Math 52201–Complex Analysis: Problem Set 6

Gill

Due: Friday, March 9

Problem 1

Suppose that f is an entire function, and that there exist positive constants A, B, α, not necessarily integers, such that

$$|f(z)| \leq A + B|z|^\alpha, \qquad z \in \mathbb{C}$$

Prove that f is a polynomial. What is the largest possible (integer) value of its degree?

Problem 2

Suppose that f is an entire function, and that there exist positive constants B and α such that

$$|f(z)| \leq Be^{|z|^\alpha}, \qquad z \in \mathbb{C} \tag{1}$$

(a) If $f(z) = \sum_{n=0}^{\infty} c_n z^n$, prove that

$$\limsup \frac{n \log n}{\log \frac{1}{|c_n|}} \leq \alpha. \tag{2}$$

Suggestion: For each n, pick a good r. Then use a problem from the (relatively near) future.

(b) If α is a positive integer, then $f(z) = e^{z^\alpha}$ is entire. Show that this f satisfies (1), and that equality holds in (2). Thus (2) is sharp for integral α. Later, we may show that (2) is sharp for all positive real α.

Problem 3

Suppose that $f(z)$ is holomorphic in $\mathbb{D} = \mathbb{D}(0,1)$.

(a) Prove that if $\sup_{\mathbb{D}} |f| \leq 1$, then $|c_n| \leq 1$ for each $n \geq 0$. The functions $f_n(z) = z^n$ show that these inequalities are sharp for each n.

(b) Suppose that $f \in H(\mathbb{D})$ satisfies

$$|f(z)| \leq (1 - |z|)^{-\alpha}, \qquad z \in \mathbb{D}. \tag{3}$$

Find a constant $C(\alpha)$, which may depend on α but not on n and f, such that

$$|c_n| \leq C(\alpha)(n+1)^\alpha, \qquad n \geq 0. \tag{4}$$

Even better, find an absolute constant C which works for all n, f, and α.

Problem 4

The Riemann zeta function is defined in $D = \{s \in \mathbb{C} : \Re s > 1\}$ by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

(a) Show that the series converges, and represents a holomorphic function in D.

(b) State and prove a series representation for $\zeta^{(k)}$, for $k \in \mathbb{Z}^+$.

Problem 5 Do not turn in.

Prove the following variant of Morera's Theorem: If $f \in C^1(D)$, where D is an open set, and if $\int_{\partial\Delta} f(z) = dz = 0$ for all disks Δ with $\overline{\Delta} \subset D$, then $f \in H(D)$. You may use Green's Theorem or Pompeiu's Formula. Can you relax the hypothesis from $f \in C^1(D)$ to $f \in C(D)$?

Problem 6 Do not turn in.

For $x \in (0, \infty)$, let $g(x) = x^{-\beta} e^{x^\alpha}$. There is a unique point $x_0 \in (0, \infty)$ at which g achieves a minimum value on $(0, \infty)$. Find x_0 and $g(x_0)$.

Problem 7 Do not turn in.

For each of the following functions, determine the order of the zero at $z_0 = 0$.

$$\text{(a)} z^2(e^{z^3} - 1) \qquad \text{(6)} 6\sin(z^4) + z^4(z^8 - 6).$$

Problem 8 Do not turn in.

In (4), is the exponent α best possible? Or does there exist $\beta < \alpha$ such that (4) holds with β in place of α, for all f satisfying (3) an for all n?