgins [2] like Howie and Pride [3] bypasses the need to assign any name for these maps by proceeding directly to the closely related notion of a diagram (See below.). For the purposes of this paper, it is essential to discuss feathers or regular maps as well as diagrams. The usefulness of these maps and the related diagrams has also been discovered independently by Kasincev [6,7,8]. More recently, they have also been discovered by Power [9] who calls them pasting schemes; unlike the other authors who work with semigroups, Power uses these maps for a construction in category theory.

Given that $R = F$, the next three lemmas are easily seen to be true for $F$ and hence for $R$ also. As well as being useful observations, they are instrumental in the proof that $R \subseteq F$. For proofs, see [10], [11,pp. 286,287] or [2,pp. 75,76]

**Lemma 1.3 (Remmers).** If $M$ is a map in $R$, then there are no positive closed walks in $M$.

**Lemma 1.4 (Remmers).** Suppose that $M$ is a map in $R$. If $v$ is a vertex of $M$, then there is a positive walk from the initial vertex of $M$ to the terminal vertex of $M$ which contains $v$. If $e$ is an edge of $M$, then there is a positive walk from the initial vertex of $M$ to the terminal vertex of $M$ which contains $e$. 

![Diagram of a map M in F_j with a source v, a sink v, and edges e and e.]
A region $D$ of the map $\mathcal{M}$ is **appended on the top side of** $\mathcal{M}$ if $\omega_D$ is a segment of $\omega_M$ and is **appended on the bottom side of** $\mathcal{M}$ if $\alpha_D$ is a segment of $\alpha_M$. A region $D$ is an **appended region of** $\mathcal{M}$ if it is appended on either side of $\mathcal{M}$.

**Lemma 1.5 (Remmers).** If $\mathcal{M} \in \mathcal{R}$ and has at least one region, then there are regions $B$ and $T$ of $\mathcal{M}$ which are appended on the bottom and top sides of $\mathcal{M}$, respectively. If $\mathcal{M}$ has at least two regions, then we can choose regions $B$ and $T$ which are distinct.

Suppose that $\mathcal{M}$ is a feathery map and that $D$ is an appended region on the bottom side of $\mathcal{M}$. Write $\alpha_M = \lambda_0 \alpha_D \lambda_1$. We allow $\lambda_0$ and/or $\lambda_1$ to be empty. We define $\mathcal{M} - D$ to be the feathery submap of $\mathcal{M}$ which has boundary walk $\lambda_0 \omega_D \lambda_1 \omega_M^{-1}$. Equivalently, $\mathcal{M} - D$ is the feathery submap of $\mathcal{M}$ which consists of all of the regions of $\mathcal{M}$ except $D$, all of the edges of $\mathcal{M}$ except those on $\alpha_D$, and all of the vertices of $\mathcal{M}$ except those which are interior vertices on $\alpha_D$.

Similarly, if $D$ is appended on the top side of $\mathcal{M}$, but is not appended on the bottom side of $\mathcal{M}$, write $\omega_M = \mu_0 \omega_D \mu_1$. Then $\mathcal{M} - D$ is the feathery submap of $\mathcal{M}$ which has boundary walk $\alpha_M (\mu_0 \alpha_D \mu_1)^{-1}$. The reader should note that we have made a rather arbitrary choice about which side of $D$ to include as part of $\mathcal{M} - D$ in the case where $D$ is appended on both sides of $\mathcal{M}$.

More generally, suppose that $S = \{D_1, D_2, \ldots, D_q\}$ is any set of appended regions of $\mathcal{M}$. Write $\alpha_i$ and $\omega_i$ for $\alpha_{D_i}$ and $\omega_{D_i}$. We may renumber so that

1. For $0 \leq p < q$, $D_1, D_2, \ldots, D_p$ are appended on the bottom side of $\mathcal{M}$,
2. $D_{p+1}, \ldots, D_q$ are appended on the top side of $\mathcal{M}$, but not on the bottom side of $\mathcal{M}$,
3. $\alpha_M = \lambda_0 \alpha_1 \lambda_1 \alpha_2 \ldots \alpha_p \lambda_p$, and
4. $\omega_M = \mu_p \omega_{p+1} \mu_{p+1} \ldots \omega_q \mu_q$

where the $\lambda_i$ and $\mu_i$ are positive or empty walks. Then we define $\mathcal{M} - S$ to be the feathery submap of $\mathcal{M}$ which has boundary walk

$$(\lambda_0 \omega_1 \lambda_1 \omega_2 \ldots \omega_p \lambda_p)(\mu_p \alpha_{p+1} \mu_{p+1} \ldots \alpha_q \mu_q)^{-1}.$$ 

Observe that when $S = \{D\}$, then $\mathcal{M} - S = \mathcal{M} - D$.

If $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k$ are feathers, then we define $\mathcal{M}_1 \mathcal{M}_2 \ldots \mathcal{M}_k$ to be the feather that is obtained by identifying the terminal vertex of $\mathcal{M}_i$ with the initial vertex of $\mathcal{M}_{i+1}$ for $1 \leq i \leq k - 1$. We will also use the notation $\bigvee_{i=1}^k \mathcal{M}_i$ for $\mathcal{M}_1 \mathcal{M}_2 \ldots \mathcal{M}_k$, and the notation $\mathcal{M}_1 \uplus \mathcal{M}_2$ for $\mathcal{M}_1 \mathcal{M}_2$. In some applications, we may have feathers $\mathcal{M}_i$ indexed by a set whose order is irrelevant and we will write $\bigvee_{\tau} \mathcal{M}_\tau$ and take this to mean any one of several possible feathers depending upon the order chosen.

The vertex $v$ in the feather $\mathcal{M}$ is a **separating vertex in** $\mathcal{M}$ if $\mathcal{M} - v$ is disconnected. Every separating vertex in $\mathcal{M}$ must be on both the top and bottom side of $\mathcal{M}$. A feather is **nonseparable** if it has no separating vertices. A **block** of a feather is a maximal nonseparable submap. A block of a feather is itself a feather. For any feather $\mathcal{M}_i$, we may write $\mathcal{M} = \mathcal{M}_1 \mathcal{M}_2 \ldots \mathcal{M}_k$ where the $\mathcal{M}_i$ are all of the blocks of $\mathcal{M}$ and their common vertices are all of the separating vertices of $\mathcal{M}$.

For any map $\mathcal{M}$, the set $W^+$ of positive walks in $\mathcal{M}$ is a partial groupoid (i.e. concatenation is a partially defined associative binary operation) generated by the
edges of $\mathcal{M}$. If $S$ is any semigroup, a function $\phi$ from $W^+$ to $S$ is a \textbf{labelling of $\mathcal{M}$ with values in $S$} when $\phi$ is a partial homomorphism from $W^+$ to $S$. (I.e. $\phi(\pi \sigma) = \phi(\pi)\phi(\sigma)$ whenever the concatenation $\pi \sigma$ is defined.) Of course, $\phi$ is determined once it is defined on the edges which generate $W^+$ and any choice of labels for the edges will produce a well-defined labelling of $\mathcal{M}$ with values in $S$.

A \textbf{diagram over the semigroup} $S$ is a pair $(\mathcal{M}, \phi)$ where $\mathcal{M}$ is a map and $\phi$ is a labelling with values in $S$. If $\mathcal{N}$ is a submap of $\mathcal{M}$, it will not result in any confusion if we also use the notation $\phi$ for the restriction, $\phi|_{\mathcal{N}}$, of $\phi$ to $\mathcal{N}$. For this paper, $S$ will always be the free semigroup $F_X$ on some set $X$, and we will refer to $(\mathcal{M}, \phi)$ as a \textbf{diagram over $X$}. Often the function $\phi$ is obvious from context and we refer simply to a diagram $\mathcal{M}$. Having suppressed $\phi$, we then write $\overline{\mathcal{M}}$ for the label $\phi(\sigma)$ on a positive walk $\sigma$. A vertex in a map $\mathcal{M}$ is \textbf{superfluous} if its indegree and its outdegree are both 1. Throughout this paper, the maps used will have enough superfluous vertices that we can label each edge by a letter of $X$ rather than a longer word of $F_X$. We use $|u|$ for the length of a word in the free semigroup $F_X$, and we use $|\sigma|$ for the length of a positive walk in $\mathcal{M}$. Since we label each edge by a letter, we will always have $|\sigma| = |\overline{\sigma}|$. This convention on superfluous vertices is useful in this paper. In several of the other papers using derivation diagrams, it is useful to assume instead that maps have no superfluous vertices and conclude only that $|\sigma| \leq |\overline{\sigma}|$.

If $Q$ is a two-sided diagram or if $Q$ is a two-sided region in a diagram, then the \textbf{boundary label of $Q$} is the ordered pair $(\overline{\alpha}_Q, \overline{\omega}_Q)$ of labels of the sides of $Q$.

Suppose that $(X; R)$ is a semigroup presentation, where $X$ is an alphabet, $F_X$ is the free semigroup on $X$, and $R \subseteq F_X \times F_X$ is a set of defining relators. Let the feather $\mathcal{M}$ be a diagram over $X$. Then $\mathcal{M}$ is a \textbf{derivation diagram over $(X; R)$} for $(\mathcal{M}, \phi)$ if at least one of $(\overline{\alpha}_D, \overline{\omega}_D)$ or $(\overline{\omega}_D, \overline{\alpha}_D)$ is in $R$ for each region $D$ of $\mathcal{M}$.

If $\mathcal{M}$ is a feather or $(\mathcal{M}, \phi)$ is a diagram, we use the notation $|\mathcal{M}|$ for the number of regions in $\mathcal{M}$. Let $(X; R)$ be some fixed semigroup presentation. A derivation diagram $(\mathcal{M}, \phi)$ over $(X; R)$ is a \textbf{minimal} derivation diagram over $(X; R)$ for $(\mathcal{M}, \phi)$ if $|\mathcal{M}| \leq |\mathcal{N}|$ whenever $(\mathcal{N}, \xi)$ is also a derivation diagram over $(X; R)$ for $(\mathcal{M}, \phi)$.

\textbf{Theorem 1.6 (Remmers).} \textit{Suppose that $(\mathcal{M}, \phi)$ is a minimal derivation diagram over $(X; R)$. If the submap $\mathcal{N}$ of $\mathcal{M}$ is in $R$, then $(\mathcal{N}, \phi)$ is a minimal derivation diagram over $(X; R)$ for $(\phi(\alpha_N), \phi(\omega_N))$.}

\textbf{Proof.} If $(\mathcal{N}, \phi)$ is not minimal, there is a derivation diagram $(\mathcal{N}', \xi)$ over $(X; R)$ having the same boundary label as $(\mathcal{N}, \phi)$. We could then replace the submap $\mathcal{N}$ of $\mathcal{M}$ by $\mathcal{N}'$ and contradict the minimality of $(\mathcal{M}, \phi)$.

A \textbf{directed walk} in a map $\mathcal{M}$ is a walk which is either a positive walk or the empty walk at some vertex of $\mathcal{M}$.

If $\mathcal{M}$ is any map in $R$, Remmers has described the following natural partial orders on the vertices of $\mathcal{M}$ and on the edges of $\mathcal{M}$:

(1) For vertices $u$ and $v$, $u < v$ if there is a positive walk in $\mathcal{M}$ from $u$ to $v$.
(2) For edges $e$ and $f$, $e < f$ if there is a directed walk from the terminal endpoint of $e$ to the initial endpoint of $f$. 
It is straightforward to verify, by induction on the number of regions, that any two vertices \( u \) and \( v \) in a feather \( \mathcal{M} \) have both a greatest lower bound, \( \text{glb}_{\mathcal{M}}(u,v) \), and a least upper bound, \( \text{lub}_{\mathcal{M}}(u,v) \). We shall make use of these orders in the following sections. For convenience of reference, we will refer to these orders on the vertices and edges as Remmers’ orders.

A derivation diagram \( (\mathcal{M}, \phi) \) is **quasiminimal** if every block \( \mathcal{N} \) of \( \mathcal{M} \) is a minimal derivation diagram for \( (\phi(\alpha_{\mathcal{N}}), \phi(\omega_{\mathcal{N}})) \). A derivation diagram \( (\mathcal{M}, \phi) \) is **strongly quasiminimal** if \( (\mathcal{M}, \phi) \) is quasiminimal and for each block \( \mathcal{N} \) of \( \mathcal{M} \), every quasiminimal derivation diagram \( (\mathcal{N}', \phi') \) for \( (\phi(\alpha_{\mathcal{N}}), \phi(\omega_{\mathcal{N}})) \) has only one block. In the following theorem, the equivalence of i), ii), and iii) is due to Remmers.

**Theorem 1.7.** Suppose that \( \langle X; R \rangle \) presents a semigroup \( S \) and that \( w \) and \( w' \) are words on the alphabet \( X \). Then the following are equivalent:

i) The words \( w \) and \( w' \) represent the same element of \( S \).

ii) There is a derivation diagram over \( \langle X; R \rangle \) for \( (w, w') \).

iii) There is a minimal derivation diagram over \( \langle X; R \rangle \) for \( (w, w') \).

iv) There is a quasiminimal derivation diagram over \( \langle X; R \rangle \) for \( (w, w') \).

v) There is a strongly quasiminimal derivation diagram over \( \langle X; R \rangle \) for \( (w, w') \).

**Proof.** i) \( \Rightarrow \) ii) If \( w \) and \( w' \) represent the same element of \( S \), we may use any sequence of elementary transitions from \( w \) to \( w' \) to construct a map in the set \( \mathcal{F} \) labelled as a derivation diagram over \( \langle X; R \rangle \) for \( (w, w') \).

ii) \( \Rightarrow \) iii) If the derivation diagram \( (\mathcal{M}, \phi) \) is not minimal, then there is a derivation diagram \( (\mathcal{N}, \xi) \) for \( (w, w') \) with \( |\mathcal{N}| < |\mathcal{M}| \). If \( (\mathcal{N}, \xi) \) is not minimal, we repeat this: since \( \mathcal{M} \) has only a finite number of regions, we must eventually reach a minimal diagram.

iii) \( \Rightarrow \) iv) By Theorem 1.6, the blocks of a minimal derivation diagram are minimal.

iv) \( \Rightarrow \) v) Let \( k \) be the smaller of \( |w| \) and \( |w'| \). Then no derivation diagram over \( \langle X; R \rangle \) for \( (w, w') \) can have more than \( k \) blocks. Suppose \( (\mathcal{M}, \phi) \) is a quasiminimal derivation diagram over \( \langle X; R \rangle \) for \( (w, w') \). If \( (\mathcal{M}, \phi) \) is not strongly quasiminimal, then for some block \( \mathcal{N} \) of \( \mathcal{M} \), there is a quasiminimal derivation diagram \( (\mathcal{N}', \phi') \) over \( \langle X; R \rangle \) for \( (\phi(\alpha_{\mathcal{N}}), \phi(\omega_{\mathcal{N}})) \), where the map \( \mathcal{N}' \) has more than one block. We may replace the submap \( \mathcal{N} \) of \( \mathcal{M} \) by \( \mathcal{N}' \) to obtain a quasiminimal derivation diagram \( (\mathcal{M}', \xi) \) for \( (w, w') \) where \( \mathcal{M}' \) has more blocks than \( \mathcal{M} \). If \( (\mathcal{M}', \xi) \) is not strongly quasiminimal, we repeat this process, but we must eventually reach a strongly quasiminimal derivation since \( k \) is a bound for the number of blocks in \( \mathcal{M}' \).

ii), iii), iv), or v) \( \Rightarrow \) i) We may use any derivation diagram for \( (w, w') \) to construct a sequence of elementary transitions from \( w \) to \( w' \).

**Example 1.8.** Let \( \langle X; R \rangle \) be the semigroup presentation

\[
(a, b, c; \ abb = cbb, \ bba = bbc, \ aba = cbc).
\]

Then the diagram \( \mathcal{M} \) below is a strongly quasiminimal derivation diagram. It is not a minimal derivation diagram since the diagram \( \mathcal{M}' \) is also a derivation diagram over \( \langle X; R \rangle \) for \( (bbabb, bbcbcbb) \).
Example 1.9. Let \( \langle X; R \rangle \) be the semigroup presentation
\[
\langle a, b, c, d, e; \ ae = bcd, \ ea = bcd, \ ab = dc, \ cd = ba \rangle.
\]
Then in following figure both \( M_1 \) and \( M_2 \) are minimal (and quasiminimal) derivation diagrams for \( abcd = dcba \), but only \( M_2 \) is strongly quasiminimal.

