Math 5220 –Complex Analysis: Problem Set 4

Gill

Due: Friday, February 16

From $e^{2i\theta} = (\cos\theta + i\sin\theta)^2$ and $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$, by equating real and imaginary parts, one can derive formulas such as

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \quad \text{and} \quad \cos^2\theta = \frac{1}{2}\cos 2\theta + \frac{1}{2}.$$

Problem 1

Express $\cos 5\theta$ and $\sin 5\theta$ as polynomials in $\cos\theta$ and $\sin\theta$. Express $\cos^5\theta$ in the form $\sum_{j=0}^{5} a_j \cos j\theta$.

Problem 2

Find an algebraic expression for $\cos\frac{\pi}{12}$.

Problem 3

(a) Find formulae for $\Re\cos z, \Im\cos z$, and $|\cos z|$. Render them as simple as you can.

(b) Find all solutions of $\cos z = 0$ and of $\cos z = 3i$. To describe the latter, you may, for $t \in \mathbb{R}$, denote by $\sinh^{-1} t$ the unique real number s such that $\sinh s = t$. In both cases, sketch the solutions in the plane.

(c) The images of horizontal and of vertical lines under the mapping $w = \cos z$ are curves familiar from calculus. Figure out what they are and make appropriate sketches in the z and w planes.

Problem 4

Let e and π be the numbers defined in Theorem 10. Prove that $2 < e < 3$ and $2 < \pi < 4$. Your proofs must be analytic. Estimates obtained via processing power will not constitute sufficient proof.