A SOLVABLE CONJUGACY PROBLEM FOR FINITELY PRESENTED C(3) SEMIGROUPS

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Abstract. In his thesis, Remmers [R], introduced semigroup derivation diagrams and used them to prove that the word problem for finitely presented C(3) semigroups was solvable. In this article we introduce the annular analog of semigroup derivation diagrams and use them to demonstrate the solution to a conjugacy problem for finitely presented C(3) semigroups.

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0. Introduction

As part of his thesis, Remmers [R] proved that if superfluous vertices are removed, then for any semigroup derivation diagram, over a C(3) semigroup, the number of edges along the diagram's left boundary is the same as the number of edges along its right boundary. The solution to the word problem for such finitely presented semigroups immediately followed. A complete proof can be found in the textbook [H], and we will often cite this source. In an analogous fashion, we will demonstrate that if we remove superfluous vertices, then in any feathered coannular diagram, over a C(3) semigroup, the number of edges along its inner boundary is the same as the number of edges along its outer boundary. The solution to the conjugacy problem will then be shown to immediately follow for such finitely presented semigroups. In Section 1 we construct feathered coannular diagrams and show how they model conjugacy, i.e. the equivalence relation determined by the transitive closure of the relation $xy \sim_p yx$. In Section 2 we provide all the geometric arguments needed to solve the conjugacy problem.

1. Feathered Coannular Diagrams and Conjugacy

The annular and coannular maps we first introduce below are meant only to serve as vehicles to introduce notation and provide a context for the construction of feathered coannular maps, a proper subclass of coannular maps as well as the featured maps of this article. Other than this and except for a few innocent comparative remarks, annular maps and coannular maps will play no roles in any proof. Those roles will be reserved for feathered coannular maps.

A map $\mathfrak{M}$ is a finite connected 2-complex in the plane. If p is a walk in the one-skelton of $\mathfrak{M}$, then $|p|$ will denote the number of edges traversed along the walk p, $Vert(p)$ will denote the set of vertices incident with p, and $|Vert(p)|$ will denote the number of vertices incident with p. So when p is a path, a simple cycle, or a

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walk that returns to multiple vertices then $|Vert(p)| = |p| + 1$, $|Vert(p)| = |p|$, or $|Vert(p)| < |p|$, respectively.

$\mathfrak{M}$ is **annular** if the complement of $\mathfrak{M}$ in the plane has exactly two components. Since $\mathfrak{M}$ is finite, one of these two components is bounded, while the other is unbounded. We will refer to the bounded component of the complement of an annular map $\mathfrak{M}$ as the **hole** of $\mathfrak{M}$. For any annular map $\mathfrak{M}$, there will be closed walks in the one-skeleton of $\mathfrak{M}$ which bound the hole of $\mathfrak{M}$ and closed walks in the one-skeleton of $\mathfrak{M}$ which bound $\mathfrak{M}$ together with its hole. A closed walk of minimal length which bounds the hole of $\mathfrak{M}$ is an inner boundary of $\mathfrak{M}$. A closed walk of minimal length which bounds $\mathfrak{M}$ together with its hole is an outer boundary of $\mathfrak{M}$. Cyclic permutations of inner boundaries are also inner boundaries and cyclic permutations of outer boundaries are also outer boundaries. We make a choice for an inner boundary and borrowing notation from [GuS] we denote it as $\mathbf{inn}(\mathfrak{M})$. Similarly we make a choice for an outer boundary and denote it by $\mathbf{out}(\mathfrak{M})$. We denote the initial and terminal vertex of $\mathbf{inn}(\mathfrak{M})$ by $v_{\mathbf{inn}(\mathfrak{M})}$ and similarly we denote the initial and terminal vertex of $\mathbf{out}(\mathfrak{M})$ by $v_{\mathbf{out}(\mathfrak{M})}$.

We assign a direction of flow to each edge in $\mathfrak{M}$'s one-skeleton, so that in this fashion we can think of the one-skeleton as also being a directed graph and we will denote this directed graph by $\Gamma_{\mathfrak{M}}$. So when we say p is a directed walk, it's to be understood that p is a walk in $\Gamma_{\mathfrak{M}}$. If D is a two-cell of $\mathfrak{M}$, we will say that D is **two-sided** if D has a boundary walk of the form pq^{-1} where p and q are directed walks. If p and q are minimal then we will call them **sides** of D. If vertex v_0 is the initial vertex of p then we also deem it as the initial vertex of the two-sided two-cell D and denote it as v_{0_D}. Similarly if vertex v_1 is the terminal vertex of p then we also deem it as the terminal vertex of D and denote it as v_{1_D}. Following Higgins' textbook [H, page 70], a vertex v in a directed graph Γ is a **source** if for any other vertex w in Γ there is a directed path p that begins at v and terminates at w. A **sink** is the dual concept. A vertex v in a directed graph is a **transmitter** if v has indegree 0 and is a **receiver** if v has outdegree 0. A vertex v is a **singularity** if it is either a transmitter or a receiver. We won't be concerned with sources and sinks in this work. We mention them only because transmitters and receivers are called sources and sinks in some places in the literature.

We will call $\mathfrak{M}$ a **coannular map** if $\mathfrak{M}$ is an annular map without singularities, whose two-cells are all two-sided, where both $\mathbf{inn}(\mathfrak{M})$ and $\mathbf{out}(\mathfrak{M})$ are simple directed cycles, and both $\mathbf{inn}(\mathfrak{M})$ and $\mathbf{out}(\mathfrak{M})$ are directed in the plane with counterclockwise orientations.

Since all the two-cells will be two-sided from this point forward in the article, we may simply refer to them as cells. Now for each coannular map $\mathfrak{M}$ we assign a counterclockwise orientation to $\mathfrak{M}$'s hole, the unbounded component, and each cell D of $\mathfrak{M}$. So if D is a cell of $\mathfrak{M}$ with sides p and q and the walk $v_{0_D} \cdot p \cdot v_{1_D} \cdot q^{-1} \cdot v_{0_D}$ traces out the boundary of D in a counterclockwise sense then we will call the side p the **positive side** of D and we will call side q the **negative side** of D. Of course, if the walk $v_{0_D} \cdot p \cdot v_{1_D} \cdot q^{-1} \cdot v_{0_D}$ traces out the boundary of D in a clockwise sense then we would call p the **negative side** of D and q the **positive side** of D. Henceforth we will denote the positve side of a cell D by β_D and its negative side by α_D. So if a directed edge e lies in $\mathfrak{M}$'s interior and lies on α_D for some cell D then because the plane is orientable e would also lie on β_E for some cell E.

We borrow the following notation from Remmers [R] also found in [H]. Let e be a directed edge along α_D or β_D. We then call e an **initial** edge of α_D or β_D, respectively, if the initial vertex of e is also the initial vertex v_{0_D} of D. We call e a **terminal** edge of α_D or β_D, respectively, if the terminal vertex of e is also the terminal vertex v_{1_D} of D. We call e an **intermediate** edge of α_D or β_D, respectively, if e is neither an initial edge nor a terminal edge of α_D or β_D, respectively.