Lemma 1.10. If \( M \in R \) and \( \mathcal{N} \) is a submap of \( M \) which is also in \( R \) and which contains every appended region of \( M \), then \( \mathcal{N} \) contains every region of \( M \).

Proof. Observe first that it will suffice to prove the lemma in the case where \( M \) has only one block. We assume that we are in this case. Let \( \delta_1 \) be a positive or empty path from the initial vertex of \( M \) to the initial vertex of \( \mathcal{N} \) and let \( \delta_2 \) be a positive or empty path from the terminal vertex of \( \mathcal{N} \) to the terminal vertex of \( M \). Then the closed walk \( \delta_1 \omega_\mathcal{N} \delta_2 (\omega_M)^{-1} \) is the closed two-sided boundary for a two-sided submap \( \mathcal{T} \) of \( M \). It follows from Corollary 1.2 that \( \mathcal{T} \in R \). If there are regions in \( T \), then by Lemma 1.5, there is a region \( D \) of \( T \) which is appended on the top side of \( T \). The region \( D \) is then also an appended region on the top side of \( M \). By hypothesis, \( D \) is then a region of \( \mathcal{N} \), but this is impossible because \( \omega_\mathcal{N} \) is the bottom side of \( T \). We conclude that \( \mathcal{T} \) is regionless and that \( \delta_1 \omega_\mathcal{N} \delta_2 = \omega_M \). A similar argument shows that \( \delta_1 \alpha_\mathcal{N} \delta_2 = \alpha_M \). Since \( M \) has only one block and \( \delta_1 \) and \( \delta_2 \) are on both the bottom and top sides of \( M \), the paths \( \delta_1 \) and \( \delta_2 \) are empty. Then \( \omega_\mathcal{N} = \omega_M \) and \( \alpha_\mathcal{N} = \alpha_M \). We conclude that \( \mathcal{N} = M \) in the case where \( M \) has only one block.

We introduce the notion of coherent pairs of regions in a feathery map. These will be useful later. Since our inductive construction of feathery maps starts at the bottom (rather than at the top) our terminology for coherent pairs will also reflect a “start-at-the-bottom” bias.

An ordered pair \( (D_1, D_2) \) of regions in a feathery map \( M \) is a left-right coherent pair if there are positive walks \( \omega_1, \mu, \) and \( \alpha_2 \) in \( M \) such that \( \omega_{D_1} = \omega_1 \mu \) and
The ordered pair \((D_1, D_2)\) of regions in a feathery map \(M\) is a **matched coherent pair** if \(\omega D_1 = \alpha D_2\), an **expansive coherent pair** if \(\omega D_1\) is a proper segment of \(\alpha D_2\), and a **contractive coherent pair** if \(\alpha D_2\) is a proper segment of \(\omega D_1\).

**Example 1.11.** The pair of regions on the left is an lr coherent pair and the pair of regions on the right is an rl coherent pair.

\[
\begin{array}{cc}
\omega, & \mu \\
D_1 & D_2 \\
\alpha_2 & \alpha_2 \\
\end{array}
\]

The pair on the left is matched, the pair in the middle is expansive, and the pair on the right is contractive.

In the next three maps, the regions \(D_1\) and \(D_2\) do not form a coherent pair.

A **coherent pair** of regions is an ordered pair of regions that is left-right, right-left, matched, expansive, or contractive coherent. We note that if \((D_1, D_2)\) is a coherent pair of regions in a feathery map \(M\) then (1) there is a feathery submap \(N\) of \(M\) which consists of just \(D_1, D_2\) and the edges and vertices on the boundaries of these two regions, and (2) there is at least one edge \(e\) which is in both \(\omega D_1\) and \(\alpha D_2\).

Conversely, if (1) and (2) hold, then by considering the provenance of the initial and terminal endpoints of \(N\), we see that \((D_1, D_2)\) must be a coherent pair.

**Lemma 1.12.** Suppose that \(M\) is a feathery map and that \(D_0\) is any region of \(M\).

1. Either \(D_0\) is appended on the bottom side of \(M\) or else there is a region \(D_-\) of \(M\) such that \((D_-, D_0)\) is a coherent pair.
2. Either \(D_0\) is appended on the top side of \(M\) or else there is a region \(D_+\) of \(M\) such that \((D_0, D_+)\) is a coherent pair.

**Proof.** We prove (1) by induction on \(|M|\); the proof of (2) is dual. If \(M\) has only one region that region is necessarily appended on the bottom side of \(M\). Assume that \(M > 1\). If \(D_0\) is appended on the bottom side of \(M\) we are done, so assume that \(D_0\) is not an appended region on the bottom side of \(M\). By Lemma 1.5, there is a region \(D\) of \(M\) which is appended on the bottom side of \(M\). By induction, the lemma is true for \(D_0\) as a region of \(M - D\). If \((D_-, D_0)\) is a coherent pair in \(M - D\) then this is also a coherent pair in \(M\). If \(D_0\) is appended on the bottom side of \(M - D\), then \((D, D_0)\) is a coherent pair in \(M\).
2 Alphabetically Constructed Relators

For natural numbers \( s < \ell \), an \((s,\ell)\) map is a map in \( \mathbb{R} \) in which each region has a short side with \( s \) edges and a long side with \( \ell \) edges. Given an alphabet \( X \) and words \( w_s \) and \( w_\ell \) on \( X \) having respective lengths \( s \) and \( \ell \), the relator \( w_s = w_\ell \) is an \((s,\ell)\) relator and the semigroup presentation \( \langle X; w_s = w_\ell \rangle \) is an \((s,\ell)\) presentation.

Assume that \( X \) is a finite or countable, ordered alphabet: \( X = \{x_1, x_2, \ldots \} \), with \( x_1 < x_2 \ldots \). A word \( w \) on \( X \) is said to be alphabetically constructed if whenever \( x_j \) occurs in \( w \) and \( i < j \), then at least one occurrence of \( x_i \) in \( w \) precedes the first occurrence of \( x_j \). An \((s,\ell)\) relator \( w_s = w_\ell \) is alphabetically constructed if the concatenated word \( w_sw_\ell \) is.

Example 2.1. On the ordered alphabet \( \{a, b, c, d, e\} \), the words \( abac, ab^2aba^3cab, \) and \( a^2babacadb \) are alphabetically constructed, but \( acab, bcd, \) and \( acad \) are not.

Proposition 2.2. If \( Y \) is any finite alphabet and \( w \) is any word on \( Y \), then there is an order on \( Y \) for which \( w \) is alphabetically constructed. For the letters of \( Y \) which occur in \( w \), this order is unique.

Proof. For letters \( x \) and \( y \) which occur in \( w \), let \( x < y \) if the first occurrence of \( x \) in \( w \) precedes the first occurrence of \( y \) in \( w \). All letters that do not occur in \( w \) follow those that do occur.

Corollary 2.3. If a semigroup has an \((s,\ell)\) presentation \( \langle Y; w_s = w_\ell \rangle \), then there is an order on \( Y \) for which \( w_s = w_\ell \) is alphabetically constructed.

Let \( x_1 < x_2 < x_3 \ldots \) be a given order for the alphabet \( X \). Then any other order \( x_{\sigma 1} < x_{\sigma 2} < x_{\sigma 3} \ldots \) corresponds to a permutation \( \sigma \) of \( X \). The permutation \( \sigma \) in turn induces an automorphism, which we will also denote by \( \sigma \), of the free semigroup \( F_X \) via \( (x_{i_1}x_{i_2} \ldots x_{i_n})\sigma = x_{\sigma i_1}x_{\sigma i_2} \ldots x_{\sigma i_n} \).

Proposition 2.4. If \( w \) is a word on \( X \) which is alphabetically constructed for the order \( x_{\sigma 1} < x_{\sigma 2} < x_{\sigma 3} \ldots \), then \( w\sigma^{-1} \) is alphabetically constructed for the order \( x_1 < x_2 < x_3 \ldots \). If \( w \) is also alphabetically constructed for the order \( x_{\tau 1} < x_{\tau 2} < x_{\tau 3} \ldots \), then \( w\tau^{-1} = w\sigma^{-1} \).

Proof. Write \( w\sigma^{-1} = x_{i_1}x_{i_2} \ldots x_{i_{\ell}} x_{j_1} \ldots x_{j_r} \). Then

\[
  w = (w\sigma^{-1})\sigma = x_{\sigma i_1}x_{\sigma i_2} \ldots x_{\sigma i_{\ell}}x_{\sigma j_1} \ldots x_{\sigma j_r},
\]

hence \( x_i \) precedes \( x_j \) in \( w\sigma^{-1} \) if and only if \( x_{\sigma i} \) precedes \( x_{\sigma j} \) in \( w \). If \( r \) distinct letters occur in \( w \), then \( \sigma i \in \{\sigma 1, \sigma 2, \ldots, \sigma r\} \), so \( 1 \leq i \leq r \). By Proposition 2.2, \( \sigma i = \tau i \) for \( 1 \leq i \leq r \), hence

\[
  w = (w\sigma^{-1})\sigma = x_{\sigma i_1}x_{\sigma i_2} \ldots x_{\sigma i_n} x_{\tau i_1}x_{\tau i_2} \ldots x_{\tau i_n} = (w\sigma^{-1})\tau,
\]

and \( w\tau^{-1} = w\sigma^{-1} \).

For a fixed order \( x_1 < x_2 < x_3 \ldots \) of \( X \), define a function \( A \) on the set \( F_X \times F_X \) by \( A(w, w') = (w\sigma, w'\sigma) \) for any permutation \( \sigma \) of \( X \) such that the word \( (ww')\sigma \) is alphabetically constructed for the given order on \( X \).
Corollary 2.5. For any \((s, \ell)\) presentation \(\langle X; (w_s, w_\ell) \rangle\) on an ordered alphabet \(X\), the presentation \(\langle X; A(w_s, w_\ell) \rangle\) has an alphabetically constructed \((s, \ell)\) relator, presents the same semigroup as \(\langle X; (w_s, w_\ell) \rangle\), and is effectively obtained from the presentation \(\langle X; (w_s, w_\ell) \rangle\).

The following combinatorial observation is mostly peripheral to the main results of this paper, but it will help illustrate some later remarks.

Proposition 2.6. If \(S_{n,r}\) is the number of alphabetically constructed words of length \(n\) on \(r\) distinct letters, then \(S_{n,r}\) is the Stirling number of the second kind.

Proof. Define a partition of the \(n\) positions in the word by assigning two positions to the same subset if and only if the same letter occurs in both positions. Stirling numbers of the second kind count the number of partitions that are possible.\([1]\)

An obvious consequence of Proposition 2.6 is that for \(n = s + \ell\), the number of alphabetically constructed \((s, \ell)\) relators is the Bell number \(\sum_{r=1}^{\infty} S_{n,r}\).\([1]\)

If \(S\) is any set and \(\rho\) is an equivalence relation on \(S\), then for elements \(s\) and \(t\) of \(S\), we use the notations \(s\rho t\) and \((s, t) \in \rho\) interchangeably. We use the usual notation \([s]_{\rho} = \{t \mid s\rho t\}\) and \(S/\rho = \{[s]_{\rho} \mid s \in S\}\). We write \([s]_{\rho}\) for \([s]_{\rho}\) whenever context renders this unambiguous.

When \(S\) is a finite ordered set and \(\rho\) is an equivalence relation on \(S\), we define an induced order on \(S/\rho\) by \([x] < [y]\) if the first element in \([x]\) is less than the first element in \([y]\). Now fix natural numbers \(s\) and \(\ell\) with \(s < \ell\) and let \(Z = \{z_1, z_2, \ldots, z_{s+\ell}\}\) be an auxiliary ordered set. We regard \(Z\) as a set of positions available for letters in \((s, \ell)\) relator. Given a relation \(\rho\) on \(Z\), we regard \(Z/\rho\) as an initial segment of an ordered alphabet \(X\). In more detail, suppose that \(r = (w_s, w_\ell)\) is an \((s, \ell)\) relator on \(X\) in which \(m\) distinct letters of \(X\) occur. Write \(w_s = y_1 y_2 \cdots y_s\) and \(w_\ell = y_{s+1} y_{s+2} \cdots y_{s+\ell}\) where each \(y_i\) is a letter of \(X\). Define a relation \(\rho_r\) on \(Z\) by \(\rho_r = \{(z_j, z_k) \mid y_j\) and \(y_k\) in \(r\) are the same letter of \(X)\). Then \(\rho_r\) is an equivalence relation on \(Z\) having \(m\) distinct equivalence classes. Conversely, given an equivalence relation \(\rho\) on \(Z\), define words \(w_s\) and \(w_\ell\) on the ordered alphabet \(Z/\rho\) by \(w_s = [z_1] [z_2] \cdots [z_s]\), and \(w_\ell = [z_{s+1}] [z_{s+2}] \cdots [z_{s+\ell}]\). Then the relator \(r(\rho) = (w_s, w_\ell)\) is an alphabetically constructed relator for the induced order on the alphabet \(Z/\rho\) and the distinct letters of \(r(\rho)\) correspond to the \(\rho\)-equivalence classes. Partly for notational convenience, we regard \(Z/\rho\) as an initial segment of the ordered alphabet \(X\) and write \(x_i\) for the \(i^{th}\) letter in the induced order on \(Z/\rho\).

Proposition 2.7. If \(\rho\) is an equivalence relation on \(Z\), then \(\rho = \rho_r(\rho)\). If \(r\) is an \((s, \ell)\) relator on \(X\), then the number of distinct letters of \(X\) that occur in \(r\) is the same as the number of letters in \(Z/\rho_r\). If we use the induced order on \(Z/\rho_r\) to identify \(Z/\rho_r\) with the initial segment of \(X\), then \(r(\rho_r) = A(r)\). Thus there is a bijection between alphabetically constructed \((s, \ell)\) relators on \(X\) and equivalence relations on \(Z\).

Proof. For the first statement, \((z_i, z_j) \in \rho_r(\rho)\) if and only if \([z_i]\) and \([z_j]\) in the relator \(r(\rho)\) are the same letter of the alphabet \(Z/\rho\); the latter occurs if and only if \((z_i, z_j) \in \rho\). For the second statement, each \(\rho_r\)-equivalence class corresponds to
some alphabetically constructed relator \( r \) with \( r(\rho) \) for an alphabetically constructed relator \( r \). For a permutation \( \sigma \) of \( X \) and a \((s, \ell)\) relator \( r = (w_s, w_\ell) \), write \( r\sigma \) for the \((s, \ell)\) relator \((w_s, w_\ell)\). Then for any such \( \sigma \) and any \( r \), \( \rho_r = \rho_{r\sigma} \), since \( \rho_r \) records only the positions at which the same letters occur. In particular, \( \rho_r = \rho_{\rho(A)} \), so \( r(\rho_r) = r(\rho_{\rho(A)(r)}) = A(r) \).

Suppose that \( r \) is an alphabetically constructed \((s, \ell)\) relator, that \( \rho \) is the equivalence relation on \( Z \) which corresponds to \( r \), and that \( x \) is a letter of \( X \) which occurs in \( r \). Using reasonably standard notation, we define the restriction of \( \rho \) to \( x \) by

\[
\rho_{\mid x} = \{ (z_i, z_j) \mid [z_i]_\rho = [z_j]_\rho = x \} \cup \{ (z_k, z_k) \}
\]

Then it is easy to see that \( \rho_{\mid x} \) is an equivalence relation on \( Z \), that \( \rho_{\mid x} \subseteq \rho \), and that \( \rho = \cup_{x \in X} \rho_{\mid x} \). We will also find it useful to define \( J(x, r) \) to be the set of indices for letters of \( r \) which have value \( x \). That is, \( J(x, r) = \{ i \mid x_i = x \} \). If \( x \) is the index set of \( \rho \) in \( S \).

