Math 5220–Complex Analysis: Problem Set 8

Gill

Due: Friday April 6, 2018

Problem 1

Suppose that f is holomorphic in a neighborhood of 0, and that f has a zero of order $n \geq 2$ at 0. True or false: There exist arbitrarily small values of ϵ for which $f|_{\mathbb{D}(0,\epsilon)}$ is, taking multiplicity into acccount, an n-to-1 mapping of $\mathbb{D}(0,\epsilon)$ onto $f(\mathbb{D}(0,\epsilon))$. Suggestion: Study squares of simple functions.

Problem 2

Suppose that f is meromorphic in $\mathbb{C}^*$. Prove that f is rational.

Problem 3

Suppose that f is holomorphic in some domain which contains $\overline{\mathbb{D}}(0,1)$, and that $\sup_{|z|=1}|f| < 1$.

a. Prove that the equation $f(z) = z^n$ has exactly n solutions in $\mathbb{D}$, counting multiplicity. Meaning, for $n = 1$ show that f has a fixed point, and so on.

b. True or false? The same conlcusion holds if instead of $\sup_{|z|=1}|f| < 1$ we assume $\sup_{|z|=1}|f| \leq 1$.

Problem 4

Let f and g be entire functions. Suppose that there exists a constant C such that $|f(z)| \leq C|g(z)|$ for all $z \in \mathbb{C}$. True or false? There exists a constant C_1 such that $f(z) = C_1 g(z)$ for all $z \in \mathbb{C}$.

Problem 5

A holomorphic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in the unit disk $\mathbb{D}$ is called a Bloch functon if

$$\|f\|_{\mathcal{B}} := \sup_{|z|<1}(1-|z|^2)|f'(z)| < \infty.$$

a. Find a constant C_1 such that

$$\sup_{n\geq 1}|a_n| \leq C_1\|f\|_{\mathcal{B}}.$$

Thus, Bloch functions have bounded coefficients.

b. Find a constant C_2 such that

$$\|f\|_{\mathcal{B}} \leq C_2 \sup_{|z|<1}|f(z)|.$$

Thus, bounded holomorphic functions in $\mathbb{D}$, aka H^∞ functions, are Bloch functions.

Problem 6 **You may substitute this problem for one of Problems 1-4, but not 5**

Prove that the lacunary series $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ is a Bloch function, but is not in H^∞.