A vertex v of $\mathfrak{M}$ is **superfluous** if its degree in $\mathfrak{M}$ is 2 and the directed edge e_1 with terminal vertex v is distinct from the directed edge e_2 with initial vertex v. If v is superfluous then we could "coalesce" v with e_1 and e_2 so as to form a single edge e whose initial vertex is the initial vertex of e_1 and whose terminal vertex is the terminal vertex of e_2. We then assign a positive direction to the new edge e that is consistent with both of the positive directions initially assigned to e_1 and e_2. In this way we say we **removed** the superfluous vertex v from $\mathfrak{M}$. Clearly if the superfluous vertex v is removed from $\mathfrak{M}$ then the resulting map is still coannular.

We now present a process for appending the negative side of a cell to the outer boundary of coannular map that will form the basis for the construction of **feathered coannular maps**. So let $\mathfrak{M}'$ be a coannular map and let D be a cell in the plane that is not a cell of the original complex $\mathfrak{M}'$ but where α_D is either a proper directed subpath of one of $\mathbf{out}(\mathfrak{M}')$'s cyclic permutations or else α_D is itself one of $\mathbf{out}(\mathfrak{M}')$'s cyclic permutations, where β_D is either a simple directed path or a simple directed cycle when α_D is, and where the interiors of both D and β_D lie in the unbounded region of $\mathfrak{M}'$'s complement. If this is the case, then we say D has been **appended** to $\mathfrak{M}'$ and we denote the new complex consisting of $\mathfrak{M}'$ and D as $\mathfrak{M}'\#D$. It is easy to see that no singularities are introduced after appending D to $\mathfrak{M}'$ and that the inner and outer boundaries of $\mathfrak{M}'\#D$ are directed cycles oriented counterclockwise. So $\mathfrak{M}'\#D$ is also a coannular map. The two examples, in Figure 1 on the next page, where $\mathfrak{M}'$ is a simple directed cycle C, will make this appending operation much clearer.

On the left side of Figure 1, the initial coannular map $\mathfrak{M}'$ is a simple directed cycle C, where $|C| = 5$. For the appended cell D we have $|\alpha_D| = 2$, so that in $C\#D$ we see that α_D is a proper subpath of one of $\mathbf{out}(C)$'s cyclic permutations. On the right side of Figure 1, the initial coannular $\mathfrak{M}'$ is again the same simple directed cycle C. For the appended cell E we have $|\alpha_E| = 5$, so that in $C\#E$ we now see that α_E is one of $\mathbf{out}(C)$'s cyclic permutations.

The examples of Figure 1 lead to our formal definition of feathered coannular maps. They are our annular version of the simply connected feathery maps constructed in [CJ], hence the reason for the modifier feathered. They are also equivalent to the annular maps constructed by Guba and Sapir in [GuS] but they use these maps for purposes which are different from ours.

Definition 1.1. *A coannular map is a feathered coannular map $\mathfrak{C}$ if*

(1) $\mathfrak{C}$ *is a simple directed cycle* C_0

or (2) $\mathfrak{C} = ((\cdots((C_0\#D_1)\#D_2)\#\cdots)\#D_{n-1})\#D_n$, *where each* D_i *is a two-sided two-cell.*

Remark 1.1. It is clear by construction that if an interior edge e of $\mathfrak{C}$ lies on some α_D then it also lies on some β_E where E and D must be distinct. In other words no

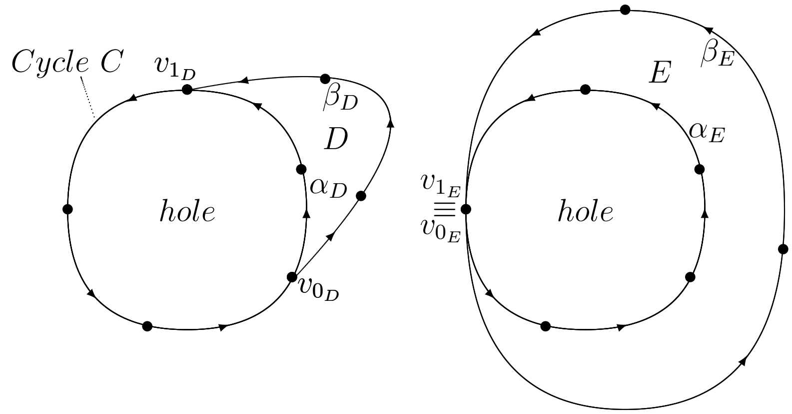

FIGURE 1. In the left figure α_D is a proper simple subpath of $\mathbf{out}(C)$ or one of its cyclic permutations. The vertices v_{0_D} and v_{1_D} are distinct. In the right figure α_E is either $\mathbf{out}(C)$ or one of its cyclic permutations. The vertices v_{0_E} and v_{1_E} are identified to one another.

edge e in a feathered coannular map can ever be traversed twice as one completely traces out a minimal boundary walk along the boundary of a cell that e lies on.

Remark 1.2. We had indicated at the beginning of Section 1 that the class of feathered coannular maps is a proper subclass of coannular maps. To verify this, we ask the reader to simply draw an annulus in the plane and orient both its boundaries counterclockwise, then draw a directed edge e with initial vertex v_0 on the inner boundary and terminal vertex v_1 on the outer boundary and such that the interior of e lies within the interior of the annulus. This annular complex is clearly a coannular map $\mathfrak{M}$ whose only cell D has as its negative side $\alpha_D \equiv v_0 \cdot \mathbf{inn}(\mathfrak{M}) \cdot e \cdot v_1$ and has as its positive side $\beta_D \equiv v_0 \cdot e \cdot \mathbf{out}(\mathfrak{M}) \cdot v_1$. Hence edge e is traversed twice as one traces out the boundary of D but this cannot happen for feathered coannular maps according to Remark 1.1 above.

Besides appending cells to the outer boundaries of maps we could just as easily append them to inner boundaries. We make precise this dual appending construction to the inner boundary in what follows. So let $\mathfrak{M}'$ be a coannular map and let E be a cell in the plane that is not a cell of the original complex $\mathfrak{M}'$ but where β_E is either a proper directed subpath of one of $\mathbf{inn}(\mathfrak{M}')$'s cyclic permutations or else β_D is itself one of $\mathbf{inn}(\mathfrak{M}')$'s cyclic permutations, where α_D is either a simple directed path or a simple directed cycle when β_D is, and where the interiors of both D and α_D lie in $\mathfrak{M}'$'s hole. If this is the case, then we also say E has been appended to $\mathfrak{M}'$ and we denote this new complex consisting of $\mathfrak{M}'$ and E as $E\#\mathfrak{M}'$. Clearly $E\#\mathfrak{M}'$ is also a coannular map and clearly we obtain the following proposition.

Proposition 1.1. *If $\mathfrak{C}$ is a feathered coannular map then $\mathfrak{C}$ is either*

(1) a simple directed cycle C_0

or *(2)* $\mathfrak{C} = E_n\#(E_{n-1}\#(...(E_2\#(E_1\#C_0))...))$, *where each E_i is a two-sided two-cell.* □

Remmers in [R] made a keen observation concerning how the directions of edges vary about vertices in the underlying maps of semigroup derivation diagrams, vary in the sense of when they point toward a vertex and point away from the same vertex. He had shown that as one makes a "complete circle" about any such vertex then the changes in edge directions one encounters from pointing toward the vertex to pointing away from the same vertex occurs at most twice. Precisely, Remmers defined a vertex v in a map $\mathfrak{M}$ to be **hyperbolic** if there exists a subgraph $\Gamma'_{\mathfrak{M}}$ of $\mathfrak{M}$'s directed one-skeleton $\Gamma_{\mathfrak{M}}$ such that the degree of v in $\Gamma'_{\mathfrak{M}}$ is four and if there exists both a small open neighborhood U in the plane about v and a homoemorphism f of the plane such that f preserves the directions assigned to $\mathfrak{M}$'s edges and such that the image of $\Gamma'_{\mathfrak{M}} \cap U$ under f is that found in Figure 2 below. Remmers then demonstrated that the underlying maps of semigroup derivation diagrams admit no hyperbolic vertices.