Example 2.8. For the \((2, 5)\) relator \( a^2 = bcabc \), we have \( J(a, r) = \{ 1, 2, 5 \} \), \( J(b, r) = \{ 3, 6 \} \), and \( J(c, r) = \{ 4, 7 \} \). Also,

\[
\rho_{\mid b} = \{ (z_3, z_6), (z_6, z_3) \} \cup \{ (z_k, z_k) \}_{1 \leq k \leq 7}, \quad \rho_{\mid c} = \{ (z_4, z_7), (z_7, z_4) \} \cup \{ (z_k, z_k) \}_{1 \leq k \leq 7}
\]

and \( \rho_{\mid a} = \{ (z_i, z_j) \}_{i, j \in J(a, r)} \cup \{ (z_k, z_k) \}_{1 \leq k \leq 7} \)

The set of equivalence relations on \( Z \) has a standard lattice structure given by \( \rho_1 \leq \rho_2 \) if \( \rho_1 \subseteq \rho_2 \), \( \rho_1 \land \rho_2 = \rho_1 \cap \rho_2 \), and \( \rho_1 \lor \rho_2 \) is the smallest equivalence relation that contains the set \( \rho_1 \cup \rho_2 \). Since we have a bijection between the equivalence relations on \( Z \) and the set of alphabetically constructed \((s, \ell)\) relators, we may regard the latter set as a lattice which is isomorphic to the lattice of equivalence relations.

Proposition 2.9. For \( i = 1, 2 \), let \( r_i \) be an alphabetically constructed \((s, \ell)\) relator, let \( X_i \) be the initial segment of \( X \) consisting of letters that occur in \( r_i \), let \( \rho_i \) be the equivalence relation on \( Z \) that corresponds to \( r_i \), and let \( S_i \) be the semigroup presented by \((X_i; r_i)\). Identify \( X_i \) with \( Z/\rho_i \). Then \( r_1 \leq r_2 \) if and only if the rule \( ([z_j]_{\rho_1})f = [z_j]_{\rho_2} \) determines a well-defined homomorphism from \( S_1 \) onto \( S_2 \).

Proof. Observe that \( r_1 \leq r_2 \iff \rho_1 \subseteq \rho_2 \iff f \) is a well-defined function on the alphabet \( Z/\rho_1 \). Once \( f \) is well-defined on \( Z/\rho_1 \), it is clear that \( r_1f = r_2 \) and that \( f \) is onto.

Example 2.10. If \( S_1 \) is presented by \( \langle a, b, c; ab = abcab \rangle \), and \( S_2 \) is presented by \( \langle a, b, c; a^2 = a^2b^2 \rangle \), then \( r_1 \leq r_2 \) and setting \( af = a, bf = a, \) and \( cf = b \) determines a well-defined homomorphism from \( S_1 \) onto \( S_2 \).
3 Achievable Relators

We wish to define, for each \((s, \ell)\) map \(M\), a naturally occurring alphabetically constructed \((s, \ell)\) relator induced by \(M\). This relator will be called the relator achieved by \(M\). Basically, we regard the map as a template to which we fit the most intricate relator possible.

An edge of a feathery map \(M\) is an interior edge of \(M\) if it is on the boundary of two regions of \(M\). Write \(\mathcal{I}\) or when necessary, \(\mathcal{I}_M\), for the set of interior edges of the feathery map \(M\).

Suppose that \(M\) is an \((s, \ell)\) map, that \(D\) is a region in \(M\), and that \(e\) is an edge of \(D\). Using Remmers’ ordering of the edges of \(M\), we define the index of \(e\) with respect to \(D\) to be \(i\) if \(e\) is the \(i^{th}\) edge on the short side of \(D\) and to be \(s + i\) if \(e\) is the \(i^{th}\) edge on the long side of \(D\). This merely mirrors our order for the set \(Z\) of positions for letters in \((s, \ell)\) relators.

Let \(E\) be any subset of \(Z\). We define a relation \(R_E\) on \(Z\) by \(R_E = \{(z_i, z_j) \mid \text{There are regions } D \text{ and } D' \text{ of } M \text{ having a common edge } e \in E \text{ such that the index of } e \text{ with respect to } D \text{ is } i \text{ and the index of } e \text{ with respect to } D' \text{ is } j\}\). Let \(\rho_E\) be the equivalence relation on \(Z\) that is generated by \(R_E\). Then the \((s, \ell)\) relator \(r(\rho_E)\) is the relator that is achieved by the subset \(E\) of interior edges of \(M\). We will abbreviate \(r(\rho_E)\) by \(r_E\). Generally, we will be interested in the case where \(E = \mathcal{I}\). In this case, we will use the notation \(R_M, \rho_M\), and \(r_M\). The relator that is achieved by \(M\) is defined to be \(r_M\). When the \((s, \ell)\) map \(M\) is clear from context, we may further abbreviate to \(R, \rho, \) and \(r\). If the \((s, \ell)\) relator \(r\) is the relator that is achieved by some \((s, \ell)\) map, then we say that \(r\) is an achievable relator.

**Example 3.1.** A routine calculation shows that the relator \(a^2 = a^2b\) is the relator achieved by the following \((2, 3)\) map.

Let \(M\) be a map in \(R = F\). An edge is constrained if it is on the boundary of at least one region. An edge of \(M\) is unconstrained if it is not on the boundary of any region of \(M\). Every unconstrained edge in \(M\) is a block of \(M\) and a block of \(M\) that contains a region of \(M\) can contain no unconstrained edges of \(M\).

Let \(U_M\) be the set of unconstrained edges of \(M\), and let \(C = \{(D, e) \mid \text{D is a region of } M \text{ and } e \text{ is on the boundary of } D\}\). A bilabelling of \(M\) with values in the semigroup \(S\) is a function \(\phi\) from \(U \cup C\) to \(S\). In this paper, \(S\) will always be the free semigroup on a finite ordered alphabet \(X\). A bilabelling is consistent if \(\phi(D_1, e) = \phi(D_2, e)\) whenever \(e\) is an interior edge on the boundary of two regions \(D_1\) and \(D_2\). It is clear that every labelling of \(M\) induces a consistent bilabelling, and that every consistent bilabelling of \(M\) induces a labelling.

Let \(M\) be an \((s, \ell)\) map. Let \(r = (w_s, w_\ell)\) be an \((s, \ell)\) relator on an ordered alphabet \(X\). Write \(w_s = y_1y_2\ldots y_s\) and \(w_\ell = y_{s+1}y_{s+2}\ldots y_{s+\ell}\). We say that a bilabelling \(\phi\) corresponds to \(r\) if \(\phi(D, e) = y_e\) whenever the index of \(e\) with respect to \(D\) is \(i\). (On \(U\), \(\phi\) may have arbitrary values in \(X\).) Since the consistency of bilabelling depends on the interior edges, if one bilabelling that corresponds to \(r\) is consistent, then every bilabelling that corresponds to \(r\) is consistent.
Proposition 3.2. Suppose that \( \mathcal{M} \) is an \((s, \ell)\) map, that \( r_\mathcal{M} \) is the relator that is achieved by \( \mathcal{M} \), and that \( r \) is an alphabetically constructed relator which is equal to or greater than \( r_\mathcal{M} \) in the lattice of alphabetically constructed relators. Then any bilabelling of \( \mathcal{M} \) which corresponds to \( r \) is consistent. Conversely, suppose that \( r \) is the least alphabetically constructed relator such that every bilabelling of \( \mathcal{M} \) which corresponds to \( r \) is consistent. Then \( r \) is the relator that is achieved by \( \mathcal{M} \).

Proof. Throughout the proof we write \( \rho \) for \( \rho_r \) and \( \rho_\mathcal{M} \) for \( \rho_{r_\mathcal{M}} \). We also write \( r = (w_s, w_\ell) \) where \( w_s = y_1y_2\ldots y_s \) and \( w_\ell = y_{s+1}y_{s+2}\ldots y_{s+\ell} \).

Suppose that \( r_\mathcal{M} \leq r \). Let \( e \) be any interior edge of \( \mathcal{M} \) and suppose that \( e \) has index \( i \) with respect to some region \( D \) and has index \( j \) with respect to some region \( D' \). Then \((z_i, z_j) \in \rho_\mathcal{M} \). Since \( r_\mathcal{M} \leq r \) by hypothesis, we have \( \rho_\mathcal{M} \subseteq \rho \). It follows that \((z_i, z_j) \in \rho \), so that \( y_i = y_j \) as required.

Recall that \( \mathcal{R}_\mathcal{M} = \{(z_i, z_j) \mid \text{There are regions } D \text{ and } D' \text{ of } \mathcal{M} \text{ having a common edge } e \text{ such that the index of } e \text{ with respect to } D \text{ is } i \text{ and the index of } e \text{ with respect to } D' \text{ is } j\} \). Observe that \( \mathcal{R}_\mathcal{M} \subseteq \rho \) whenever \( r \) is an alphabetically constructed relator for which the corresponding bilabellings are consistent. If \( r \) is the least alphabetically constructed relator for which the corresponding bilabellings of \( \mathcal{M} \) are consistent, then \( \rho \) is the least equivalence relation on \( \mathbb{Z} \) containing \( \mathcal{R}_\mathcal{M} \). Hence \( \rho \) is \( \rho_\mathcal{M} \).

Example 3.3. The relator \( a^2 = b^2a \) is not achievable. If it were achieved by some map \( \mathcal{M} \), it would induce a consistent bilabelling on \( \mathcal{M} \). Since there are just the two occurrences of the letter \( b \) in the relator, there would be some edge \( e \) in \( \mathcal{M} \) which has index three with respect to a region \( D_1 \) and index four with respect to a region \( D_2 \). Consider the rightmost such edge \( e \) and let \( f \) and \( g \) be the edges of \( D_1 \) and \( D_2 \), respectively, which follow \( e \). The edge \( f \) must also have label \( b \). Since \( f \) and \( g \) have a common initial vertex, it follows from the geometric properties of \((s, \ell)\) maps that \( f \) must be the initial edge on the side of some region \( D_3 \) of \( \mathcal{M} \). Since \( f \) has label \( b \), it must be the initial edge on the long side of \( D_3 \). Then \( f \) has index four with respect to \( D_1 \) and index three with respect to \( D_3 \); this contradicts our assumption that \( e \) is the rightmost such edge.

Example 3.4. There are 52 alphabetically constructed \((2, 3)\) relators. We list and number these and illustrate the lattice of alphabetically constructed \((2, 3)\) relators. Only 26 of these relators are achievable. The achievable relators are marked by stars. The verification that the starred relators are achievable follows as in Example 3.1 using \((2, 3)\) maps having 2, 3, or 4 regions. The verification that the remaining
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(2,3) relators are not achievable is generally less complicated than Example 3.3.

\begin{align*}
\star 1 &\quad aa = aaa & \star 14 &\quad ab = bab & 27 &\quad ab = acc & 40 &\quad ab = ccb \\
\star 2 &\quad aa = aab & \star 15 &\quad ab = bba & 28 &\quad ab = bac & \star 41 &\quad ab = ccc \\
\star 3 &\quad aa = aba & \star 16 &\quad ab = bbb & 29 &\quad ab = bbc & \star 42 &\quad aa = bcd \\
\star 4 &\quad aa = abb & \star 17 &\quad aa = abc & \star 30 &\quad ab = bca & 43 &\quad ab = acd \\
\star 5 &\quad aa = baa & 18 &\quad aa = bac & \star 31 &\quad ab = bcb & \star 44 &\quad ab = bcd \\
\star 6 &\quad aa = bab & 19 &\quad aa = bbc & 32 &\quad ab = bcc & 45 &\quad ab = cad \\
\star 7 &\quad aa = bba & \star 20 &\quad aa = bca & 33 &\quad ab = caa & 46 &\quad ab = cbd \\
\star 8 &\quad aa = bbb & \star 21 &\quad aa = bcb & \star 34 &\quad ab = cab & 47 &\quad ab = ccd \\
\star 9 &\quad ab = aaa & 22 &\quad aa = bcc & 35 &\quad ab = cac & \star 48 &\quad ab = cda \\
10 &\quad ab = aab & 23 &\quad ab = acc & 36 &\quad ab = cba & 49 &\quad ab = cdb \\
\star 11 &\quad ab = aba & \star 24 &\quad ab = abc & 37 &\quad ab = cbb & \star 50 &\quad ab = cdc \\
12 &\quad ab = abb & \star 25 &\quad ab = aca & 38 &\quad ab = cbc & 51 &\quad ab = cdd \\
\star 13 &\quad ab = baa & 26 &\quad ab = acb & 39 &\quad ab = cca & \star 52 &\quad ab = cde.
\end{align*}

The lattice of alphabetically constructed (2,3) relators

Lemma 3.5. Suppose that $\mathcal{M}$ and $\mathcal{M}'$ are $(s,\ell)$ maps and that $\mathcal{M} \subseteq \mathcal{M}'$. Then $r_M \leq r_{M'}$. Conversely, suppose that $r$ and $r'$ are achievable $(s,\ell)$ relators with $r \leq r'$ and that $r = r_M$. Then there is an $(s,\ell)$ map $\mathcal{M}'$ with $r' = r_{M'}$ and $\mathcal{M} \subseteq \mathcal{M}'$. 
Proof. We use again the definition $R_M = \{(z_i, z_j) \mid \text{There are regions } D \text{ and } D' \text{ of } M \text{ having a common edge } e \text{ such that the index of } e \text{ with respect to } D \text{ is } i \text{ and the index of } e \text{ with respect to } D' \text{ is } j\}$. Since $\mathcal{M} \subseteq \mathcal{M}'$, we have $R_M \subseteq R_{M'}$, and $\rho_M \subseteq \rho_{M'}$. Conversely if $r = r_M$ and $r'$ is achievable with $r \leq r'$, then there is an $(s, \ell)$ map $N$ with $r' = r_N$. Let $\mathcal{M}'$ be $\mathcal{M} \cup \mathcal{N}$, the $(s, \ell)$ map constructed by identifying the initial vertex of $\mathcal{N}$ with the terminal vertex of $\mathcal{M}$. Since $r \leq r'$, we will have that $R_M \subseteq \rho_M = \rho_r \subseteq \rho_{r'} = \rho_{r'}$. It follows that $R_M \cup R_{M'}$ is contained in $\rho_N$, hence $\rho_{M'} = \rho_{r'}$, and $r_{M'} = r'$.

**Example 3.6.** Given $r \leq r'$ where $r$ and $r'$ are achievable $(s, \ell)$ relators and an $(s, \ell)$ map $\mathcal{M}$ with $r' = r_{M'}$, we cannot, in general, find a submap $M$ of $\mathcal{M}'$ with $r = r_M$. With $r = (ab, cd^2 dc)$ and $r' = (a^2, bc^2 eb)$, it is easy to see that $r$ and $r'$ are achieved by the following maps $\mathcal{M}$ and $\mathcal{M}'$, that $r \leq r'$, but that $r$ is not achieved by any submap of $\mathcal{M}'$.

\[\text{M} \quad \text{M}'\]

**Proposition 3.7.** If $r$ and $r'$ are achievable $(s, \ell)$ relators, then their join, $r \lor r'$ is also achievable.

*Proof.* Suppose that $r$ is achieved by $\mathcal{M}$ and that $r'$ is achieved by $\mathcal{M}'$. Then $\rho_{r \lor r'}$ is the smallest equivalence relation on $Z$ which contains $\rho_M \cup \rho_{M'}$, and hence is the smallest equivalence relation on $Z$ which contains $R_M \cup R_{M'}$. Let $\mathcal{M}_\lor$ be $\mathcal{M} \cup \mathcal{M}'$, the $(s, \ell)$ map constructed by identifying the initial vertex of $\mathcal{M}'$ with the terminal vertex of $\mathcal{M}$. Then $R_{\mathcal{M}_\lor} = R_M \cup R_{M'}$, so $r \lor r'$ is achieved by $\mathcal{M}_\lor$.

**Example 3.8.** While the join of two achievable relators is achievable, their meet is not in general an achievable relator. If $r$ is the $(1, 7)$ relator $(a, abcabca)$ and $r'$ is the $(1, 7)$ relator $(a, bcabc)$, then $r$ and $r'$ are achieved by the respective maps $\mathcal{M}$ and $\mathcal{M}'$ below. A straightforward calculation of $\rho_r$, $\rho_{r'}$, and $\rho_r \land \rho_{r'}$ shows that

\[\rho_r \land \rho_{r'} = \{(z_4, z_7), (z_7, z_4), (z_5, z_8), (z_8, z_5)\} \cup \{(z_i, z_i)\}\]

so that $r \land r'$ is $a = bdefde$ which is easily shown to be not achievable.

\[\text{M} \quad \text{M}'\]

If $r$ is an alphabetically constructed $(s, \ell)$ relator, let $L_r$ be the set of achievable $(s, \ell)$ relators which are equal to or less than $r$. We define the **paradigm** of $r$, $P(r)$ to be the join of the finite set $L_r$. Since the relators of $L_r$ are achievable, $P(r)$ is
an achievable relator. Since \( r \) itself is clearly an upper bound for \( L_r \), we also have that \( P(r) \leq r \). Every achievable relator is its own paradigm.

