Math 5220–Complex Analysis: Problem Set 9

Gill

Due: Friday April 13, 2018

Problem 1

For $f \in H(\mathbb{D}(0, R))$, define the *maximum modulus* $M(r) = M(r, f)$ of f by

$$M(r) = \sup_{|z|=r} |f(z)|, \qquad 0 \le r < R.$$

Prove: $M(r)$ is an nondecreasing function on $[0, R)$, and is strictly increasing unless f is constant.

Problem 2

Let $f, g \in H(\mathbb{D})$. Suppose that $f(0) = g(0)$, that g is 1-1, and that $f(\mathbb{D}) \subset g(\mathbb{D})$. Prove:

$$\text{(a)}|f'(0)| \le |g'(0)|, \qquad \text{and} \qquad M(r, f) \le M(r, g), \text{ for all } r \in (0, 1).$$

This is called the *principle of subordination.* f is said to be subordinate to g. For $g(z) = z$ it reduces to Schwarz's lemma.

(Hint: Be poetic; "compose in verse".)

Problem 3

Let

$$f(z) = \frac{1}{1 - z^2} + \frac{1}{3 - z}.$$

Compute the following five Laurent series,

(a) In powers of z to represent f in a neighborhood of 0.
(b) In powers of z to represent f in a certain nondegenerate annulus.
(c) In powers of z to represent f in a neighborhood of ∞.
(d) In powers of $z - 1$ to represent f in a neighborhood of $z_0 = 1$.
(e) In powers of $z - 1$ to represent f in a neighorhood of ∞.
In each case, describe the region of convergence.

Problem 4

Suppose that f is holomorphic in a neighborhood of z_0 and that f has a zero of order m at z_0. Prove that there exists a neighborhood V of z_0 and a function g holomorphic in V such that $f = g^m$ in V.

Problem 5

Prove *Hurwitz's Theorem*: If $\{f_n\}$ is a sequence of holomorphic 1-1 functions in a domain D which converge locally unifoirmly in D to a function f, then f is either 1-1 or is a constant.

Possible beginning of proof: If f is constant we are done. Suppose f is not constant. Suppose f is not 1-1. Then ....

Problem 6 Do not turn in

Let

$$f(z) = \log(1+z), \qquad g(z) = (1+z)^{\alpha}, \qquad \alpha > 0,$$

be the single valued holomorphic branches of those functions in $\mathbb{D}$ which satisfy $f(0) = 0$, $g(0) = 1$. Obtain the power series representations of f and g around $z_0 = 0$, and compute their radii of convergence. The series for g involves expressions which start off $\alpha(\alpha-1), \ldots$

Problem 7 Do not turn in

Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are holomorphic in $\mathbb{D}(0, R)$. Then so is fg, and we can write

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad z \in \mathbb{D}(0, R).$$

Find a formula for c_n in terms of a's and b's, and prove that it is correct. One way: Obtain a formula for $(fg)^{(n)}$.