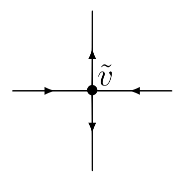

FIGURE 2. A degree four vertex $\tilde{v}$, the image of v under f. The edges alternate between pointing toward $\tilde{v}$ and pointing away from $\tilde{v}$.

Proposition 1.2. *Feathered coannular maps admit no hyperbolic vertices.*

Proof. The proof follows easily by the definition of feathered coannnular maps and by induction on the number of their cells. ☐

Remark 1.3. It can also be shown that coannular maps admit no hyperbolic vertices.

We now define our diagrams. First recall that a semigroup presentation $P = \langle X | r_1 = s_1, r_2 = s_2, r_3 = s_3, ...\rangle$ consists of a positive **alphabet**, namely the set X, and collection of equalities, namely the $r_i = s_i$'s. Each such equality $r_i = s_i$ is a **defining relation** and the individual r_i's and s_k's are **relators**. By the semigroup S defined by P we mean the quotient semigroup $F[X]/\rho$ where $F[X]$ is the free semigroup generated by P's alphabet X and ρ is the congruence generated by P's collection of defining relations. So in all of what follows, when we introduce a semigroup S, it will always be understood that S is defined in this fashion by some semigroup presentation P. So that P's alphabet X, collection of defining relations, and relators now become S's alphabet, collection of defining relations, and relators. Now let $\mathfrak{M}$ be a coannular map whose directed edges are labeled by non-empty words from the free semigroup $F[X]$ generated by S's alphabet X. So if p is a directed walk in $\Gamma_{\mathfrak{M}}$, then $\overline{p}$ will denote the word one reads as one traverses p and $|\overline{p}|$ will denote the number of letters comprising the word $\overline{p}$. If for each cell D of $\mathfrak{M}$ we have that the equality $\overline{\alpha_D} = \overline{\beta_D}$ or the equality $\overline{\beta_D} = \overline{\alpha_D}$ is one of S's defining relations then we will call such a labeled coannular map a **coannular diagram** $\mathfrak{M}$ over S. If such a labeled map is a feathered coannular map $\mathfrak{C}$ then we will call it a **feathered coannular diagram** $\mathfrak{C}$ over S.

We assume the reader is familiar with semigroup derivation diagrams, a thorough treatment can be found in the textbook [H]. We present a brief survey of them in what follows and then explain their connection to feathered coannular diagrams. A semigroup derivation diagram $\mathfrak{F}$ over a semigroup S is a labeled, connected, simply-connected two-complex in the plane whose underlying one-skeleton $\Gamma_{\mathfrak{F}}$ is a directed graph, where $\mathfrak{F}$'s interior contains no singularities, where $\mathfrak{F}$'s boundary is two-sided as well as all of its two-cells, and whose edges and boundaries along its two-sided two-cells are labeled in the same fashion as those in coannular diagrams. As a consequence, $\mathfrak{F}$ contains both a unique source vertex $v_{0_{\mathfrak{F}}}$ and a unique sink vertex $v_{1_{\mathfrak{F}}}$ that both lie on $\mathfrak{F}$'s boundary. If we denote the negative side of $\mathfrak{F}$ by $\alpha_{\mathfrak{F}}$ and its positive side by $\beta_{\mathfrak{F}}$, then both $\alpha_{\mathfrak{F}}$ and $\beta_{\mathfrak{F}}$ begin at $v_{0_{\mathfrak{F}}}$ and both terminate at $v_{1_{\mathfrak{F}}}$. The key fact about semigroup derivation diagrams is that two words u and w are equal in S if and only if there is a semigroup derivation diagram $\mathfrak{F}$ over S such $\overline{\alpha}_{\mathfrak{F}} \equiv u$ and $\overline{\beta}_{\mathfrak{F}} \equiv w$. Another important fact about semigroup derivation diagrams, this time what they share in common with feathered coannular diagrams, is that they can also be constucted by inductively appending cells. But in their case, the negative sides of cells are appended only to simple directed paths along the positive sides of the diagrams where those positive sides are themselves simple directed paths in contrast to feathered coannular diagrams where the negative sides of cells are inductively appended either to simple directed paths or simple directed cycles along outer boundaries, see [H] or [CJ] for the details. We are now in a position to make a few observations connecting semigroup derivation diagrams to feathered coannular diagrams that we gather in Propositions 1.3 and 1.4 below. But we need one more definition before we can begin their proofs. If x and y are words, in the alphabet X, at least one of which is non-empty and u and w are two non-empty words where $u \equiv xy$ and $yx \equiv w$ then we say u and w are **cyclic permutations** of each other.

Proposition 1.3. *Let $\mathfrak{F}$ be a semigroup derivation diagram over a semigroup S then there exists a feathered coannular diagram $\mathfrak{C}$ over S such that*

(1) $\mathfrak{C}$ contains the same number of cells as $\mathfrak{F}$,

(2) $|\mathbf{inn}(\mathfrak{C})| = |\alpha_{\mathfrak{F}}|$ and $|\mathbf{out}(\mathfrak{C})| = |\beta_{\mathfrak{F}}|$,

and (3) $\overline{\mathbf{inn}(\mathfrak{C})}$ and $\overline{\mathbf{out}(\mathfrak{C})}$ are cyclic permutations of $\overline{\alpha_{\mathfrak{F}}}$ and $\overline{\beta_{\mathfrak{F}}}$, respectively.

Proof. If we identify $\mathfrak{F}$'s source vertex $v_{0_{\mathfrak{F}}}$ with its sink vertex $v_{1_{\mathfrak{F}}}$ and name this identification as vertex v then we obtain a labeled annular map $\mathfrak{M}$ in the plane with no singularities, whose two-cells are two-sided, whose inner and outer boundaries can be traced out by simple directed cycles, and whose inner and outer boundaries share at least one vertex in common, namely vertex v. It's easy to see that one can identify $v_{0_{\mathfrak{F}}}$ with $v_{1_{\mathfrak{F}}}$ so that not only are $\mathbf{inn}(\mathfrak{M})$ and $\mathbf{out}(\mathfrak{M})$ both oriented counterclockwise but that their directed edges are those belonging to $\alpha_{\mathfrak{F}}$ and $\beta_{\mathfrak{F}}$, respectively. Now by the discussion above concerning how semigroup diagrams can be constructed by appropriately appending cells, it's easy to convince oneself that $\mathfrak{M}$ is in fact a feathered coannular diagram $\mathfrak{C}$. It's now clear that we obtain (1), (2), and (3) of the proposition's conclusion. $\square$

The next proposition "undoes" the identification process of the previous. We obtain similar conclusions but we insert one extra, numbered as (4) concerning the absence of superfluous vertices.