For further illustrations of the paradigm for a relator, we return to the \((2,3)\) relators of Example 3.4. There, we claimed that \( r_4 : aa = abb \) is not achievable. Inspecting the lattice, we see that \( L_{r_4} = \{ r_{17}, r_{42}, r_{44}, r_{52} \} \), so that \( P(r_4) = r_{17} : \alpha \). Similarly, for \( t_{43} : ab = acd \), we have \( L(t_{43}) = \{ r_{52} \} \) so that \( P(r_{43}) = r_{52} : ab = cde \).

**Theorem 3.9.** Suppose that \( r \) is an alphabetically constructed \((s, \ell)\) relator and that \( P(r) \) is the paradigm of \( r \). If \( M \) is an \((s, \ell)\) map and some bilabelling of \( M \) which corresponds to \( r \) is consistent, than any bilabelling of \( M \) which corresponds to \( P(r) \) is consistent.

**Proof.** Let \( r_M \) be the \((s, \ell)\) relator which is achieved by \( M \). Since a bilabelling of \( M \) which corresponds to \( r \) is consistent, \( r_M \leq r \), by Proposition 3.2. Therefore, the achievable relator \( r_M \) is in the set \( L_r \) of achievable relators which are equal to or less than \( r \). It follows that \( r_M \leq P(r) \), since \( P(r) \) is the join of \( L_r \). Using Proposition 3.2 again, bilabellings of \( M \) which correspond to \( P(r) \) are consistent.

**Theorem 3.10.** Suppose that \( r \) and \( r' \) are alphabetically constructed \((s, \ell)\) relators with \( r' \leq r \). Suppose that \( M \) is an \((s, \ell)\) map having no unconstrained edges and that \( \phi \) and \( \phi' \) are labels on \( M \) such that \((M, \phi)\) is a derivation diagram over \((X; r)\) for the pair \((u, v)\) of words on \( X \) and \((M, \phi')\) is a derivation diagram over \((X; r')\) for the pair \((u', v')\). If \((M, \phi)\) is a minimal derivation diagram, then \((M, \phi')\) is a minimal derivation diagram also.

**Proof.** We use proof by contradiction. Suppose that \((M, \phi')\) is not minimal. Then there is a derivation diagram \((N, \zeta')\) over \((X; r')\) for \((u', v')\) with \(|N| < |M|\). It will suffice to show that we can define a label \( \zeta \) on \( N \) such that \((N, \zeta)\) is a derivation diagram over \((X; r)\) for \((u, v)\), since this will contradict the minimality of \((M, \phi)\).

Since \((N, \zeta')\) is a diagram over \((X; r')\), \( \zeta' \) is induced by a consistent bilabelling which corresponds to \( r' \). By Proposition 3.5, \( r' \geq r_N \). Since \( r \geq r' \), every bilabelling of \( N \) which corresponds to \( r \) will be consistent also. Suppose first that \( N \) has no unconstrained edges. Then there is only one bilabelling of \( N \) which corresponds to \( r \) and it induces a label \( \zeta \) on \( N \). We need to show that \((N, \zeta)\) is a derivation diagram for \((u, v)\). By symmetry, it will suffice to show that \( \zeta(\omega_N) = u \). Because \( \phi'(r_M) = u' = \zeta'(\omega_N) \), \( \alpha_M \) and \( \alpha_N \) are paths of the same length, and we only need to show that \( \zeta(f) = \phi(e) \) if \( f \) is the \( k \text{th} \) edge on \( \alpha_N \) and \( e \) is the \( k \text{th} \) edge on \( \alpha_M \). Suppose that the edge \( e \) has index \( i \) with respect to some region \( D_1 \) of \( M \) and that \( f \) has index \( j \) with respect to some region \( D_2 \) of \( N \). Because \( \phi'(r_M) = u' = \zeta'(\omega_N) \), we have that \( \phi'(\omega_M) = u' = \zeta'(\omega_N) \). Both \((M, \phi')\) and \((N, \zeta')\) are derivation diagrams over \((X; r')\), hence \((z_i, z_j) \in \rho_{r'} \). This insures that the labels of \( \phi(e) \) and \( \zeta(f) \) are the same letter of \( X \).

When \( N \) has unconstrained edges, we still induce \( \zeta \) from a consistent bilabelling which corresponds to \( r \), but we must specify how \( \zeta \) is defined on the unconstrained edges of \( N \). If \( f \) is an unconstrained edge of \( N \), then \( f \) occurs on both the bottom side, \( \omega_N \), and the top side, \( \omega_N \), of \( N \). Suppose \( f \) is the \( j \text{th} \) edge which occurs on \( \omega_N \), and is the \( k \text{th} \) edge which occurs on \( \omega_N \). Let \( e_b \) be the \( j \text{th} \) edge on the
bottom side $\alpha_M$ of $M$, and let $e_i$ be the $i^{th}$ edge on the top side $\omega_M$ of $M$. Then $\phi'(e_b) = \zeta'(f)$ since $\phi'(\alpha_M) = u' = \zeta'(\alpha_X)$, and similarly, $\phi'(e_i) = \zeta'(f)$ since $\phi'(\omega_M) = v' = \zeta'(\omega_X)$. Using $\phi'(e_b) = \phi'(e_i)$ and $r' \leq r$, we argue as above that $\phi(e_b) = \phi(e_i)$. We may then define $\zeta(f) = \phi(e_b) = \phi(e_i)$, and it will follow that $\zeta(\alpha_X) = u$ and $\zeta(\omega_X) = v$.

**Corollary 3.11.** Let $r$ and $r'$ be alphabetically constructed $(s, \ell)$ relators with $r' \leq r$. Suppose that $(M, \phi)$ is a derivation diagram over $\langle X; r \rangle$ for the pair $(u, v)$ of words on $X$ and $(M, \phi')$ is a derivation diagram over $\langle X; r' \rangle$ for the pair $(u', v')$.

If $(M, \phi)$ is a quasiminimal derivation diagram, then $(M, \phi')$ is a quasiminimal derivation diagram also.

**Proof.** We may write $M = M_1M_2 \ldots M_k$, where each $M_i$ is a block of $M$. Each $M_i$ is either a single unconstrained edge of $M$, or else $M_i$ contains no unconstrained edges and $(M_i, \phi)$ is a minimal derivation diagram over $\langle X; r_i \rangle$. When $M_i$ is a single edge, clearly $(M_i, \phi')$ is a minimal derivation diagram over $\langle X; r'_i \rangle$. By Theorem 3.10, $(M_i, \phi')$ is also minimal over $\langle X; r' \rangle$ when $(M_i, \phi)$ is minimal over $\langle X; r \rangle$ and $M_i$ contains no unconstrained edges.

**Theorem 3.12.** Let $r$ be an alphabetically constructed $(s, \ell)$ relator on an ordered alphabet $X$. If we know the paradigm $P(r)$ for $r$ and we have an algorithm which solves the word problem for the semigroup presentation $(X ; P(r))$ then the word problem for the semigroup presentation $(X ; r)$ is also solvable.

**Proof.** Suppose that $u$ and $v$ are words on the alphabet $X$. Write $S$ for the semigroup presented by $\langle X ; r \rangle$ and write $P$ for the semigroup presented by $\langle X ; P(r) \rangle$. By Theorem 1.7, it will suffice to exhibit an algorithm which determines whether or not there is a quasiminimal derivation diagram over $\langle X ; r \rangle$ for $(u, v)$.

We may assume here that the alphabet $X$ is finite. We list all pairs of words $(u', v')$ on $X$ such that $|u'| = |u|$ and $|v'| = |v|$. The number of such pairs is finite. For each such pair, we may use the algorithm which solves the word problem for $\langle X ; P(r) \rangle$ to determine whether or not $u'$ and $v'$ represent the same element of $P$. If $u'$ and $v'$ do represent the same element of $P$, we may use the algorithm for $\langle X ; P(r) \rangle$ to find all possible quasiminimal derivation diagrams $(M, \phi')$ over $\langle X ; P(r) \rangle$ for $(u', v')$. For any such $(M, \phi')$, we have $r_M \leq P(r) \leq r$, so by Proposition 3.2, every bilabelling of $M$ which corresponds to $r$ will be consistent. For each $(M, \phi')$ we obtain derivation diagrams over $\langle X ; r \rangle$ using the bilabelling of $M$ which corresponds to $r$ and using all possible labels on any unconstrained edges of $M$.

We claim that there is a quasiminimal derivation diagram over $\langle X ; r \rangle$ for $(u, v)$ if and only if some diagram in the list of the preceding paragraph is a derivation diagram for $(u, v)$. Clearly if one of the preceding diagrams is a derivation diagram for $(u, v)$ then there are quasiminimal derivation diagrams for $(u, v)$ and we may effectively find one. Conversely, suppose that there is a quasiminimal derivation diagram $M$ over $\langle X ; r \rangle$ for $(u, v)$. Then by Theorem 3.9, the bilabelling of $M$ which corresponds to $P(r)$ is consistent and we will obtain a derivation diagram $(M, \phi')$ over $\langle X ; P(r) \rangle$ for some pair of words $(u', v')$ with $|u'| = |u|$ and $|v'| = |v|$. By Corollary 3.11, this is a quasiminimal derivation diagram, hence the quasiminimal derivation diagram for $(u, v)$ will occur in the list of the previous paragraph.
We wish to improve upon Theorem 3.12 by showing that we always can effectively compute the paradigm \( P(r) \) of an alphabetically constructed \((s, \ell)\) relator \(r\).

4. Local Computations: The Base Case

If \(r\) is an achievable \((s, \ell)\) relator, let \(\rho = r_\rho\) be the corresponding equivalence relation on the auxiliary alphabet \(Z\). Let \(x\) represent some fixed \(\rho\)-equivalence class. Then the \((s, \ell)\) map \(M\) is an \(x\)-section for \(r\) if

\[
(\dagger) \quad \rho|x \subseteq \rho_M \subseteq \rho
\]

Example 4.1. The relator \(r : a^2 = bcb\) is an achievable \((2, 3)\) relator where \(b = [z_3]_\rho = [z_3]_\rho\) and \(J(b, r) = \{3, 5\}\). The following \((2, 3)\) map is a \(b\)-section for \(r\).

![Diagram](image)

Remarks. (1) The inclusion \(\rho|x \subseteq \rho_M\) in \((\dagger)\) requires that \([z_i]_{\rho_M} = [z_j]_{\rho_M}\) whenever \(x = [z_i]_\rho = [z_j]_\rho\). In this sense, an \(x\)-section for \(r\) “achieves the \(x\)-part of \(r\).” Note however, that if \([z_i]_\rho = x\), the equivalence class \([z_i]_{\rho_M}\) will, in general, be represented by some element of the ordered alphabet \(X\) which follows \(x\), rather than by \(x\) itself. In the example above, \(r_M\) is \(ab = cdc\) and \([z_3]_{\rho_M} = [z_5]_{\rho_M} = c\).

(2) The inclusion \(\rho_M \subseteq \rho\) is equivalent to \(r_M \leq r\). By Proposition 3.2, this is equivalent to requiring that \(M\), with the natural labelling induced by \(r\), is always a derivation diagram over the presentation \(<X; r>\). Thus an \(x\)-section for \(r\) is a derivation diagram over \(<X; r>\) which achieves the \(x\)-part of \(r\).

An \(x\)-section \(M\) for \(r\) is \(x\)-inceptive for \(r\) if \(|M| \leq |N|\) whenever \(N\) is also an \(x\)-section for \(r\).

If an \(x\)-inceptive map \(M\), with \(\rho > 0\), has unconstrained edges, then we can concatenate those blocks of \(M\) which do contain interior edges to construct a map which is \(x\)-inceptive for \(r\) and which has no unconstrained edges. In the case where \(|J(x, r)| = 1\), all \(x\)-inceptive maps for \(r\) are regionless, and any walk \(W_k \in F_0\) is \(x\)-inceptive for \(r\).

In this section we show (Theorem 4.20) that if \(|J(x, r)| = 2\) and \(M\) is an \(x\)-inceptive map for \(r\), then \(|M| \leq \ell\). This is the base case for an induction on \(|J(x, r)|\). In the next section, we show that if \(|J(x, r)| = j\) and \(M\) is an \(x\)-inceptive map for \(r\), then \(|M| \leq (j-1)\ell\).

Lemma 4.2. Suppose that \(r\) is an achievable \((s, \ell)\) relator and let \(X_2 = \{x \in X : |J(x, r)| \geq 2\}\). For each \(x \in X_2\), let \(M_x\) be an \(x\)-section for \(r\). Then the concatenation \(\cup_{x \in X_2} M_x\) is an \((s, \ell)\) map which achieves \(r\).

Proof. It will suffice to show that \(\rho = \cup_{x \in X_2} \rho_{M_x}\). Since each \(M_x\) is an \(x\)-section for \(r\), we have \(\rho_{M_x} \subseteq \rho\) for each \(x \in X_2\), and \(\cup_{x \in X_2} \rho_{M_x} \subseteq \rho\). Earlier (at the end of section 2), it was remarked that \(\rho = \cup_{x \in X} \rho_{|x|}\). Since \(\rho_{|x|}\) is the identity relation on \(Z\) when \(|J(x, r)| = 1\), we can strengthen this to \(\rho = \cup_{x \in X_2} \rho_{|x|}\). Then \(\rho = \cup_{x \in X_2} \rho_{|x|} \subseteq \cup_{x \in X_2} \rho_{M_x}\).
Lemma 4.3. If $r$ is an achievable $(s, \ell)$ relator, $x$ occurs in $r$, and $\mathcal{M}$ is an $x$-inceptive $(s, \ell)$ map for $r$, then $\mathcal{M}$ is an inceptive $(s, \ell)$ map for $r_M$.

Proof. Let $\mathcal{N}$ be any $(s, \ell)$ map that achieves $r_M$. We need to show that $|\mathcal{M}| < |\mathcal{N}|$. Since $r_N = r_M$, we have $\rho_N = \rho_M$ and $\rho|_x \subseteq \rho_N \subseteq \rho$ because $\rho|_x \subseteq \rho_M \subseteq \rho$. Thus $\mathcal{N}$ is an $x$-section for $r$ and the conclusion follows.

Example 4.4. By Lemma 4.3, every $(s, \ell)$ map $\mathcal{M}$ which is $x$-inceptive (for some $x$ and some $r$) is inceptive for the relator which it achieves. Being $x$-inceptive is more restrictive than being inceptive. The following map is an inceptive $(3, 5)$ map for the relator $abc = cabca$, but it is not $x$-injective for $x = a, b, c$.

Lemma 4.5. Suppose that $r$ is an achievable $(s, \ell)$ relator, that $x$ occurs in $r$ and that $\mathcal{M}$ is $x$-inceptive for $r$. Let $i \in J(x, r)$ and $\bar{x} = [z_i]_{\rho_M}$. Then $\mathcal{M}$ is also $\bar{x}$-inceptive for $r_M$.

Proof. Let $\mathcal{N}$ be any $\bar{x}$-section for $r_M$. It will suffice to show that $\mathcal{N}$ is also an $x$-section for $r$. Since $\mathcal{M}$ is an $x$-section for $r$, we have that $\rho_M \subseteq \rho$. A routine argument shows that $\rho_{r|_x} = \rho_{r_M|_{\bar{x}}}$. Since $\mathcal{N}$ is an $\bar{x}$-section for $r_M$, we have

$$\rho_{r|_x} = \rho_{r_M|_{\bar{x}}} \subseteq \rho_N \subseteq \rho_M \subseteq \rho.$$ 

Example 4.6. The following map is an $a$-section for the achievable $(1, 6)$-relator $a = (aba)^2$ and it is $a$-inceptive for $r_M$: $a = abaaca$, but it is not inceptive or $a$-inceptive for $a = (aba)^2$ which can be achieved by $(1, 6)$ maps with only four regions.

