Math 5220–Complex Analysis: Problem Set 7

Gill

Due: Friday March 23

Problem 1

Let R_n be the square centered at the origin with vertices $\pm\pi(n+1/2) \pm i\pi(n+1/2)$. Prove that, with $z = x + iy$,

$$\sup_{n\in\mathbb{Z}^+} \sup_{z\in\partial R_n} \frac{e^y}{|\sin z|} < \infty$$

Problem 2

Let f be an entire function. Suppose that there is a constant C such that

$$|f(z)| \le Ce^{|y|}, \qquad \text{for all } z = x + iy \in \mathbb{C}. \tag{1}$$

Prove the *interpolation formula*

$$\frac{d}{dz}\frac{f(z)}{\sin z} = -\sum_{n=-\infty}^{\infty} \frac{(-1)^n f(n\pi)}{(z-n\pi)^2}, \qquad z \in \mathbb{C}\setminus\pi\mathbb{Z}.$$

Verify that the series converges absolutely for each $z \notin \pi\mathbb{Z}$, and that the convergence is locally uniform in $\mathbb{C}\setminus\pi\mathbb{Z}$.

The example $f(z) = \sin 2z$ shows that some growth condition such as (1) is necessary for the interpolation formula to hold. Taking $f(z) = \cos z$ in problem 2, one finds

$$\csc^2 \pi z = \pi^{-2} \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}, \qquad z \in \mathbb{C}\setminus\pi\mathbb{Z}. \tag{2}$$

Problem 3

Find constants a, b, c such that, as $z \to 0$,

$$\csc^2 z = az^{-2} + b + cz^2 + O(z^4)$$

where $f = g + O(h)$ means that $(f-g)/h$ is bounded as $z \to 0$.

Problem 4

Use the results of (2) to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Using the argument above, one can show that for every positive even integer m,

$$\zeta(m) = \sum_{n=1}^{\infty} n^{-m}$$

has the form $\pi^m q_m$, where q_m is rational.

Famous unsolved problem: is $\zeta(3) = \pi^3 q$, for some rational q?

Problem 5

Find $\zeta(4)$.

Problem 6

Prove carefully, using the method of residues, that

$$\int_{-\infty}^{\infty} \cos xt \frac{1}{1+x^2} dx = \pi e^{-|t|}, \qquad t \in \mathbb{R}.$$