Proposition 1.4. *Let $\mathfrak{C}$ be a feathered coannular diagram over a semigroup S that shares a vertex v on both $\mathbf{inn}(\mathfrak{C})$ and $\mathbf{out}(\mathfrak{C})$, then there exists a semigroup derivation diagram $\mathfrak{F}$ over S such that*

(1) $\mathfrak{F}$ contains the same number of cells as $\mathfrak{C}$,

(2) $|\alpha_{\mathfrak{F}}| = |\mathbf{inn}(\mathfrak{C})|$ and $|\beta_{\mathfrak{F}}| = |\mathbf{out}(\mathfrak{C})|$,

(3) $\overline{\alpha_{\mathfrak{F}}}$ and $\overline{\beta_{\mathfrak{F}}}$ are cyclic permutations of $\overline{\mathbf{inn}(\mathfrak{C})}$ and $\overline{\mathbf{out}(\mathfrak{C})}$, respectively,

and (4) if it is the case that $\mathfrak{C}$ contains no superfluous vertices then neither does $\mathfrak{F}$.

Proof. If we "cut" the feathered coannular diagram $\mathfrak{C}$ at the vertex v common on both $\mathbf{inn}(\mathfrak{C})$ and $\mathbf{out}(\mathfrak{C})$ so as to "split" v into two separate vertices say v_0 and v_1, then we obtain in this manner a labeled, connected, simply-connected two-complex in the plane with no interior singularities and whose boundary as well as all of its two-cells are two-sided. In other words we have "undone" the identification process of the previous proposition. This new labeled two-complex is therefore a semigroup derivation diagram $\mathfrak{F}$ where v_0 and v_1 play the roles of $\mathfrak{F}$'s source and sink vertices, respectively. Its sides $\alpha_{\mathfrak{F}}$ and $\beta_{\mathfrak{F}}$ correspond to $\mathbf{inn}(\mathfrak{C})$ and $\mathbf{out}(\mathfrak{C})$, respectively. It's now clear that we obtain (1), (2), and (3) of the proposition's conclusion. It is also easy to see that we obtain conclusion (4) as well. □

The following definition for conjugacy can be found in [KM]. We say two non-empty words u and w are **primarily conjugate** to each other in S and write $u \sim_p w$ if there exist words x and y, allowing for one of x or y to be empty but not both, such that $u = xy$ and $w = yx$ in S. We say u is **conjugate** to w in S and write $u \sim w$ if there exists a finite sequence of non-empty words $u_1, u_2, ..., u_n$ such that $n \geq 2$, $u \equiv u_1$, $w \equiv u_n$, and $u_i \sim_p u_{i+1}$ for $1 \leq i < n$. Clearly $\sim$ is an equivalence relation on S. We note that if u and w are cyclic permutations of each other then clearly they are also primarily conjugate to each other as well.

We now demonstrate how feathered coannular diagrams model the equivalence relation $\sim$.

Theorem 1.1. *Two non-empty words u and w are conjugate to each other in $S \Leftrightarrow$ there exists a feathered coannular diagram $\mathfrak{C}$ over S such that u is a cyclic permutation of $\overline{\mathbf{inn}(\mathfrak{C})}$ and w is a cyclic permutation of $\overline{\mathbf{out}(\mathfrak{C})}$.*

Proof. $\Rightarrow$ Let $u_1, u_2, ..., u_n$ be a finite sequence of non-empty words where $n \geq 2$, $u \equiv u_1$, $w \equiv u_n$, and $u_i \sim_p u_{i+1}$ for $1 \leq i < n$. Consider the first two words in this sequence, u_1 and u_2. We therefore have $u_1 = xy$ and $yx = u_2$ in S for words x and y, not both empty. Hence there must exist two semigroup derivation diagrams $\mathfrak{F}_1$ and $\mathfrak{F}_2$ such that $u_1 \equiv \overline{\alpha_{\mathfrak{F}_1}}$, $xy \equiv \overline{\beta_{\mathfrak{F}_1}}$, $yx \equiv \overline{\alpha_{\mathfrak{F}_2}}$, and $u_2 \equiv \overline{\beta_{\mathfrak{F}_2}}$. Correspondingly, by Proposition 1.3's conclusion (3), there are two feathered coannular diagrams $\mathfrak{C}_1$ and $\mathfrak{C}_2$ such that u_1 and xy are cyclic permutations of $\overline{\mathbf{inn}(\mathfrak{C}_1)}$ and $\overline{\mathbf{out}(\mathfrak{C}_1)}$, respectively, and where yx and u_2 are cyclic permutations of $\overline{\mathbf{inn}(\mathfrak{C}_2)}$ and $\overline{\mathbf{out}(\mathfrak{C}_2)}$, respectively. Since $\overline{\mathbf{out}(\mathfrak{C}_1)}$ and $\overline{\mathbf{inn}(\mathfrak{C}_2)}$ are both cyclic permutations of xy, we can then identify $\mathbf{out}(\mathfrak{C}_1)$ with $\mathbf{inn}(\mathfrak{C}_2)$ in the obvious fashion to form a new feathered coannular diagram $\mathfrak{C}^1$ where clearly u_1 is a cyclic permutation of $\overline{\mathbf{inn}(\mathfrak{C}^1)}$ and u_2 is a cyclic permutation of $\overline{\mathbf{out}(\mathfrak{C}^1)}$. Similarly for each subsequent pair of words u_i and u_{i+1} of the sequence, construct a feathered coannular diagram $\mathfrak{C}^i$ where u_i is a cyclic permutation of $\overline{\mathbf{inn}(\mathfrak{C}^i)}$ and u_{i+1} is a cyclic permutation of $\overline{\mathbf{out}(\mathfrak{C}^i)}$. Hence we have a sequence of feathered coannular diagrams $\mathfrak{C}^1$, $\mathfrak{C}^2$, ..., $\mathfrak{C}^{n-1}$ where

$\overline{\mathbf{out}(\mathfrak{C}^i)}$ is a cyclic permutation of $\overline{\mathbf{inn}(\mathfrak{C}^{i+1})}$ for $1 \leq i \leq n-2$. Lastly form a feathered coannular diagram $\mathfrak{C}$ composed from this sequence by identifying, in the obvious fashion, $\mathbf{out}(\mathfrak{C}^i)$ with $\mathbf{inn}(\mathfrak{C}^{i+1})$ for $1 \leq i \leq n-2$. Since u and w are cyclic permutations of $\overline{\mathbf{inn}(\mathfrak{C}^1)}$ and $\overline{\mathbf{out}(\mathfrak{C}^{n-1})}$, respectively, then clearly u and w are also cyclic permutations of $\overline{\mathbf{inn}(\mathfrak{C})}$ and $\overline{\mathbf{out}(\mathfrak{C})}$, respectively.

$\Leftarrow$ We induct on n the number of appended cells D_1, D_2, ..., D_n such that $\mathfrak{C} = (C_0\#D_1)\#D_2)\#...)\#D_{n-1})\#D_n$, where C_0 is a simple directed cycle. If $n = 0$ then $\mathfrak{C}$ is the simple directed cycle C_0. Hence $\mathbf{inn}(C_0)$ and $\mathbf{out}(C_0)$ are cyclic permutations of each other and therefore the label $\overline{\mathbf{inn}(C_0)}$ is a cyclic permutation of the label $\overline{\mathbf{out}(C_0)}$. Now by assumption u and w are a cyclic permutations $\overline{\mathbf{inn}(C_0)}$ and $\overline{\mathbf{out}(C_0)}$, respectively. Hence u and w must be cyclic permutations of each other and therefore u and w are also primarily conjugate to each other. So the result follows for the case $n = 0$.