Example 4.7. The following three $(1, 9)$ maps $\mathcal{M}_1, \mathcal{M}_2, \text{and} \mathcal{M}_3$ are all $a$-inceptive for the $(1, 9)$ relator $a = baccdbbac$. Since $r_{M_1}$ is $a = bacdefgh$, $r_{M_2}$ is $a = bacdebac$, and $r_{M_3}$ is $a = bacdebbac$, we see that different $x$-inceptive maps may achieve the $x$-part of $r$ in distinctly different ways.
Let $M$ be an $(s, \ell)$ map and $e$ an interior edge of $M$. The edge $e$ is an $\{i,j\}$-edge if $e$ has index $i$ with respect to $D_1$ and has index $j$ with respect to $D_2$, where $e$ is on the common boundary of regions $D_1$ and $D_2$. We allow the possibility that $i = j$. If $e$ is an $\{i,i\}$-edge for some $i$, we will say that $e$ is reflective. (In a slightly different context, Remmers [10], see [11, p.293] or [2], defines such an edge to be an interface.) If $r$ is an achievable $(s, \ell)$ relator and $i \in J(x, r)$, we will say that an $\{i,i\}$-edge $e$ is $x$-reflective.

**Lemma 4.8.** Suppose that $r$ is an achievable $(s, \ell)$ relator, that the letter $x$ occurs in $r$, that $J(x, r) = \{i_1, i_2\}$, where $i_1 \neq i_2$, and that $M$ is an $x$-inceptive $(s, \ell)$ map for $r$. Then

1. $M$ has exactly two appended regions and every $\{i_1, i_2\}$-edge of $M$ is on their common boundary, and
2. there is exactly one interior edge of $M$ which is an $\{i_1, i_2\}$-edge.

**Proof.** By the definition of an $x$-section, $M$ must have at least one edge which is an $\{i_1, i_2\}$-edge. Since this edge is an interior edge, we must have $|M| \geq 2$, and hence by Lemma 1.5, $M$ has at least two appended regions. Suppose that $T$ is any region which is appended on the top side of $M$. If $(z_{i_1}, z_{i_2}) \in p_{M-T}$, then $M-T$ is also an $x$-section for $r$: this would contradict our hypothesis that $M$ is $x$-inceptive for $r$. Thus every $\{i_1, i_2\}$-edge of $M$ must occur on the bottom side of every region which is appended on the top side of $M$. This is impossible if there is more than one region that is appended on the top side of $M$. Similarly, there is exactly one appended region on the bottom of $M$ and every $\{i_1, i_2\}$-edge of $M$ occurs on the top side of this region. Thus the first claim of the lemma is verified.

Suppose then that $e$ and $f$ are distinct $\{i_1, i_2\}$-edges in $M$. We show that $i_1 < i_2$ and $i_2 < i_1$, a contradiction. Let $T$ be appended on the top side of $M$ and let $B$ be appended on the bottom side of $M$, so that both $e$ and $f$ occur on both $\alpha_T$ and $\omega_B$. Assume, without loss of generality, that $e$ precedes $f$ on $\alpha_T$ and hence on $\omega_B$ as well. Choose notation for the indices so that $e$ has index $i_1$ with respect to $B$. Then $f$, by default, must have index $i_2$ with respect to $B$. Since $e$ precedes $f$ on $\omega_B$, we have $i_1 < i_2$. Since $e$ and $f$ are both $\{i_1, i_2\}$-edges, $e$ has index $i_2$ with respect to $T$ and $f$ has index $i_1$ with respect to $T$. Since $e$ precedes $f$ on $\alpha_T$, we have $i_2 < i_1$.

Several of the next lemmas will consider the size and structure of an $x$-inceptive $(s, \ell)$ map, $M$, for $r$ when $|J(x, r)| = 2$. We find bounds for $|M|$ depending upon the values of $i_1$ and $i_2$. It will be efficient to use the conclusion of Lemma 4.8 and to extend the notation in its proof as a common starting point for all of these lemmas.

Let $r$ be an achievable $(s, \ell)$ relator and $x$ a letter that occurs exactly twice in $r$. Let $J(x, r) = \{i_1, i_2\}$, where $i_1 < i_2$. Let $M$ be an $x$-inceptive $(s, \ell)$ map for $r$. By Lemma 4.8, let $e$ be the unique $\{i_1, i_2\}$-edge in $M$ and let $B$ and $T$ be the regions of $M$ which are appended on the bottom and top sides, respectively. Write $\alpha_T = \mu_1 \gamma_1 e_\gamma 2 \mu_2$ and $\omega_B = \nu_1 \gamma_1 e_\gamma 2 \nu_2$, where $\gamma_1 e_\gamma 2$ is the largest common walk on the boundary of $B$ and $T$ containing $e$. Let $v$ be the terminal vertex of $\gamma_1 e_\gamma 2$. Express $\alpha_M$ as $\alpha_1 \alpha_2 \alpha_3$ and express $\omega_M$ as $\omega_1 \omega_T \omega_2$. We allow the possibility that, for $i = 1, 2$, any of $\alpha_i$, $\omega_i$, $\mu_i$, $\nu_i$, or $\gamma_i$ might be an empty walk. Let $M_r$ (for $M_{right}$) be the feathery submap of $M$ which is bounded by $\nu_2 \alpha_2 (\mu_2 \omega_2)^{-1}$. Note that $v$
is the initial vertex of \( \mathcal{M}_r \). Let \( \mathcal{M}_{\text{left}} \) be the feathery submap of \( \mathcal{M} \) bounded by \( \alpha_1 \nu_1 (\omega_1 \mu_1)^{-1} \). All of our arguments will treat \( \mathcal{M}_r \). The corresponding arguments for \( \mathcal{M}_{\text{left}} \) are dual.

Observe that if \( \mu_2 \) is empty, then \( \mathcal{M}_r \) can have no regions. Otherwise, some region of \( \mathcal{M}_r \) that is appended on the top side of \( \mathcal{M}_r \) would also be appended on the top side of \( \mathcal{M} \), contradicting Lemma 4.8. Similarly, if \( \nu_2 \) is empty, then \( |\mathcal{M}_r| = 0 \), and if either \( \mu_1 \) or \( \nu_1 \) is empty, then \( |\mathcal{M}_{\text{left}}| = 0 \).

**Lemma 4.9.** Suppose that \( r \) is an achievable \((s, \ell)\) relator, that the letter \( x \) occurs in \( r \), that \( J(x, r) = \{ i_1, i_2 \} \), where \( i_1 < i_2 \), and that \( \mathcal{M} \) is an \( x\)-inceptive \((s, \ell)\) map for \( r \).

1. If \( i_2 \leq s \), then \( |\mathcal{M}| = 2 \).
2. If \( i_1 > s \), and \( i_2 - i_1 \leq s \), then \( |\mathcal{M}| = 2 \).

**Proof.** Assume without loss of generality that \( e \) has index \( i_1 \) with respect to \( B \) and has index \( i_2 \) with respect to \( T \). We show that \( \mu_2 \) is empty and hence \( |\mathcal{M}_r| = 0 \). A dual argument shows that \( \nu_1 \) is empty also. If \( e \) has index \( i_2 \) with respect to \( B \), then the same line of reasoning shows that both \( \mu_1 \) and \( \nu_2 \) are empty.

Let \( f \) be the edge on the top side of \( B \) which has index \( i_2 \) with respect to \( B \). Since \( e \) has index \( i_2 \) with respect to \( T \) (and \( |J(x, r)| = 2 \)), \( f \) cannot be an edge of \( \gamma_2 \). Write \( \nu_2 = \nu' f \nu'' \). In case (1), both \( e \) and \( f \) are on the short side of \( B \), and \( \nu' \) is a segment of \( \omega_B \) containing neither \( e \) nor \( f \), hence \( |\nu'| \leq s - 2 \). In case (2), we have \( |\nu'| \leq |\gamma_2 \nu'| \leq i_2 - i_1 + 1 < s \).

Suppose first that the edge \( f \) is on the top side of \( \mathcal{M} \). Let \( \omega' \) be the segment of \( \omega_M \) from the terminal vertex of \( T \) to the initial vertex of \( f \). Then both \( \nu' \) and \( \mu_2 \omega' \) are directed walks from \( v \) to the initial vertex of \( f \). Hence \( \nu'(\mu_2 \omega')^{-1} \) bounds a feathery submap of \( \mathcal{M} \).

Since \( |\nu'| < s \), this submap cannot contain any regions, hence \( \nu' = \mu_2 \omega' \). By our choice of \( \gamma_2 (\gamma_1 \gamma_2) \) is a maximal walk common to \( \omega_B \) and \( \alpha_T \) containing \( e \), any initial edges of \( \mu_2 \) and \( \nu_2 = \nu' f \nu'' \) must be distinct. We conclude that \( \mu_2 \) is empty.

Now suppose that \( f \) is an interior edge of \( \mathcal{M} \). We show that this must lead to a contradiction. If \( f \) is an interior edge of \( \mathcal{M} \), then \( f \) occurs on the bottom side
of some region $D$ of $M$. Because $e$ is the unique $\{i_1, i_2\}$-edge of $M$, $f$ must be an $\{i_2, i_3\}$-edge. Write $\alpha_D = \alpha'g\alpha''$ where the edge $g$ has index $i_1$ with respect to $D$. Since $e$ and $f$ occur on $\omega_B$ and $|J(x, r)| = 2$, $g$ cannot be an edge of $\nu'$ which is a segment of $\omega_B$.

By Lemma 1.4, we can find a directed walk, $\tau$ in the map $M_r$ from the initial vertex $v$ of $M_r$ to the initial vertex of $D$. Then $\nu'(\tau\alpha'g\alpha'')^{-1}$ bounds a feather submap of $M_r$. Since $|\nu'| < s$, this submap is regionless and $g$ is an edge of $\nu'$. This contradiction shows that $f$ cannot be an interior edge of $M$.

**Example 4.10.** Case (2) in the preceding lemma was included because it required little more than the argument for case (1). We will consider below the general case of an $x$-inceptive map $M$ for $r$ when $J(x, r) = \{i_1, i_2\}$ and $s < i_1 < i_2$. For now, we give a fairly simple example which shows that $M$ can have more than two regions if we omit the hypothesis that $i_2 - i_1 < s$. The $(1, 5)$ relator $a = bcacb$ is an achievable relator on the alphabet $\{a, b, c\}$. The map below is $c$-inceptive and is inceptive for this relator. Several generalizations of this example are possible.

**Lemma 4.11.** Suppose that $r$ is an achievable $(s, \ell)$ relator, that the letter $x$ occurs in $r$, that $J(x, r) = \{i_1, i_2\}$, where $i_1 \leq s < i_2$, and that $M$ is an $x$-inceptive $(s, \ell)$ map for $r$. Then no edge of $M$ is $x$-reflective.

**Proof.** We may assume without loss of generality that $e$ has index $i_1$ with respect to $B$ and has index $i_2$ with respect to $T$, so that $e$ is on the short side of $B$ (since $i_1 \leq s$) and is on the long side of $T$ (since $s < i_2$). Here, an $x$-reflective edge can be only an $\{i_1, i_2\}$-edge or an $\{i_2, i_3\}$-edge. No edge on $B$ can be $x$-reflective: $e$ is an $\{i_1, i_2\}$-edge and the edge which has index $i_2$ with respect to $B$ is on the bottom side of $M$ and is not an interior edge. Similarly, no edge on $T$ can be $x$-reflective. Since interior edges of $M$ are either interior edges of $M_r$ or interior edges of $M_{left}$, or else occur on $\alpha_T$ or $\omega_B$, by symmetry, it will suffice to show that no interior edge of $M_r$ is $x$-reflective. (In the next lemma, using the current hypotheses, we will show that $M_r$ has no interior edges.)

We want to regard the edges of $M$ as partially ordered using the order from section 1 where $g < f$ if there is a directed walk in $M$ from the terminal vertex of $g$ to the initial vertex of $f$. Say that an $x$-reflective edge, $f$, of $M_r$ is left-primitive if there is no $x$-reflective edge $g$ of $M_r$ which precedes $f$ in $M_r$. If $M_r$ contains any $x$-reflective edge $f$, then we can always obtain a left-primitive edge by repeatedly replacing $f$ by such a predecessor $g$ until we obtain an $x$-reflective edge which is
left-primitive. We will assume that there is at least one \( x\)-reflective edge in \( M_r \) and we will contradict our hypothesis that \( M_r \) is \( x\)-inceptive.

For a (momentarily indeterminant) natural number \( i \), let \( f_i \) be a left-primitive edge in \( M_r \) which occurs on the top side of a region \( D_{i-1} \) of \( M_r \) and on the bottom side of a region \( D_i \) of \( M_r \). Inductively, for \( j < i \), let \( f_j \) be the edge on the bottom side of \( D_{j+1} \) which has index in \( J(x, r) \). At some point, the edge \( f_j \) must occur on the bottom side of \( M \) : when this occurs, we determine \( i \) so that \( j = 0 \) and \( f_0 \) occurs on the bottom side of \( M \). When \( f_j \) is not on the bottom side of \( M \), \( f_j \) is on the top side of some region \( D_j \) of \( M_r \) and we choose \( f_{j-1} \) to be the edge on the bottom side of \( D_j \) which has index in \( J(x, r) \). Similarly, for \( j \geq i \), let \( f_{j+1} \) be the edge on the top side of \( D_j \) which has index in \( J(x, r) \). If \( f_{j+1} \) is not on the top side of \( M_r \) (and of \( M \)), then \( f_{j+1} \) is on the bottom side of some region \( D_{j+1} \) of \( M \).

Eventually, we must arrive at an edge \( f_n \) which is on the top side of \( M_r \). We have edges \( f_j \) for \( 0 \leq j \leq n \), and for some \( i \) with \( 0 < i < n \), the edge \( f_i \) is left-primitive. Write the bottom side, \( \alpha_{D_j} \), of \( D_j \) as \( \alpha_j' f_j \alpha_j'' \), and write the top side, \( \omega_{D_j} \), of \( D_j \) as \( \omega_j' f_j \omega_j'' \). For \( 1 \leq j \leq n \), let \( \tau_j \) be a directed walk in \( M_r \) from the initial vertex \( v \) of \( M_r \) to the initial vertex of \( D_j \). Observe that we can do this in such a way that \( (\tau_j \omega_j')(\tau_{j+1} \alpha_{j+1}'')^{-1} \) is a counterclockwise walk in \( M_r \). (That is, if necessary, we can exchange segments of \( \tau_j \) with segments of \( \tau_{j+1} \) so that \( (\tau_j \omega_j')(\tau_{j+1} \alpha_{j+1}'')^{-1} \) is a counterclockwise walk in \( M_r \), and we can start this with \( \tau_0 \) and \( \tau_1 \) and work our way up.) Similarly, for \( 1 \leq j \leq n \), let \( \sigma_j \) be a directed walk in \( M_r \) from the terminal vertex of \( D_j \) to the terminal vertex of \( M_r \), chosen so that \( (\omega_j'' \sigma_j)(\alpha_{j+1}' \sigma_{j+1}'')^{-1} \) is a counterclockwise walk. Let \( \tilde{\sigma}_1 \) denote the terminal segment of \( \alpha_M \) whose initial vertex is the terminal vertex of \( f_0 \), then \( \tilde{\sigma}_1 = \alpha_1'' \sigma_1 \); otherwise \( \tilde{\sigma}_1 (\alpha_1'' \sigma_1)^{-1} \) would contain a region of \( M \) appended on the bottom side of \( M \). Similarly, \( \sigma_n \) must be a terminal segment of \( \omega_M \).

We show concurrently, by induction on \( j \) for \( i \leq j < n \), that \( \alpha_{j+1}' = \omega_j' \) and that \( f_j \) is left-primitive. A similar induction, from \( i \) backwards to \( j = 0 \), shows that \( \alpha_{j+1}' = \omega_j' \) and that \( f_j \) is left-primitive for \( 0 < j \leq i \).

For the base step of the induction, we are given that \( f_i \) is left-primitive. We show that:
(a) given that $f_j$ is left-primitive, we have $\alpha_{j+1}' = \omega_j$ and $\tau_j = \tau_{j+1}$, and
(b) given that $f_j$ is left-primitive and $\alpha_{j+1}' = \omega_j$, then $f_{j+1}$ is left-primitive.

For (a), let $N_j'$ be the feathery submap of $M_r$ bounded by $(\tau_j \omega_j') (\tau_{j+1} \alpha_{j+1}')^{-1}$. It will suffice to show that $|N_j'| = 0$ since this guarantees that $\omega_j' = \tau_{j+1} \alpha_{j+1}'$. Because $f_{j+1}$ has the same index with respect to both $D_j$ and $D_{j+1}$, $\omega_j' = |\alpha_{j+1}'|$. It thus follows from $|N_j'| = 0$, that $\omega_j' = \alpha_{j+1}'$ and $\tau_j = \tau_{j+1}$. We will assume that $|N_j'| > 0$ and find a region $E$ of $M_r$ which is appended to $M$ on the top side of $M$, contradicting Lemma 4.8.

If $|N_j'| > 0$, let $D$ be a region of $N_j$ which is appended on the top side of $N_j$. Let $g$ be the edge on the top side of $D$ whose index with respect to $D$ is in $J(x, r)$. The edge $g$ must occur as an edge of $\tau_{j+1}$ rather than in $\alpha_{j+1}'$, i.e., the edge $f_j$ is the edge on the bottom side of $D_{j+1}$ whose index with respect to $D_{j+1}$ is in $J(x, r)$. Further, $g$ cannot be an $\{i_1, i_2\}$-edge by the uniqueness of $e$, and $g$ cannot be $n$-reflective because $f_j$ is left-primitive and $g$ precedes $f_j$ in the positive walk $\tau_{j+1} \alpha_{j+1}' f_j$. Since $g$ cannot be an interior edge of $M$, it must occur on the to side of $M$. Let $\lambda$ be the terminal segment of $\omega_{\alpha_0}$ whose initial vertex is the terminal vertex of $g$. Let $\tau'$ be the terminal segment of $\tau_{j+1}$ whose initial vertex is the terminal vertex of $g$. Let $\mathcal{P}_j$ be the feathery submap of $M_r$ bounded by $\tau' \omega_{D_{j+1}} \sigma_{j+1} \lambda^{-1}$. Then $|\mathcal{P}_j| > 0$, because $D_{j+1}$ is appended to $\mathcal{P}_j$ on its bottom side. Then any region $E$ which is appended on the top side of $\mathcal{P}_j$ is also appended on the top side of $M$, producing the expected contradiction.

For (b), we assume by induction that $f_j$ is left-primitive and that $\alpha_{j+1}' = \omega_j'$. We need to show that $f_{j+1}$ is left-primitive. We show that if we suppose that $f_{j+1}$ is not left-primitive, then we can contradict Lemma 4.8 by finding a region $E_2$ of $M_r$, which is appended on the bottom side of $M$.

Suppose that $g$ is an $x$-reflective edge of $M_r$ which precedes $f_{j+1}$. Let $\xi_1$ be a directed walk in $M_r$ from the initial vertex $v$ of $M_r$ to the initial vertex of $g$ and let $\xi_2$ be a directed walk in $M_r$ from the terminal vertex of $g$ to the initial vertex of $f_{j+1}$. Because $f_{j+1}$ is left-primitive, the walk $\xi_2$ cannot intersect $\tau_{j+1} \omega_j'$. We can choose the walk $\xi_1$ so that $\tau_{j+1} \omega_j' (\xi_1 g \xi_2)^{-1}$ is a counterclockwise walk in $M_r$. Let $\mathcal{Q}_j$ be the feathery submap of $M_r$ that is bounded by this walk. Then $|\mathcal{Q}_j| > 0$, because $g$ occurs on its topside but cannot occur on its bottom side. Let $D$ be a region of $\mathcal{Q}_j$ which is appended to the bottom side of $\mathcal{Q}_j$ and let $g_2$ be the edge on the bottom side of $D$ whose index with respect to $D$ is in $J(x, r)$. Then $g_2$ can be neither an $\{i_1, i_2\}$-edge nor an $x$-reflective edge, so $g_2$ must occur on the bottom side of $M$. Arguing as in part (a), let $\tau''$ be the terminal segment of $\tau_{j+1}$ whose initial endpoint is the terminal endpoint of $g_2$, and let $\lambda_2$ be the terminal segment of $\alpha_0$ whose initial endpoint is the terminal endpoint of $g$. Then the submap of $M_r$ bounded by $\lambda_2 (\tau'' \omega_{D_{j+1}} \sigma_{j+1})^{-1}$ must contain a region $E_2$ of $M_r$ which is appended to $M$ on the bottom side of $M$. This completes the inductive proof of (a) and (b).

If we have shown that if there were $x$-reflective edges in $M_r$, then we would have, for some $n \geq 2$, left-primitive edges $f_j$, for $0 < j < n$. Structurally, these edges occur vertically between an edge $f_0$ on the bottom side of $M$ and an edge $f_n$ on the top side of $M$. Further, each left-primitive $f_j$ would be on the top side of a region $D_j$ and on the bottom side of a region $D_{j+1}$, where all of these regions have a common vertical vertex. Let $N'$ be the submap of $M_r$ bounded by $\alpha_1' f_0 \alpha_n' \sigma_1 (\omega_n f_n \omega_n' \sigma_n)^{-1}$. 


By Lemma 4.8, the edge $f$ be the edge on the bottom side of $T$ and the edge on the top side of $B$. Assume that $M$ is an initial segment of the long side of $D_n$ when $\alpha'_1$ is an initial segment of the short side of $D_1$ and $\omega'_n$ is an initial segment of the short side of $D_n$ when $\alpha'_1$ is an initial segment of the long side of $D_1$. Suppose without loss of generality that $\alpha'_1$ is an initial segment on the long side of $D_1$. We replace the submap $N'$ (having at least three regions $D_1, D_2, D_3$) in $M$ by a single region $D'$. We identify the initial segment of length $|\alpha'_1|$ on the long side of $D'$ with $\alpha'_1$ and we identify the initial segment of length $|\omega'_1|$ on the short side of $D'$ with $\omega'_1$. Let $M'$ be the result of thus replacing $N'$ in $M$ with $D'$. The indices of edges on $\alpha'_1$ and $\omega'_1$ are not changed by this substitution, so $\rho_{M'} \subseteq \rho_M \subseteq \rho_r$. Since the edge $e$ also occurs in $M'$, $M'$ is an $x$-section for $r$. This contradicts our choice of $M$ as $x$-inceptive.

If $N$ is even, then $\omega'_n$ is an initial segment of the long side of $D_n$ when $\alpha'_1$ is an initial segment of the long side of $D_1$, and $\omega'_n$ is an initial segment of the short side of $D_n$ when $\alpha'_1$ is an initial segment of the short side of $D_1$. We obtain a map $M'$ by deleting $N'$ and identifying $\alpha'_1$ with $\omega'_1$. Suppose that $e_1$ is an edge on $\alpha'_1$ which is identified with an edge $e_n$ on $\omega'_n$ in $M'$. If the index of $e_1$ with respect to $D_1$ in $M$ is $j_0$, then $j_0$ is also the index of $e_n$ with respect to $D_n$ in $M$. If $e_1$ and $e_n$ are both interior edges of $M$, then $e_1$ has index $j_1$ with respect to some region and $e_n$ has index $j_n$ with respect to some region. Since $(z_{j_0}, z_{j_1})$ and $(z_{j_0}, z_{j_n})$ are in the equivalence relation $\rho_M$, $(z_{j_1}, z_{j_n})$ is in $\rho_M$ also. If either $e_1$ or $e_n$ is a boundary edge in $M$, then their identification contributes nothing to $\rho_{M'}$. We again have $\rho_{M'} \subseteq \rho_M \subseteq \rho_r$ and that $M'$ is also an $x$-section for $r$, contradicting our choice of $M$.

**Lemma 4.12.** Suppose that $r$ is an achievable $(s, \ell)$ relator, that the letter $x$ occurs in $r$, that $J(x, r) = \{i_1, i_2\}$, where $i_1 \leq s < i_2$, and that $M$ is an $x$-inceptive $(s, \ell)$ map for $r$. Then $|M| \leq 4$.

It will suffice to prove that $M_r$ (and dually, $M_{\text{left}}$) has at most one region. For this draft, we provide two different proofs.

**First Proof.** Assume that $|M_r| \geq 2$ and that $B_r$ and $T_r$ are distinct regions of $M_r$ which are appended on the bottom and top sides of $M_r$, respectively. Let $f$ be the edge on the top side of $B_r$ whose index with respect to $B_r$ is in $J(x, r)$. Let $g$ be the edge on the bottom side of $T_r$ whose index with respect to $T_r$ is in $J(x, r)$. By Lemma 4.8, the edges $f$ and $g$ cannot be $\{i_1, i_2\}$-edges. By Lemma 4.11, they...
cannot be $x$-reflective. Hence $f$ must occur on the top side of $M$ and $g$ must occur on the bottom side of $M$. If $f$ precedes $\omega_T$, $\omega_M$, then $T_r$ is appended to the top side of $M$, contradicting Lemma 4.8. Similarly $\alpha_B$, must precede $g$ on $\alpha_M$.

Let $\theta_1$ be a directed walk on $\alpha_M$ from the terminal vertex of $\alpha_B$ to the initial vertex of $g$ and let $\theta_2$ be a directed walk on $\omega_M$ from the terminal vertex of $\omega_T$ to the initial vertex of $f$. Write $\alpha_T = \alpha'\alpha''$ and $\alpha_B = \alpha'f\alpha''$. Then $\alpha''\theta_2f\omega''\theta_1g$ is a positive closed walk in $M$ which is impossible by Lemma 1.3.

Second Proof. Let $D$ be any region of $M_r$ whose initial vertex is the initial vertex $v$ of $M_r$. (By our choice of maximality of $\gamma_1 \gamma_2$, there must be such a region $D$ if $|M_r| > 0$.) Let $f_0$ and $f_1$ be the edges on the bottom and top sides of $D$ whose indices with respect to $D$ are in $J(x, r)$. Write $\alpha_D = \alpha_f \omega''$ and $\omega_D = \omega'f_1\omega''$. By Lemmas 4.8 and 4.11, $f_0$ must occur on the bottom side of $M$ as an edge of $\alpha_2$ in $\alpha_M = \alpha_1\alpha_B\alpha_2$ and $f_1$ must occur on the top side of $M$ as an edge of $\omega_2$ in $\omega_M = \omega_1\omega_T\omega_2$. Write $\alpha_2 = \alpha_3f_0\alpha_4$ and $\omega_2 = \omega_3f_1\omega_4$. Let $\sigma$ be any directed walk from the terminal vertex of $D$ to the terminal vertex of $M_r$. Recall that $M_r$ is bounded by $\nu_2\alpha_2(\mu_2\omega_2)^{-1}$, where $\nu_2$ and $\mu_2$ are terminal segments of $\omega_B$ and $\alpha_T$, respectively. Let $M_{tm}, M_{tp}, N_{tm}$, and $N_{tp}$ be the feathery submaps of $M_r$ bounded by $\nu_2\alpha_2(\alpha')^{-1}, \omega'\omega_2\omega_3^{-1}, \alpha_2(\alpha'\sigma)^{-1}$, and $\omega''\sigma(\omega_2)^{-1}$, respectively. Then any region of $M_r$, other than $D$, must be in one of these four submaps. We show that they are all regionless.

If $|M_{bd}| > 0$, then some region of $M_{bd}$ is appended to $M_{bd}$ on the top side of $M_{bd}$. Some edge on the top side of this appended region will have its index with respect to the appended region in $J(x, r)$. But $f_0$ is the only edge on the bottom side of $D$ whose index with respect to $D$ is in $J(x, r)$. We conclude that $|M_{bd}| = 0$, and similarly that $|M_{tp}| = 0$.

If $|N_{bd}| > 0$, then some region of $N_{bd}$ would be appended to $N_{bd}$ on the bottom side of $N_{bd}$ and appended to $M$ on the bottom side of $M$. This contradicts Lemma 4.8, so $|N_{bd}| = 0$. Similarly, $|N_{tp}| = 0$.

We define ‘up’-words, $\lceil_i u$, on the ordered alphabet $X = \{x_1, x_2, \ldots, x_n\}$. For $0 \leq i < j \leq n$, let $\lceil_i u = x_{i+1}x_{i+2}\ldots x_{j}$. If $j \leq i$, then $\lceil_i u$ is the empty word. Observe that $\lceil_i u = j - i$ if $i \leq j$ and that $(\lceil_i u)(\lceil_k u) = \lceil_{i + k} u$ if $i \leq j \leq k$.

Example 4.13. The following $(3, 5)$ map is both inceptive and $b$-inceptive for the $(3, 5)$ relator $aba = cdbed$. If $D$ is either of the appended regions of $M$, then $M – D$ is not inceptive. Since $|M| = 4$, we cannot, in general, improve on the bound given in the lemma above. If we replace $a, b, c$, and $d$ with upwords on an ordered
alphabet $X$, we obtain similar $(s, \ell)$ maps with $\ell - s$ an even number and $|s| \geq 3$, where $\mathcal{M}$ is $x_i$-inceptive for every letter $x_i$ that occurs in the word $b$.

For an alphabetically contracted $(s, \ell)$ relator $r$ and a letter $x$ that occurs in $r$, say that $x$ is restricted to the long side of $r$ if $i \in J(x, r) \Rightarrow i > s$, and that $x$ is restricted to the short side of $r$ if $i \in J(x, r) \Rightarrow i \leq s$. Typically, a letter might occur in both sides of a relator rather than being restricted to one side or the other.

**Lemma 4.14.** Suppose that $r$ is an alphabetically constructed $(s, \ell)$ relator, that the letter $x$ occurs in $r$, and that $x$ is restricted to either the short side or the long side of $r$. Let $i_1, i_2 \in J(x, r)$ with $i_1 \neq i_2$ and let $\mathcal{N}$ be a derivation diagram over $\langle X; r \rangle$ which contains no $\{i, j\}$-edges for $i, j \in J(x, r)$ except possibly $x$-reflective ones.

Let $e_1$ be an $\{i_1, i_1\}$-edge in $\mathcal{N}$ which occurs on the bottom side of a region $D_1$ of $\mathcal{N}$ and on the top side of a region $D_2$ of $\mathcal{N}$. Then the edge on the bottom side of $D_1$ which has index $i_2$ with respect to $D_1$ is also the edge on the top side of $D_2$ which has index $i_2$ with respect to $D_2$.

**Proof.** We may assume without loss of generality that $i_1 < i_2$. We use induction on $|\mathcal{N}|$. When $|\mathcal{N}| = 1$, there are no $\{i_1, i_1\}$-edges and the lemma is vacuously true. When $|\mathcal{N}| = 2$, either there are no $\{i_1, i_1\}$-edges or else we must have $\alpha_{D_1} = \omega_{D_2}$ in order that $\mathcal{N}$ is two-sided.

Assume then that $|\mathcal{N}| > 2$. If $\mathcal{N}$ has an appended region $D$ which is neither $D_1$ nor $D_2$, then the lemma is true by induction for $\mathcal{N} - D$ and is true then also for $\mathcal{N}$. By this, we may assume that $D_1$ is the only appended region on the top side of $\mathcal{N}$ and that $D_2$ is the only appended region on the bottom side of $\mathcal{N}$. Let $f_1$ be the edge of $D_1$ which has index $i_2$ with respect to $D_1$ and let $f_2$ be the edge of $D_2$ which has index $i_2$ with respect to $D_2$. We need to show that $f_1$ is $f_2$. A consequence is that this edge is an interior edge. As a step in the proof, we next show that at least one of $f_1$ or $f_2$ must be an interior edge.

Assume, without loss of generality, that $f_1$ is not an interior edge. Then $f_1$ must occur on the bottom side of $\mathcal{N}$. An easy argument shows that $f_1$ must occur after $\alpha_{D_1}$ on $\alpha_{\mathcal{N}}$. (Otherwise, we will have a directed walk from $f_1$ to $e_1$ as well as a directed walk from $e_1$ to $f_1$). Write $\alpha_{\mathcal{N}} = \lambda_1 \alpha_{D_2} \lambda_2 f_1 \lambda_3$, $\alpha_{D_1} = \phi_1 e_1 \phi_2 f_1 \phi_3$ and $\omega_{D_2} = \theta_1 \phi_1 \theta_2 f_2 \theta_3$. 
Let $S$ be the feathery submap of $\mathcal{N}$ bounded by $\theta_2 f_2 \theta_3 \lambda_2 \phi_2^{-1}$. Since $\mathcal{N}$ is a derivation diagram over $\langle X; r \rangle$ and no edges of $\phi_2$ are labelled by $x$, the edge $f_2$ cannot occur on the top side of $S$. Hence, $f_2$, occurs on the bottom side of some region of $S$ and is an interior edge of $\mathcal{N}$, as required.