Suppose $n > 0$, so $\mathfrak{C} = \mathfrak{C}'\#D$ for some feathered coannular diagram $\mathfrak{C}'$ containing $n-1$ cells. First observe that there must exist a simple directed path q, possibly empty, along $\mathbf{out}(\mathfrak{C})$ such $\beta_D \cdot q$ is a cyclic permutation of $\mathbf{out}(\mathfrak{C})$. By assumption w is a cyclic permutation of $\overline{\mathbf{out}(\mathfrak{C})}$. Hence w is a cyclic permutation of $\overline{\beta_D} \cdot \overline{q}$. Hence $w \sim_p \overline{\beta_D} \cdot \overline{q}$. Since either the equality $\overline{\alpha_D} = \overline{\beta_D}$ or the equality $\overline{\beta_D} = \overline{\alpha_D}$ is one of S's defining relations, we then have that $\overline{\beta_D} \cdot \overline{q} = \overline{\alpha_D} \cdot \overline{q}$ in S. Hence $w \sim_p \overline{\alpha_D} \cdot \overline{q}$. Now clearly $\overline{\alpha_D} \cdot \overline{q}$ is a cyclic permutation of $\overline{\mathbf{out}(\mathfrak{C}')}$. Hence $w \sim \overline{\mathbf{out}(\mathfrak{C}')}$. By induction we have that $\overline{\mathbf{out}(\mathfrak{C}')} \sim \overline{\mathbf{inn}(\mathfrak{C}')}$. Hence $w \sim \overline{\mathbf{inn}(\mathfrak{C}')}$. Clearly $\overline{\mathbf{inn}(\mathfrak{C}')}$ is a cyclic permutation of $\overline{\mathbf{inn}(\mathfrak{C})}$. Hence $w \sim \overline{\mathbf{inn}(\mathfrak{C})}$. Since u, by assumption, is a cyclic permutation of $\overline{\mathbf{inn}(\mathfrak{C})}$ we therefore have that $w \sim u$ and the induction is complete. □

Remark 1.4. Goldstein and Teymouri, in [GT], introduced a notion of conjugacy that properly contains that notion under review in this article. It can be shown that coannular diagrams model their notion when S's presentation P contains no cycles in either it's left or right graphs or when S's congruency classes (determined by S's defining relations) are all finite in size such as in the cases when S is a finitely presented $C(3)$ semigroup or a finitely presented $C(2)$ & $T(4)$ semigroup. For the general case, coannular diagrams can be shown to model their notion when the conjugate elements being modeled are not also *null conjugate* as defined in [CJT], a note currently under preparation.

2. A Solvable Conjugacy Problem

Recall that a word z is a **piece** if z appears as a factor in two distinct locations of a single relator or z appears as a factor in two distinct relators. Also recall that S satisfies $C(n)$ if no relator of S can be written as a product of fewer that n pieces. The following three theorems due to Remmers appear as Theorems 5.2.13, 5.2.14, and 5.2.15 in [H]. We will essentially re-translate his theorems and arguments to obtain the analogous Theorem 2.4, Corollary 2.1, and Theorem 2.5, respectively.

Theorem 2.1. *(Remmers) If $\mathfrak{F}$ is a semigroup derivation diagram, without superfluous vertices, over a semigroup S satisfying C(3) then $|\alpha_{\mathfrak{F}}| = |\beta_{\mathfrak{F}}|$.*

Theorem 2.2. *(Remmers) Let S be a C(3) semigroup, let δ be an upper bound for the lengths of S's relators, and let u and w be two non-empty words. If $u = w$ in S then $|w| \leq \delta \cdot |u|$.*

Theorem 2.3. *(Remmers) If S is a finitely presented C(3) semigroup, then S has a solvable word problem.*

In the second half of Theorem 2.4's proof, it will be more convenient for us to argue that $|Vert(\mathbf{inn}(\mathfrak{C}))|$ is the same as $|Vert(\mathbf{out}(\mathfrak{C}))|$. If we can show this, then as $\mathbf{inn}(\mathfrak{C})$ and $\mathbf{out}(\mathfrak{C})$ are simple directed cycles, we would then have the theorem's result $|\mathbf{inn}(\mathfrak{C})| = |\mathbf{out}(\mathfrak{C})|$. We introduce the following notation and briefly survey the vertex counting strategy. If v is a vertex of $\mathfrak{C} = \mathfrak{C}'\#D$ and is neither v_{0_D} nor v_{1_D} but otherwise is incident with α_D, then we say D **covers** vertex v. Now the theorem assumes that $\mathfrak{C} = \mathfrak{C}'\#D$ contains no superfluous vertices. However the vertices v_{0_D} and v_{1_D} maybe superfluous in $\mathfrak{C}'$, so in order to revert successfully to the inductive case of $\mathfrak{C}'$ they would have to be removed. After their removal, we re-append D but now observe that v_{0_D} and/or v_{1_D} may contribute new vertices along $\mathbf{out}(\mathfrak{C})$ if they had been removed as superfluous vertices along $\mathbf{out}(\mathfrak{C}')$. The strategy is to show that when v_{0_D} and v_{1_D} contribute new vertices along $\mathbf{out}(\mathfrak{C})$ then D must cover at least as many vertices as were newly contributed. So we will argue that the net effect from appending cells, in the crucial case, is that $|Vert(\mathbf{out}(\mathfrak{C}))|$ could never exceed $|Vert(\mathbf{out}(\mathfrak{C}'))|$.

Theorem 2.4. *If $\mathfrak{C}$ is a feathered coannular diagram, without superfluous vertices, over a C(3) semigroup, then $|\mathbf{inn}(\mathfrak{C})| = |\mathbf{out}(\mathfrak{C})|$.*

Proof. We induct on n the number of cells of $\mathfrak{C}$. For $n = 0$, $\mathfrak{C}$ is a simple directed cycle C. By assumption $\mathfrak{C}$ contains no superfluous vertices and therefore $|\mathbf{inn}(\mathfrak{C})| = 1 = |\mathbf{out}(\mathfrak{C})|$.

Assuming $n > 0$, we have that $\mathfrak{C} = \mathfrak{C}'\#D$ for some feathered coannular map $\mathfrak{C}'$ with $n-1$ cells. We examine two cases. Case (i): $\mathbf{inn}(\mathfrak{C})$ and $\mathbf{out}(\mathfrak{C})$ are not disjoint. Case (ii): $\mathbf{inn}(\mathfrak{C})$ and $\mathbf{out}(\mathfrak{C})$ are disjoint.

Case (i): By Proposition 1.4's conclusion (2) there exists a semigroup derivation diagram $\mathfrak{F}$ where $|\alpha_{\mathfrak{F}}| = |\mathbf{inn}(\mathfrak{C})|$ and $|\beta_{\mathfrak{F}}| = |\mathbf{out}(\mathfrak{C})|$. Since $\mathfrak{C}$ contains no superfluous vertices then, by Proposition 1.4's conclusion (4), $\mathfrak{F}$ contains none as well. Therefore by Remmers, Theorem 2.1, $|\alpha_{\mathfrak{F}}| = |\beta_{\mathfrak{F}}|$. Hence $|\mathbf{inn}(\mathfrak{C})| = |\mathbf{out}(\mathfrak{C})|$.

Case (ii): We will first show that $|\mathbf{inn}(\mathfrak{C})| \geq |\mathbf{out}(\mathfrak{C})|$. The reverse inequality will be easily obtained by dual arguments provided in the final paragraph of this proof.

So first, if v_{0_D} and v_{1_D} are identified with each other on $\mathbf{out}(\mathfrak{C})$ then the positive side of D, β_D, is $\mathbf{out}(\mathfrak{C})$. Hence $\mathbf{out}(\mathfrak{C})$ consist of a single edge, namely β_D and since $\mathbf{inn}(\mathfrak{C})$ contains at least one edge, then $|\mathbf{inn}(\mathfrak{C})| \geq |\mathbf{out}(\mathfrak{C})|$.

Let's now assume that v_{0_D} and v_{1_D} are distinct on $\mathbf{out}(\mathfrak{C})$. This is where our vertex counting comes into play. Now the degrees of v_{0_D} and v_{1_D} are both at least 3 in $\mathfrak{C}$. We examine three subcases. Subcase(a): both their degrees exceed 3 in $\mathfrak{C}$. Subcase(b): exactly one has a degree 3 while the other exceeds 3 in $\mathfrak{C}$. Subcase(c): both have degree 3 in $\mathfrak{C}$. The accompanying Figures 3, 4, and both of 5 and 6 greatly simplify and concisely capture the spirit of the geometric arguments found in subcases (a), (b), and (c), respectively.

Subcase(a): As the degrees of both v_{0_D} and v_{1_D} exceed 3, it's then clear that the sub-diagram $\mathfrak{C}'$ contains no superfluous vertices. By our inductive hypothesis $|\mathbf{inn}(\mathfrak{C}')| = |\mathbf{out}(\mathfrak{C}')|$, hence $|Vert(\mathbf{inn}(\mathfrak{C}'))| = |Vert(\mathbf{out}(\mathfrak{C}'))|$. As we are under the assumption of case(ii) that $\mathbf{inn}(\mathfrak{C})$ and $\mathbf{out}(\mathfrak{C})$ are disjoint then it is easy to convince oneself that $Vert(\mathbf{inn}(\mathfrak{C})) = Vert(\mathbf{inn}(\mathfrak{C}'))$ and therefore $|Vert(\mathbf{inn}(\mathfrak{C}))| = |Vert(\mathbf{inn}(\mathfrak{C}'))|$. Combining we now have

$$(1)\ |Vert(\mathbf{inn}(\mathfrak{C}))| = |Vert(\mathbf{inn}(\mathfrak{C}'))| = |Vert(\mathbf{out}(\mathfrak{C}'))|.$$

Clearly by re-appending D to $\mathfrak{C}'$ we see that v_{0_D} and v_{1_D} contribute no new vertices along $\mathbf{out}(\mathfrak{C})$. Hence $|Vert(\mathbf{out}(\mathfrak{C}'))| \geq |Vert(\mathbf{out}(\mathfrak{C}))|$. Combining this with (1), we obtain $|Vert(\mathbf{inn}(\mathfrak{C}))| \geq |Vert(\mathbf{out}(\mathfrak{C}))|$. Hence $|\mathbf{inn}(\mathfrak{C})| \geq |\mathbf{out}(\mathfrak{C})|$. The inequality may even be strict if D could possibly cover any vertices as in Figure 3 below.

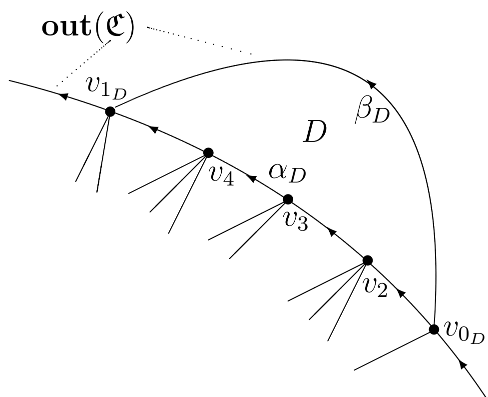

FIGURE 3. Vertices v_{0_D} and v_{1_D} contribute no new vertices along $\mathbf{out}(\mathfrak{C})$. But if D could cover vertices such as v_2, v_3, and v_4, then $|Vert(\mathbf{out}(\mathfrak{C}'))| > |Vert(\mathbf{out}(\mathfrak{C}))|$.

Subcase(b): Suppose v_{0_D} is the degree 3 vertex in $\mathfrak{C}$, the argument for v_{1_D} being the degree 3 vertex is similar and so is left to the reader. Clearly v_{0_D} is a superfluous vertex of the sub-diagram $\mathfrak{C}'$. We remove it and concatenate edge labels in the obvious manner to obtain a new feathered coannular diagram without superfluous vertices. In the interest of reducing notation we will keep the same name $\mathfrak{C}'$ for this new diagram. Just as in subcase(a) we still obtain the same equalities for the new $\mathfrak{C}'$ found in (1).

Since v_{0_D} is a degree 3 vertex in $\mathfrak{C}$ and has been removed as a superfluous vertex from the new $\mathfrak{C}'$, we see that after we re-append D to the new $\mathfrak{C}'$ then v_{0_D} contributes exactly one new vertex to $\mathbf{out}(\mathfrak{C})$. Of course if D covers at least one vertex then clearly $|Vert(\mathbf{out}(\mathfrak{C}'))| \geq |Vert\,\mathbf{out}(\mathfrak{C})|$ and by combining this with (1) we would again that $|Vert(\mathbf{inn}(\mathfrak{C}))| \geq |Vert\,\mathbf{out}(\mathfrak{C})|$ and therefore we would have obtained our desired inequality $|\mathbf{inn}(\mathfrak{C})| \geq |\mathbf{out}(\mathfrak{C})|$. We will argue by contradiction that D in fact must cover at least one vertex. So suppose D cover no vertices then, as $\mathbf{inn}(\mathfrak{C})$ and $\mathbf{out}(\mathfrak{C})$ are disjoint, α_D must nowhere be incident with $\mathbf{inn}(\mathfrak{C})$. Hence there must be some cell E of $\mathfrak{C}$ such that α_D lies along a subarc of β_E, the positive side of E, see Figure 4 on the next page. In Figure 4, edge e, incident with vertex v_{1_D}, also lies along the boundary of E. Now since the degree of v_{0_D} is 3 in $\mathfrak{C}$, it must be the case that α_D is either an intermediate or a proper terminal edge of

β_E, depending on the direction of edge e. In either case we would have that $\overline{\alpha_D}$ is a proper subword of $\overline{\beta_E}$. Hence $\overline{\alpha_D}$ would be a piece, contradicting C(3). Therefore D must cover at least one vertex.