Now, using that $f_2$ is an interior edge of $\mathcal{N}$, either $f_2$ is also an interior edge of $\mathcal{N} - D_1$ or else $f_2$ is on the top side of $\mathcal{N} - D_1$. In the latter case, $f_2$ must coincide with some edge on the bottom side of $D_1$ and the index requirements force $f_1 = f_2$.

If the former case were to occur, by induction we could apply the lemma to the edge $f_2$ as an $\{i_2, i_2\}$-edge of the derivation diagram $\mathcal{N} - D_1$. A contradiction occurs since $e_1$ must then occur on the bottom side of some region of $\mathcal{N} - D_1$.

Lemma 4.15. Suppose that $\mathcal{N}$ is an $(s, \ell)$ map, that $n_1$ regions of $\mathcal{N}$ have their long side on the top and that $n_2$ regions of $\mathcal{N}$ have their long side on the bottom. Then $|\omega_\mathcal{N}| = |\alpha_\mathcal{N}| + n_1(\ell - s) + n_2(s - \ell)$.

Proof. This follows by a routine induction on $|\mathcal{N}|$.

Lemma 4.16. Suppose that $r$ is an alphabetically constructed $(s, \ell)$ relator, that the letter $x$ occurs in $r$, and that $x$ is restricted to either the short side or the long side of $r$. Let $\mathcal{N}$ be a derivation diagram over $\langle X; r \rangle$ such that $\mathcal{N}$ contains no $\{i, j\}$-edges for $i \neq j$ and such that $x$ occurs in neither $\overline{\mathcal{N}}$ nor $\underline{\mathcal{N}}$. Then $|\mathcal{N}|$ is even and $|\alpha_\mathcal{N}| = |\omega_\mathcal{N}|$.

Proof. Let $n_1$ be the number of regions of $\mathcal{N}$ which have their long side on the top and let $n_2$ be the number of regions of $\mathcal{N}$ which have their long side on the bottom. By Lemma 4.14, any interior $x$-edge of $\mathcal{N}$ determines a pair of regions such that every $x$-edge which occurs on the boundary of either region must occur on the top side of one of the regions and and the bottom side of the other region. If $n_1 > n_2$, then some region with edges labelled $x$ on the top side cannot be paired with a region where the edges labelled $x$ are on the bottom side: when this happens, $x$ must occur in $\overline{\mathcal{N}}$. Similarly, if $n_2 > n_1$, then $x$ occurs in $\underline{\mathcal{N}}$. Since by hypothesis, neither $\overline{\mathcal{N}}$ nor $\underline{\mathcal{N}}$ contains an occurrence of $x$, we must have $n_1 = n_2$. Then $|\mathcal{N}| = n_1 + n_2$ is even and $|\alpha_\mathcal{N}| = |\omega_\mathcal{N}|$ by Lemma 4.15.

Lemma 4.17. Suppose that $r$ is an achievable $(s, \ell)$ relator, that the letter $x$ occurs in $r$, and that $J(x, r) = \{ i \}$, where $s < i$. Let $\mathcal{N}$ be a derivation diagram over $\langle X; r \rangle$ such that $x$ occurs in neither $\overline{\mathcal{N}}$ nor $\underline{\mathcal{N}}$. Then $\overline{\mathcal{N}} = \underline{\mathcal{N}}$. 

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Proof. We use induction on $|N|$ for $|N| \geq 0$. It is obvious that $\beta_N = \gamma_N$ when $|N| = 0$. If $|N| = 1$, the hypothesis that $x$ occurs in neither $\beta_N$ nor $\gamma_N$ must fail, so the conclusion follows vacuously. When $|N| = 2$, the only way that the hypothesis on occurrences of $x$ can be satisfied is if the two regions share a common long side; in this case, the conclusion is again clear. Assume then that $|N| > 2$. The following sublemma will be useful. A corollary to the sublemma is that we may assume that $N$ has only one block.

**Sublemma 4.17.1.** If $N$ contains a feathery submap $E$ with $0 < |E| < |N|$ such that $x$ occurs in neither $\beta_E$ nor $\gamma_E$, then the lemma is verified for $N$.

**Proof of the first sublemma.** By the induction hypothesis, we have that $\beta_E = \gamma_E$. Let $\tau_1$ be a directed walk from the initial vertex of $N$ to the initial vertex of $E$ and let $\tau_2$ be a directed walk from the terminal vertex of $E$ to the terminal vertex of $N$.

Let $N_{btm}$ be the feathery submap of $N$ bounded by $\alpha_N(\tau_1, \beta_E, \tau_2)^{-1}$ and let $N_{top}$ be the feathery submap of $N$ bounded by $\tau_1, \omega_E, \tau_2, \omega_N^1$. The top side of $N_{btm}$ has the same label as the bottom side of $N_{top}$, so we obtain a derivation diagram $N'$ over $\langle X; r \rangle$ when we identify the top side of $N_{btm}$ with the bottom side of $N_{top}$. Because $|E| > 0$, we have $|N'| < |N|$. By induction, the lemma is true for $N'$. Since $\alpha_N' = \alpha_N$ and $\omega_N' = \omega_N$, the lemma is verified for $N$ also.

**Sublemma 4.17.2.** With the hypotheses of the lemma, if $N$ contains an $x$-pair, then $N$ contains at least one coinitial $x$-pair and at least one coterminal $x$-pair.

**Proof of the second sublemma.** If $f$ is a leftmost $x$-reflective edge, then the $x$-pair which contains $f$ is coinitial. If $f$ is a rightmost $x$-reflective edge, then the $x$-pair which contains $f$ is coterminal.
If \( \gg D_1, E_1 \ll \) and \( \gg D_2, E_2 \ll \) are \( x \)-pairs in \( \mathcal{N} \), then the number of **coherence bonds** between \( \gg D_1, E_1 \ll \) and \( \gg D_2, E_2 \ll \) is the number of coherence pairs in \( \mathcal{N} \) among the following:

\[
(D_1, D_2), (D_1, E_2), (E_1, D_2), (E_1, E_2), (D_2, D_1), (E_2, D_1), (D_2, E_1), \text{ and } (E_2, E_1).
\]

**Sublemma 4.17.3.** If \( \mathcal{N} \) contains no proper feathery submaps with \( x \)-free boundary and \( \gg D_1, E_1 \ll \) and \( \gg D_2, E_2 \ll \) are \( x \)-pairs in \( \mathcal{N} \), then there are at most three coherence bonds between \( \gg D_1, E_1 \ll \) and \( \gg D_2, E_2 \ll \). If there are three coherence bonds between \( \gg D_1, E_1 \ll \) and \( \gg D_2, E_2 \ll \), then by duality and change of notation, we may assume that one of the following two cases occurs:

**case (a)** \( (D_1, D_2) \) is left-right coherent, \( (D_2, E_1) \) is right-left coherent, and \( (E_1, E_2) \) is left-right coherent, or

**case (b)** \( (D_1, D_2) \) is contractive, \( (D_2, E_1) \) is right-left coherent, and \( (E_1, E_2) \) is expansive.

**Proof of the third sublemma.** For any regions \( A \) and \( B \), the definition for the ordered pair \((A, B)\) to be a coherent pair requires that \( A \) and \( B \) share a common edge which is on the top side of \( A \) and the bottom side of \( B \). It follows that at most one of \((A, B)\) and \((B, A)\) can be a coherent pair. (For a proof, consider the appended regions of the smallest feathery submap of \( \mathcal{N} \) containing both \( A \) and \( B \)).

By the paragraph above, it is clear that there can be at most four coherence bonds between \( \gg D_1, E_1 \ll \) and \( \gg D_2, E_2 \ll \). We need to show that when we do have at least three coherence bonds that the fourth cannot occur. In doing this the structural assertions in the case where there are exactly three coherence bonds will be justified. Suppose then that we do have at least three coherence bonds. At least one of the coherence bonds must come either from one of \((D_1, E_2)\) and \((E_2, D_1)\) or else from \((E_1, D_2)\) and \((D_2, E_1)\). If only one coherence bond comes from these, then a second of the three coherence bonds must come from \((D_1, D_2)\) or \((D_2, D_1)\) and the third from \((E_1, E_2)\) or \((E_2, E_1)\).

By vertical duality, we may assume that one bond is contributed by \((D_1, D_2)\) or \((D_2, D_1)\) and another is contributed by \((D_2, E_1)\) or \((E_1, D_2)\). If it were the case that coherence bonds were formed both among \((D_1, E_2)\) and \((E_2, D_1)\) and also among \((D_2, E_1)\) and \((E_1, D_2)\), then any third or fourth bond would be between either \((D_1, D_2)\) and \((D_2, D_1)\) or else between \((E_1, E_2)\) and \((E_2, E_1)\).

Using vertical duality again, we can assume in this case also that one bond is contributed by \((D_1, D_2)\) or \((D_2, D_1)\) and another is contributed by \((D_2, E_1)\) or \((E_1, D_2)\): that is, using vertical duality we may assume that this happens in any case when there are at least three coherence bonds.

Since we may re-index the \( x \)-pairs by switching subscripts, we may assume that \((D_1, D_2)\) rather than \((D_2, D_1)\) is a coherent pair. Using horizontal duality, we may assume either that

\( (a) \quad (D_1, D_2) \) is a left-right coherent pair or else that
(b) \((D_1, D_2)\) is a contractive coherent pair in which the edges on \(\alpha_{D_2}\) occur to the right of the edge on \(\omega_{D_1}\) that is labelled \(x\). (In this case, it is also clear that \(\alpha_{D_2}\) is the short side of \(D_2\).) We separate the further analysis of cases (a) and (b).

**Case (a).** We have used duality above to ensure that one of \((D_2, E_1)\) or \((E_1, D_2)\) is a coherent pair in \(\mathcal{N}\). We want to next verify that, with our choices, \((D_2, E_1)\) is a right-left coherent pair. Since \((D_1, D_2)\) is a left-right coherent pair, we may write \(\omega_{D_1} = \omega \gamma\) and \(\alpha_{D_2} = \gamma \alpha\) for some positive paths \(\gamma\) and \(\alpha\). Since \(\gg D_1, E_1 \ll\) is an \(x\)-pair, we may write \(\omega = \omega_1 \omega_2 e_1 \omega_3 \omega_4\) where \(e_1\) is the common edge of \(\omega_{D_1}\) and \(\alpha_{E_1}\) which has label \(x\) and \(\omega_2 e_1 \omega_3\) is the longest common segment of \(\omega_{D_1}\) and \(\alpha_{E_1}\) which contains \(e_1\).

Let \(\alpha_1\) be the initial segment of \(\alpha_{E_1}\) which has the same terminal vertex as \(\omega_1\) and let \(\alpha_2\) be the terminal segment of \(\alpha_{E_1}\) which has the same initial vertex as \(\omega_4\). Let \(\tau_1\) be any directed path in \(\mathcal{N}\) from the terminal vertex of \(D_2\) to the terminal vertex of \(\mathcal{N}\). Observe that there is at least one directed path \(\tau_2\) in \(\mathcal{N}\) from the terminal vertex of \(E_1\) to the terminal vertex of \(\mathcal{N}\) and that we may choose one of these so that the path \((\omega_4 \alpha_{D_2} \tau_1)(\alpha_2 \tau_2)^{-1}\) is counter-clockwise and bounds a feathery submap \(\mathcal{N}_1\) of \(\mathcal{N}\). Since \(\alpha_{D_2}\) is on the bottom side of \(\mathcal{N}_1\) and edges of \(\alpha_{E_1}\) occur on the top side of \(\mathcal{N}_1\), we can see that \((D_2, E_1)\) rather than \((E_1, D_2)\) must be a coherent pair.

Next we claim that no edges of \(\omega_{D_2}\) can be edges on \(\alpha_1\): if an edge \(f\) occurred on both \(\alpha_1\) and on \(\omega_{D_2}\), we could construct a positive closed path in \(\mathcal{N}\) from \(f\) to \(e_1\) along \(\alpha_{E_1}\) and from \(e_1\) to \(f\) along \(\omega_{D_1}\) and \(\omega_{D_2}\). Thus, the coherent pair \((D_2, E_1)\) cannot be a left-right coherent pair. Since \(e_1\) is on \(\alpha_{E_1}\) and is on \(\omega_{D_1}\) rather than \(\omega_{D_2}\), the coherent pair \((D_2, E_1)\) cannot be matched or contractive. Since \(D_2\) is the bottom region of the \(\gg D_2, E_2 \ll\), the side \(\omega_{D_2}\) has the same length as \(\alpha_{E_1}\) and \((D_2, E_1)\) cannot be expansive. The remaining possibility is that \((D_2, E_1)\) is a right-left coherent pair as required. Notice that this forces \(\omega_4\) to be an empty path since \(\mathcal{N}\) contains no feathery submaps with \(x\)-free boundary.
From the illustration above, it is clear that neither \((D_1, E_2)\) nor \((E_2, D_1)\) can now be a coherent pair in \(N\). For a slightly more formal argument, first note that \((E_2, D_1)\) cannot be a coherent pair in \(N\). If that happened, edges of \(\omega E_2\) would then be on the bottom side of the feathery submap of \(N\) bounded by \((\alpha D_1, \alpha)(\omega_1 \omega_2 e_1 \omega_3 \omega D_2)^{-1}\) and the edge labelled by \(x\) on the bottom side of \(E_2\) would occur on the top side of this same submap. \((D_1, E_2)\) cannot be a matched, expansive, or contractive coherent pair because \(\omega D_1\) and \(\alpha E_2\) have the same length and share edges labelled by \(x\) with other regions. \((D_1, E_2)\) cannot be a left-right coherent pair because \((D_1, D_2)\) is a left-right coherent pair. Let \(e_2\) be the edge common to \(\omega D_2\) and \(\alpha E_2\) that is labelled by \(x\). \((D_1, E_2)\) cannot be a right-left coherent pair because this would force a positive closed path in \(N\) (along \(\omega D_1\) and \(\omega D_2\) from \(e_1\) to \(e_2\) and then along \(\alpha E_2\) and \(\omega D_1\) from \(e_2\) to \(e_1\)).

Next, we need to show that \((E_1, E_2)\) rather than \((E_2, E_1)\) is the third coherent pair and that \((E_1, E_2)\) is a left-right coherent pair.

To see that \((E_2, E_1)\) cannot be a coherent pair, first observe that, except for the edges on \(\alpha_1\), the edges on \(\alpha E_1\) are already shared with \(\omega D_1\) or \(\omega D_2\). If the terminal edge \(f\) of \(\omega E_2\) were to occur on \(\alpha_1\) then there would be a positive closed path in \(N\) from \(e_2\) to the terminal vertex of \(f\) along \(\alpha E_2\) and from the terminal vertex of \(f\) to \(e_2\) along \(\alpha E_1\) and \(\omega D_2\).

Write \(\alpha E_2\) as \(\nu_1 \nu_2 e_2 \nu_3 \nu_4\) where \(\nu_2 e_2 \nu_3\) is the longest common segment of \(\omega D_2\) and \(\alpha E_2\) which contains \(e_2\). Because \(e_2\) is on \(\alpha E_2\), \((E_1, E_2)\) cannot be a
matched coherent pair or a contractive coherent pair. If any edge $f$ of $\nu_4$ was also an edge on $\omega_{E_1}$, then there would be a positive closed path in $N$ from $e_2$ to $f$ along $\nu_3$ and $\nu_4$, and then from $f$ to $e_2$ along $\omega_{E_1}$ and $\omega_{D_2}$. Hence $(E_1, E_2)$ can be neither a right-left coherent pair nor an expansive coherent pair with $\omega_{E_1}$ a segment of $\nu_4$. We will conclude case (a) by showing that $(E_1, E_2)$ can only be a left-right coherent pair because $\nu_1$ is too short for $(E_1, E_2)$ to be an expansive coherent pair with $\omega_{E_1}$ a segment of $\nu_1$. We need to show that $|\omega_{E_1}| > |\nu_1|$.