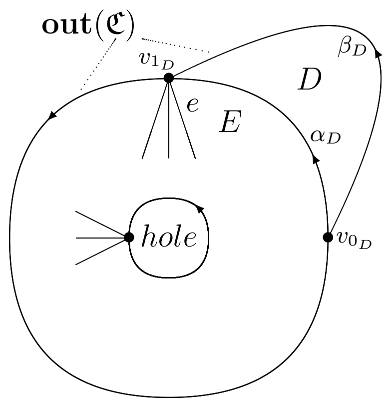

FIGURE 4. $\overline{\alpha_D}$ is a piece

Subcase(c): First, suppose that D covers no vertices. Hence, as both v_{0_D} and v_{1_D} are degree 3 vertices in $\mathfrak{C}$, it must be that α_D is an intermediate edge along the positive side β_E of some region E, by reasons similarly outlined in subcase(b). Hence $\overline{\alpha_D}$ would be a proper subword of $\overline{\beta_E}$ and therefore $\overline{\alpha_D}$ would be a piece, contradicting C(3). Therefore D must cover at least one vertex, say v_2. Clearly both v_{0_D} and v_{1_D} are superfluous vertices in the sub-diagram $\mathfrak{C}'$ and since D covers v_2 we see that we can remove both v_{0_D} and v_{1_D} from $\mathfrak{C}'$ and concatenate edge labels in the obvious manner to obtain a new feathered coannular diagram without superfluous vertices whose name we keep as $\mathfrak{C}'$ in the interest of reducing notation. As in subcases (a) and (b) we still obtain the equalities for the new $\mathfrak{C}'$ found in (1) above.

Since both v_{0_D} and v_{1_D} are degree 3 vertices in $\mathfrak{C}$ and since they have both been removed as a superfluous vertex from the new $\mathfrak{C}'$, we see that after we re-append D to the new $\mathfrak{C}'$ then v_{0_D} and v_{1_D} contribute exactly two new vertices to $\mathbf{out}(\mathfrak{C})$. Of course if D covers at least two vertices then again $|Vert(\mathbf{out}(\mathfrak{C}'))| \geq |Vert\,\mathbf{out}(\mathfrak{C})|$ and combining this with (1) we have $|Vert(\mathbf{inn}(\mathfrak{C}))| \geq |Vert\,\mathbf{out}(\mathfrak{C})|$ and therefore $|\mathbf{inn}(\mathfrak{C})| \geq |\mathbf{out}(\mathfrak{C})|$.

We will argue again by contradiction that D must cover at least two vertices. So suppose D covers only the one vertex v_2. Now vertex v_2 is either incident with $\mathbf{inn}(\mathfrak{C})$ or not. So we now break up subcase(c) into two further subcases to handle these two possibilities.

Subcase(c_1): v_2 is not incident with $\mathbf{inn}(\mathfrak{C})$. Now since neither v_{0_D}, v_{1_D}, nor v_2 is incident with $\mathbf{inn}(\mathfrak{C})$ then α_D factors as $v_{0_D} \cdot e \cdot v_2 \cdot f \cdot v_{1_D}$ where edge e lies along the positive side β_E of some cell E and where edge f lies along the positive side β_F of some cell F. We will examine the situations where E and F are distinct and when there are the same. As v_2 is not superfluous, we will also examine the situations where its degree is greater than 3 or exactly 3.

We will use Figure 5 located on page 13 as our guide for all of these situations even though Figure 5 seems to imply that E and F are distinct and that the degree of v_2 exceeds 3. So we take as our first situation that implied by Figure 5, the other situations will be handled similarly. Now the edges g and h in Figure 5 belong to the boundaries of cells E and F, respectively. It ought to be clear from Figure 5 that edge e is clearly an initial edge but not a terminal edge of α_D and again it ought to be clear that e is either a terminal edge or an intermediate edge of β_E depending on the direction of edge g. Hence $\overline{e}$ must be a piece. Similarly edge f is a terminal edge but not an initial edge of α_D and from Figure 5 again it ought to be clear that f is either an initial edge or an intermediate edge of β_F, depending on the direction of edge h. Hence $\overline{f}$ must be a piece. Therefore $\overline{\alpha_D}$ is the product of two pieces $\overline{e} \cdot \overline{f}$, contradicting C(3). Hence D covers at least two vertices. If E and F are distinct but the degree of v_2 is exactly 3, then g and h must be the same edge. Nevertheless, the same type of argument applies to produce the same contradiction. If E and F are the same but the degree of v_2 exceeds 3, then again the same type of argument produces the same contradiction. Lastly we note that that if E and F are the same then the degree of v_2 must exceed 3 for otherwise g and h would not only be the same edge but then this same edge would be traversed twice when tracing out the boundary of the cell E which by our observation in Remark 1.1 cannot happen in feathered coannular maps.

Subcase(c_2): v_2 is incident with $\mathbf{inn}(\mathfrak{C})$. See Figure 6 on the next page. We argue the same as in subcase(c_1) but because v_2 is not hyperbolic, by Proposition 1.2, and since $\mathbf{inn}(\mathfrak{M})$ and $\mathbf{out}(\mathfrak{M})$ are both directed counterclockwise we must have that edge e is proper terminal edge of β_E. Since e is also a proper initial edge of α_D then $\overline{e}$ must be a piece. Similarly edge f must be a proper intial edge of β_F and since f is also a proper terminal edge of α_D then again we must have that $\overline{f}$ is a piece. Therefore $\overline{\alpha_D}$ is he product of two pieces $\overline{e} \cdot \overline{f}$, contradicting C(3).

To obtain the reverse inequality $|\mathbf{out}(\mathfrak{C})| \geq |\mathbf{inn}(\mathfrak{C})|$, we first note, by Proposition 1.1, that $\mathfrak{C} = E\#\mathfrak{C}''$ for some feathered coannular map $\mathfrak{C}''$ containing $n-1$ cells. We then argue dually as in case(ii) subcases (a), (b), and (c) to obtain $|Vert(\mathbf{inn}(\mathfrak{C}''))| \geq |Vert(\mathbf{inn}(\mathfrak{C}))|$. Then combine this with the dual version of (1), namely

$$(1')\ |Vert(\mathbf{out}(\mathfrak{C}))| = |Vert(\mathbf{out}(\mathfrak{C}''))| = |Vert(\mathbf{inn}(\mathfrak{C}''))|$$

to obtain the desired reverse inequality.

□

Corollary 2.1. *Let S be a C(3) semigroup, let δ be an upper bound for the lengths of S's relators, and let u and w be two non-empty words. If u and w are conjugate to each other in S then $|w| \leq \delta \cdot |u|$.*

Proof. By Theorem 1.1 there is a feathered coannular diagram $\mathfrak{C}$ where u and w are cyclic permutations of $\overline{\mathbf{inn}(\mathfrak{C})}$ and $\overline{\mathbf{out}(\mathfrak{C})}$, respectively. WLOG we can assume that $\mathfrak{C}$ contains no superfluous vertices. Case(i): $\mathbf{inn}(\mathfrak{C})$ and $\mathbf{out}(\mathfrak{C})$ are not disjoint. Case(ii): $\mathbf{inn}(\mathfrak{C})$ and $\mathbf{out}(\mathfrak{C})$ are disjoint.

Case(i): Split $\mathfrak{C}$ at a vertex v common to both boundaries as in Proposition 1.4 to obtain a semigroup derivation diagram $\mathfrak{F}$ where $\overline{\alpha_{\mathfrak{F}}}$ is a cyclic permutation of $\overline{\mathbf{inn}(\mathfrak{C})}$ and therefore a cyclic permutation of u and where $\overline{\beta_{\mathfrak{F}}}$ is a cyclic permutation of $\overline{\mathbf{out}(\mathfrak{C})}$ and therefore a cyclic permutation of w. Since $\overline{\alpha_{\mathfrak{F}}} = \overline{\beta_{\mathfrak{F}}}$ in S then by

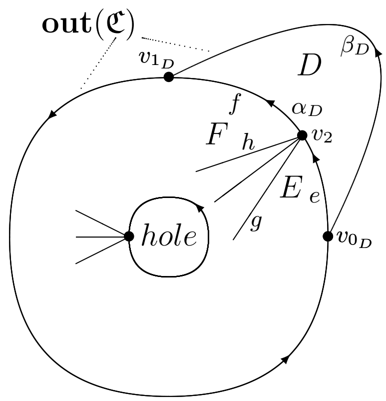

FIGURE 5. $\overline{\alpha_D}$ is a product of two pieces $\overline{e} \cdot \overline{f}$

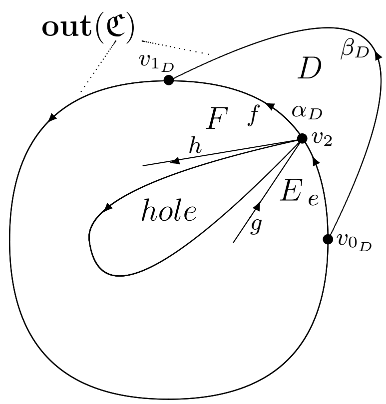

FIGURE 6. $\overline{\alpha_D}$ is a product of two pieces $\overline{e} \cdot \overline{f}$

Remmers, Theorem 2.2, we have $\left|\overline{\beta_{\mathfrak{F}}}\right| \leq \delta \cdot |\overline{\alpha_{\mathfrak{F}}}|$. Since $\overline{\alpha_{\mathfrak{F}}}$ and $\overline{\beta_{\mathfrak{F}}}$ are cyclic permutations of u and w, respectively, then $|\overline{\alpha_{\mathfrak{F}}}| = |u|$ and $\left|\overline{\beta_{\mathfrak{F}}}\right| = |w|$ and we have our result.

Case (ii): By Theorem 2.4 $|\mathbf{inn}(\mathfrak{C})| = |\mathbf{out}(\mathfrak{C})|$. Let integer $n = |\mathbf{inn}(\mathfrak{C})| = |\mathbf{out}(\mathfrak{C})|$. Hence there are two finite sequences of directed edges $e_1, e_2, ..., e_n$ and $f_1, f_2, ..., f_n$ where $\mathbf{inn}(\mathfrak{C}) \equiv v_{\mathbf{inn}(\mathfrak{C})} \cdot e_1 \cdot e_2 \cdot ... \cdot e_n \cdot v_{\mathbf{inn}(\mathfrak{C})}$ and $\mathbf{out}(\mathfrak{C}) \equiv v_{\mathbf{out}(\mathfrak{C})} \cdot f_1 \cdot f_2 \cdot ... \cdot f_n \cdot v_{\mathbf{out}(\mathfrak{C})}$. Now each such edge e_i and f_i must lie on some α_{E_i} and β_{D_i}, respectively, for some cells E_i and D_i, respectively. Therefore $\overline{e_i}$ and $\overline{f_i}$ are subwords of relators and hence $|\overline{e_i}| \leq \delta$ and $\left|\overline{f_i}\right| \leq \delta$. Therefore we have

$$\left|\overline{\mathbf{out}(\mathfrak{C})}\right| = \sum_{i=1}^{n} \left|\overline{f_i}\right| \leq n \cdot \delta = \delta \cdot |\mathbf{inn}(\mathfrak{C})| \leq \delta \cdot \left|\overline{\mathbf{inn}(\mathfrak{C})}\right|.$$

Since u and w are cyclic permutations of $\overline{\mathbf{inn}(\mathfrak{C})}$ and $\overline{\mathbf{out}(\mathfrak{C})}$, respectively, then $|u| = \left|\overline{\mathbf{inn}(\mathfrak{C})}\right|$ and $|w| = \left|\overline{\mathbf{out}(\mathfrak{C})}\right|$ and the proof is complete.

□

Corollary 2.2. *Let S be a finitely presented C(3) semigroup, let $|X|$ denote the size of its finite alphabet X, and let δ equal the length of S's longest relatior. If u is a non-empty word then there are at most $\sum_{i=1}^{\delta\cdot|u|}|X|^i$ distinct non-empty words conjugate to u.*

Proof. The sum $\sum_{i=1}^{\delta\cdot|u|}|X|^i$ gives the count of distinct non-empty words whose lengths do not exceed $\delta\cdot|u|$. By Corollary 2.1, if w is conjugate to u then $|w|\leq\delta\cdot|u|$. Hence the number of words conjugate to u cannot exceed $\sum_{i=1}^{\delta\cdot|u|}|X|^i$. □

Theorem 2.5. *If S is a finitely presented C(3) semigroup, then S has a solvable conjugacy problem.*

Proof. Let u be some non-empty element of S and let δ denote the length of the longest defining relator in the given presentation for S. If u_1 is another non-empty element of S where $u_1 = u$ in S, then u_1 is also conjugate to u in S. Hence the number of words equal to u in S cannot exceed the number of words conjugate to u in S and so by Corollary 2.2 there can be at most $\sum_{i=1}^{\delta\cdot|u|}|X|^i$ words equal to u in S. Therefore, since S's set of defining relations is finite, one could apply a standard algorithm to find all those words equal to u in S as did Remmers when he solved the word problem, Theorem 2.3. We now re-apply that same standard algorithm to find all pairs of words (x_1, y_1) such that $u = x_1y_1$ in S and allowing for one of x_1 or y_1 to be empty but certainly not both. Clearly there are only a finite number of such pairs. Next, apply the same algorithm to find all words w_1 such that $y_1x_1 = w_1$ in S for at least one of the previous pairs of (x_1, y_1)'s. The collection of all such w_1's is also clearly finite and clearly any word primarily conjugate to u must be one of these w_1's. Now re-iterate this procedure to find all such w_2's that are primarily conjugate to at least one of the preceding w_1's. Continuing this process indefinitely, we see that any word w conjugate to u must lie among the collection of such w_i's. By Corollary 2.2, we see that this procedure can produce no new words conjugate to u after the $\left(\sum_{i=1}^{\delta\cdot|u|}|X|^i\right)^{th}$ iteration. Hence the conjugacy problem is solvable. □

Corollary 2.3. *If S is a finitely presented C(n) semigroup where $n\geq 3$, then S has a solvable conjugacy problem.*

Proof. Clearly any $C(n)$ semigroup is a $C(3)$ semigroup when $n \geq 3$. □

Using Remmers' geometric methods, it was shown in [CG] that if $\mathfrak{F}$ is a minimal semigroup derivation diagram over a $C(2)$ & $T(4)$ semigroup whose set of defining relations is transitively closed, then $|\beta_{\mathfrak{F}}| \leq 2\cdot|\alpha_{\mathfrak{F}}|$ when $\mathfrak{F}$ contains no superfluous vertices. The solution to the word problem immediately followed for such finitely presented semigroups. We believe that the geometric methods developed in this article will produce identical bounds, namely $|\mathbf{out}(\mathfrak{C})| \leq 2\cdot|\mathbf{inn}(\mathfrak{C})|$, when $\mathfrak{C}$ is a minimal feathered coannular diagram, without superfluous vertices, over such a finitely presented semigroup. So we conjecture the following.

Conjecture 2.1. *If S is a finitely presented $C(2)$ & $T(4)$ semigroup whose set of defining relations is transitively closed, then S has a solvable conjugacy problem.*

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