Because $N$ contains no feathery submaps with $x$-free boundary, we can make the following observation. If $(E_1, E_2)$ is an expansive coherent pair with $\omega_{E_1}$ a segment of $\nu_1$, then $\nu_1 = \mu \omega_{E_1}$ for some directed path $\mu$ and the terminal vertex of $E_1$ is the initial vertex of $\nu_2$. Since the initial segments of $\alpha_{E_2}$ and $\omega_{D_2}$ are then $\nu_1 \nu_2 e_2$ and $\alpha_2 \nu_2 e_2$ (where $e_2$ has the same index with respect to $E_2$ and $D_2$), we see that $|\nu_1| = |\alpha_2|$. A similar argument for $e_1 \omega_3 \alpha_2$ and $e_1 \omega_3 \gamma$ shows that $|\alpha_2| = |\gamma|$. Since $(D_1, D_2)$ is a left-right coherent pair, with $\alpha_{D_2} = \gamma \alpha$, we have $|\alpha_{D_2}| > |\gamma|$. Both $\alpha_{D_2}$ and $\omega_{E_1}$ are short sides of regions, so we have $|\omega_{E_1}| = |\alpha_{D_2}| > |\gamma| = |\alpha_2| = |\nu_1|$, as required.

Case (b). Assume now that $\alpha_{D_2}$ is a segment of $\omega_{D_1}$ that occurs to the right of $e_1$ on $\omega_{D_1}$ and write $\omega_{D_1}$ as $\omega_1 \omega_2 e_1 \omega_3 \omega_4 \alpha_{D_2} \omega_5$ where $\omega_2 \omega_3 \omega_4$ is the longest common segment of $\omega_{D_1}$ and $\alpha_{E_1}$ which contains $e_1$.

As in case (a), we have already used duality to assure that one of $(D_2, E_1)$ or $(D_2, E_1)$ is a coherent pair in $N$. In the current case, it is easy to see that the coherent pair must be $(D_2, E_1)$ because $\alpha_{D_2}$ is a segment of $\omega_{D_1}$. Since $\omega_{D_2}$ and $\alpha_{E_1}$ have the same length and share edges labelled by $x$ with other regions, $(D_2, E_1)$ cannot be an expansive, contractive, or matched coherent pair. As in earlier argument, if $(D_2, E_1)$ were a left-right coherent pair, we would have a positive closed path in $N$. Thus $(D_2, E_1)$ must be a right-left coherent pair. Let $e_2$ be the edge common to $\omega_{D_2}$ and $\alpha_{E_2}$ which has label $x$. We may write $\alpha_{E_1}$ as $\alpha_1 \omega_2 e_1 \omega_3 \alpha_2$ and $\omega_{D_2}$ as $\alpha_2 \nu_2 e_2 \nu_3 \nu_4$ where $\nu_2 \nu_3 \nu_4$ is the longest common segment of $\omega_{D_2}$ and $\alpha_{E_2}$ containing $e_2$. 
As in case (a), we need to show that neither \((E_2, D_1)\) nor \((D_1, E_2)\) can be a coherent pair. Since \(\alpha_{D_1}(\omega_1\omega_2\omega_3\omega_5\omega_1^{-1})\) bounds a feathery submap of \(\mathcal{N}\) and \(e_2\) occurs on the top side of this map, no edge of \(\omega_{E_2}\) can occur on \(\alpha_{D_1}\). Thus, \((E_2, D_1)\) is not a coherent pair. Since \(\omega_{D_1}\) and \(\alpha_{E_2}\) have the same length and share edges labelled by \(x\) with other regions, \((D_1, E_2)\) cannot be a contractive, expansive or matched coherent pair. If \((D_1, E_2)\) were either a right-left coherent pair or a left-right coherent pair, we would find positive closed paths in \(\mathcal{N}\).

Next we need to show that \((E_1, E_2)\) rather than \((E_2, E_1)\) is a coherent pair in \(\mathcal{N}\). We need only observe that if the terminal edge of \(\omega_{E_2}\) were an edge on \(\alpha_1\), then we would have a positive closed path in \(\mathcal{N}\). We are now reduced to the case that \((E_1, E_2)\) is a coherent pair in \(\mathcal{N}\). Write \(\alpha_{E_2}\) as \(\mu_1\nu_2\nu_3\mu_2\). To conclude the proof of this sublemma, we will show that \(\omega_{E_1}\) is a segment of \(\mu_1\). Because \(e_2\) occurs on \(\alpha_{E_2}\), we cannot have that \((E_1, E_2)\) is matched or contractive. If the initial edge of \(\omega_{E_2}\) occurred on \(\mu_2\), we would have a positive closed path in \(\mathcal{N}\). The remaining possibilities are that \((E_1, E_2)\) is left-right coherent or that \(\omega_{E_1}\) is a segment of \(\mu_1\). In either case, we can see that (because \(\mathcal{N}\) contains no proper feathery submaps with \(x\)-free boundary) \(\nu_1\) must be empty. We now need only show that \(|\mu_1| \geq |\omega_{E_1}|\). With arguments like those in case (a), we can see that \(|\mu_1| = |\alpha_2| = |\alpha_{D_2}\omega_5| \geq |\alpha_{D_2}| = |\omega_{e_1}|\).

Sublemma 4.17.3

\textbf{Sublemma 4.17.3.} If \(\not\overline{\overline{\overline{D_1}}}, E_1 \ll \text{ is a coinitial x-pair in } \mathcal{N}, \overline{\overline{\overline{D_2}}}, E_2 \ll \text{ is an x-pair in } \mathcal{N}, \text{ and there are three coherence bonds between } \overline{\overline{\overline{\overline{D_1}}}}, E_1 \ll \text{ and } \overline{\overline{\overline{D_2}}}, E_2 \ll, \text{ then the lemma is verified for } \mathcal{N}.

If \(\overline{\overline{\overline{\overline{D_1}}}}, E_1 \ll \text{ is a coterminial x-pair in } \mathcal{N}, \overline{\overline{\overline{D_2}}}, E_2 \ll \text{ is an x-pair in } \mathcal{N}, \text{ and there are three coherence bonds between } \overline{\overline{\overline{\overline{D_1}}}}, E_1 \ll \text{ and } \overline{\overline{\overline{D_2}}}, E_2 \ll, \text{ then the lemma is verified for } \mathcal{N}.

Proof of the fourth sublemma. The two assertions are dual, so we may assume that we are in the case where \(\overline{\overline{\overline{D_1}}}, E_1 \ll \text{ is coinitial and then that we are in one of the two structural cases (a) and (b) in the conclusion of the previous sublemma.}

case (a) Let \(e_1\) be the \(x\)-reflective edge for \(\overline{\overline{\overline{D_1}}}, D_1 \ll\). Since \(\overline{\overline{\overline{D_1}}}, E_1 \ll \text{ is coinitial, we can write } \omega_{D_1} \text{ as } \mu_1\lambda_2 \text{ and } \alpha_{E_1} \text{ as } \mu_1\lambda_3 \text{ for positive paths } \mu_1, \lambda_2 \text{ and } \lambda_3 \text{ where } e_1 \text{ occurs on } \mu_1 \text{ and the initial edge of } \lambda_2 \text{ is distinct from the}
initial edge of $\lambda_3$. Since $(D_1, D_2)$ is a left-right coherent pair, we can write $\alpha_{D_2}$ as $\lambda_2 \alpha_2$ for some positive path $\alpha_2$. Since $(D_2, E_1)$ is a right-left coherent pair and $\succ D_2, E_1 \preccurlyeq$ is an $x$-pair, we can write $\omega_{D_2}$ as $\lambda_3 \mu_2 \psi_1$ where $e_2$ occurs on $\mu_2$ and $\mu_2$ is the longest common segment of $\omega_{D_2}$ and $\alpha_{E_2}$ containing $e_2$. Since $(E_1, E_2)$ is a left-right coherent pair, write $\omega_{E_1}$ as $\omega_1 \lambda_4$ and $\alpha_{E_2}$ as $\lambda_4 \mu_2 \psi_2$.

In the previous sublemma, we saw, with different notation, that $|\lambda_2| = |\lambda_3| = |\lambda_4|$ and that this common length is less than the length $s$ for short sides of regions. Hence, we may also write $\alpha_{D_1}$ as $\alpha_1 \lambda_1$ and $\omega_{E_2}$ as $\lambda_5 \omega_2$ for positive walks $\lambda_1, \alpha_1, \lambda_5$ and $\omega_2$ where $|\lambda_i| = |\lambda_j|$ for $1 \leq i, j \leq 5$.

We next argue that $\underline{\lambda_i} = \underline{\lambda_j}$ for $1 \leq i, j \leq 5$. First, $\underline{\lambda_2} = \underline{\lambda_3}$, since $\lambda_2$ and $\lambda_3$ have the same length and are terminal segments of the long sides of $D_1$ and $E_1$, respectively. Next, $\underline{\lambda_3} = \underline{\lambda_4}$ since $\lambda_3$ and $\lambda_4$ have the same length and are initial segments on the long sides of $D_2$ and $E_2$, respectively. Then, $\underline{\lambda_2} = \underline{\lambda_5}$, since $\lambda_2$ and $\lambda_5$ are initial segments on the short sides of $D_2$ and $E_2$. Finally, $\underline{\lambda_1} = \underline{\lambda_4}$, since $\lambda_1$ and $\lambda_4$ are terminal segments on the short sides of $D_1$ and $E_1$. It is similarly apparent that $\overline{\tau_1} = \overline{\omega_1}$ and $\overline{\tau_2} = \overline{\omega_2}$.

Choose any directed path $\tau_0$ from the initial endpoint of $\mathcal{N}$ to the initial endpoint of $\alpha_1, \mu_1$, and $\omega_1$. Let $\tau_1$ be a directed path from the terminal endpoint of $\psi_1$ to the terminal endpoint of $\mathcal{N}$ and choose a directed path $\tau_2$ from the terminal endpoint of $\psi_2$ to the terminal endpoint of $\mathcal{N}$ such that $(\psi_1 \tau_1)(\psi_2 \tau_2)^{-1}$ is counterclockwise. Let $\mathcal{N}_{\text{medial}}$ be the feathery submap of $\mathcal{N}$ that is bounded by $(\psi_1 \tau_1)(\psi_2 \tau_2)^{-1}$. Let $\mathcal{N}_{\text{bottom}}$ and $\mathcal{N}_{\text{top}}$ be the feathery submaps of $\mathcal{N}$ that are bounded by $\alpha_{\mathcal{N}}(\tau_0 \alpha_1 \lambda_1 \alpha_2 \tau_1)^{-1}$ and $(\tau_0 \omega_1 \lambda_5 \omega_2 \tau_2) \omega_{\mathcal{N}}^{-1}$.
Append a region $D'_2$ to the top of $N_{\text{bottom}}$, identifying the short side of $D'_2$ with $\lambda_1\alpha_2$ to obtain a feathery map $N_1$. Write the top side of $D'_2$ as $\lambda'_3\mu'_2\psi'_1$ where $\lambda'_3 = \lambda'_3$ and $\psi'_1 = \psi'_1$. Append the feathery submap $N_\text{medial}$ to the tops side of $N_1$ along $\psi'_1\tau_1$ to obtain a feathery map $N_2$. Then $\alpha_{N_4} = \omega_{N_4}$, by induction on $|N|$, so $\alpha_N = \omega_N$.

**TO BE COMPLETED**
Lemma 4.18. Suppose that \( r \) is an achievable \((s, \ell)\) relator, that the letter \( x \) occurs in \( r \), and that \( J(x, r) = \{ i_1, i_2 \} \), where \( s < i_1 < i_2 \). Let \( \mathcal{N} \) be a derivation diagram over \( \langle X; r \rangle \) which contains no \( \{i_1, i_2\}\)-edges. If neither \( \mathcal{N} \) nor \( \mathcal{N} \) contains an occurrence of \( x \), then \( \mathcal{N} = \mathcal{N} \).

Proof. Since \( J(x, r) = \{ i_1, i_2 \} \) and \( \mathcal{N} \) contains no \( \{i_1, i_2\}\)-edges, \( [z_i]_{\rho_N} \neq [z_j]_{\rho_N} \) and \( J([z_i]_{\rho_N}, r_N) = \{ i_1 \} \). Write \( \phi \) for the labelling on \( \mathcal{N} \) which corresponds to \( r \) and write \( \phi' \) for the labelling on \( \mathcal{N} \) which corresponds to \( r_N \). Since \( x \) occurs in neither \( \phi(\alpha_N) \) nor \( \phi(\omega_N) \), \( [z_i]_{\rho_N} \) cannot occur in \( \phi'(\alpha_N) \) or \( \phi'(\omega_N) \). By Lemma 4.17, we have \( \phi'(\alpha_N) = \phi'(\omega_N) \). By Lemma 3.2, \( r_N \leq r \). Define a function \( f \) from \( \langle X; r_N \rangle \) to \( \langle X; R \rangle \) by \( f([z_i]_{\rho_N}) = [z_j]_{\rho} \). By Proposition 2.9, this is a well-defined homomorphism. Since \( f\phi'(e) = \phi(e) \) for every edge \( e \), we have \( \phi(\alpha_N) = \phi(\omega_N) \).

Lemma 4.19. Suppose that \( r \) is an achievable \((s, \ell)\) relator, that the letter \( x \) occurs in \( r \), that \( J(x, r) = \{ i_1, i_2 \} \), where \( s < i_1 < i_2 \), and that \( \mathcal{M} \) is an \( x\)-inceptive \((s, \ell)\) map for \( r \). Then no edge of \( \mathcal{M} \) is \( x\)-reflective.

Proof. We return to the notation for \( \mathcal{M} \) that was introduced following Lemma 4.8.

By (horizontal) symmetry, it will suffice to prove that no edge on \( \mu_2 \) or \( \nu_2 \) is \( x\)-reflective and that no interior edge of \( \mathcal{M}_r \) is \( x\)-reflective. By (vertical) symmetry, we may assume that \( e \) has index \( i_1 \) with respect to \( T \) and has index \( i_2 \) with respect to \( B \). Let \( f \) be the edge on the bottom side of \( T \) which has index \( i_2 \) with respect to \( T \).

The only edges on the top side of \( B \) which can have label \( x \) are those which have index \( i_1 \) or \( i_2 \) with respect to \( B \). Since \( e \) has label \( i_2 \) with respect to \( B \) and \( \nu_2 \) follows \( e \) on \( \omega_B \), no edge of \( \nu_2 \) can be \( x\)-reflective or even have label \( x \).

The edge \( f \) must occur on \( \mu_2 \) rather than on \( \gamma_2 \); it has label \( x \) so it cannot follow \( e \) on the top side of \( B \). Suppose, for the sake of contradiction, that \( f \) is \( x\)-reflective in \( \mathcal{M} \). Then Lemma 4.14 applies to \( f \) and \( e \) as edges of \( \mathcal{M} \) and \( e \) must be an \( x\)-reflective edge in \( \mathcal{M} \), also. This is impossible since \( e \) is the unique \( \{i_1, i_2\}\)-edge of \( \mathcal{M} \).

We need finally to show that no interior edge of \( \mathcal{M}_r \) can be \( x\)-reflective. We show that every edge of \( \mathcal{M}_r \) which has label \( x \) must occur on the bottom side of \( \mathcal{M}_r \).

TO BE COMPLETED
Lemma 4.20. Suppose that $r$ is an achievable $(s, \ell)$ relator, that the letter $x$ occurs in $r$, that $J(x, r) = \{i_1, i_2\}$, where $s < i_1 < i_2$, and that $M$ is an $x$-inceptive $(s, \ell)$ map for $r$. Then $|M| \leq \ell$.

Theorem 4.21. Suppose that $r$ is an achievable $(s, \ell)$ relator, that the letter $x$ occurs in $r$, that $|J(x, r)| = 2$, and that $M$ is an $x$-inceptive $(s, \ell)$ map for $r$. Then $|M| \leq \ell$.

Proof. This summarizes Lemmas 4.9, 4.12, and 4.20.

5. Local Computations: The Induction Step

Theorem 5.1. Suppose that $r$ is an achievable $(s, \ell)$ relator, that the letter $x$ occurs in $r$, that $|J(x, r)| = j$, and that $M$ is an $x$-inceptive $(s, \ell)$ map for $r$. Then $|M| \leq (j - 1)\ell$.

Theorem 5.2. Suppose that $r$ is an achievable $(s, \ell)$ relator and that $M$ is an inceptive $(s, \ell)$ map for $r$. Then $|M| \leq \ell(s + \ell - 2)$.

References